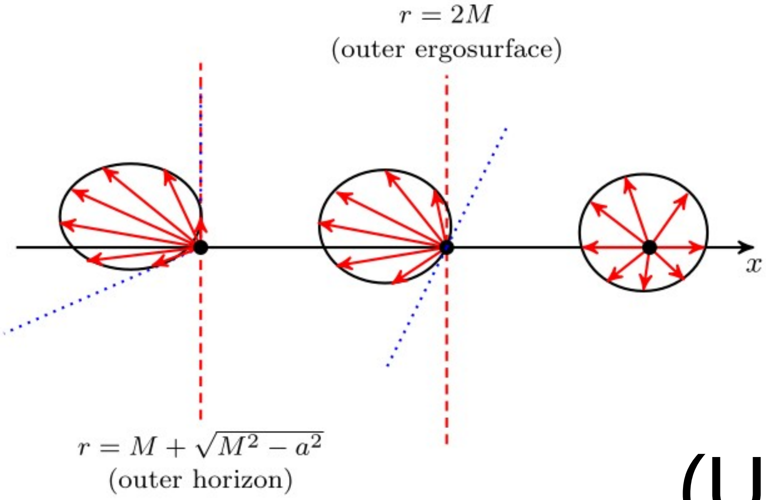
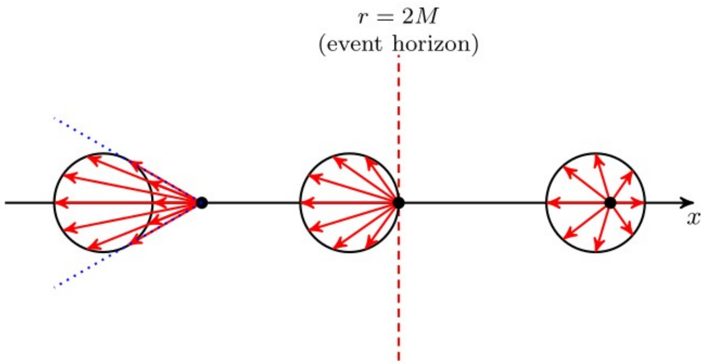
Finslerian structure of black holes



Quantum Summer 2025
Iriri, April 4th – 11th

Alberto Saa
(UNICAMP - Brazil)



Quantum Summer 2025

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Site em português

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The meeting gathers international researchers and graduate students on different aspects of cosmology, gravitation, and quantum field theory. It includes both seminars (with a focus on recent research developments) and mini-courses (with a more introductory approach that is helpful to both students and researchers from different areas).

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Kirill Alexandrovich Bronnikov

2346 J. Math. Phys. 37 (5), May 1996 0022-2488/96/37(5)/2346/6/$10.00

New no-scalar-hair theorem for black holes

Alberto Saa
Departamento de Física de Partículas, Universidade de Santiago de Compostela, 15706 Santiago de Compostela, Spain

(Received 13 October 1995; accepted for publication 2 February 1996)

A new no-hair theorem is formulated which rules out a very large class of non-minimally coupled finite scalar dressing of an asymptotically flat, static, and spherically symmetric black hole. The proof is very simple and based on a covariant method for generating solutions for nonminimally coupled scalar fields starting from the minimally coupled case. Such a method generalizes the Bekenstein method for conformal coupling and other recent ones. We also discuss the role of the finiteness assumption for the scalar field. © *1996 American Institute of Physics.* [S0022-2488(96)00905-1]

Regular Phantom Black Holes

K. A. Bronnikov[1,2,*] and J. C. Fabris[3,†]

[1]Center for Gravitation and Fundamental Metrology, VNIIMS, 46 Ozyornaya Street, Moscow 119361, Russia
[2]Institute of Gravitation and Cosmology, PFUR, 6 Miklukho-Maklaya Street, Moscow 117198, Russia
[3]Departamento de Física, Universidade Federal do Espírito Santo, Vitória, 29060-900, Espírito Santo, Brazil
(Received 24 November 2005; published 27 June 2006)

We study self-gravitating, static, spherically symmetric phantom scalar fields with arbitrary potentials (favored by cosmological observations) and single out 16 classes of possible regular configurations with flat, de Sitter, and anti–de Sitter asymptotics. Among them are traversable wormholes, bouncing Kantowski-Sachs (KS) cosmologies, and asymptotically flat black holes (BHs). A regular BH has a Schwarzschild-like causal structure, but the singularity is replaced by a de Sitter infinity, giving a hypothetic BH explorer a chance to survive. It also looks possible that our Universe has originated in a phantom-dominated collapse in another universe, with KS expansion and isotropization after crossing the horizon. Explicit examples of regular solutions are built and discussed. Possible generalizations include *k*-essence type scalar fields (with a potential) and scalar-tensor gravity.

DOI: 10.1103/PhysRevLett.96.251101 PACS numbers: 04.70.Bw, 95.35.+d, 98.80.-k

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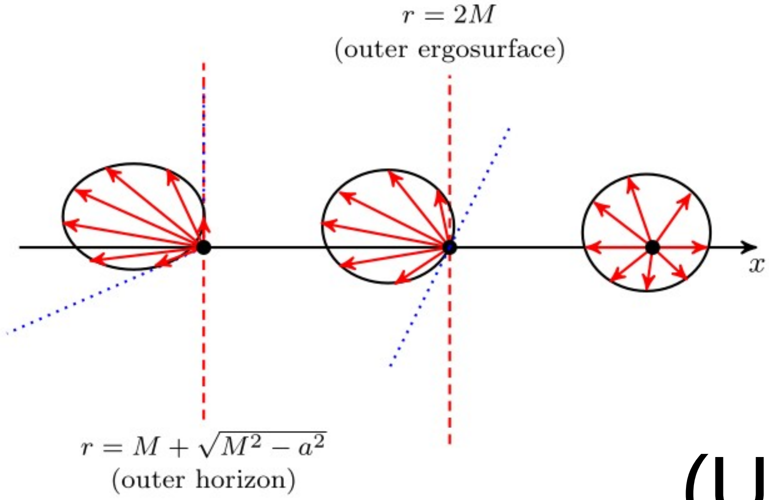
Finslerian structure of black holes



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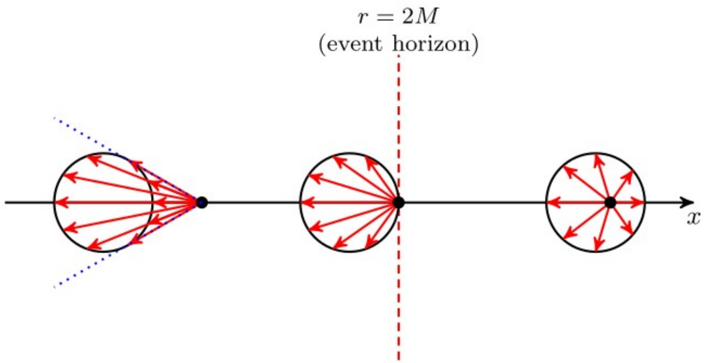

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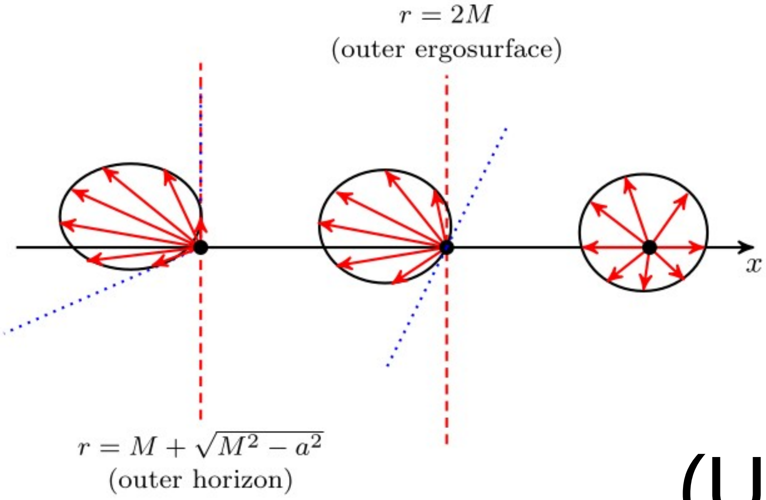

Quantum Summer 2025
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(UNICAMP - Brazil)

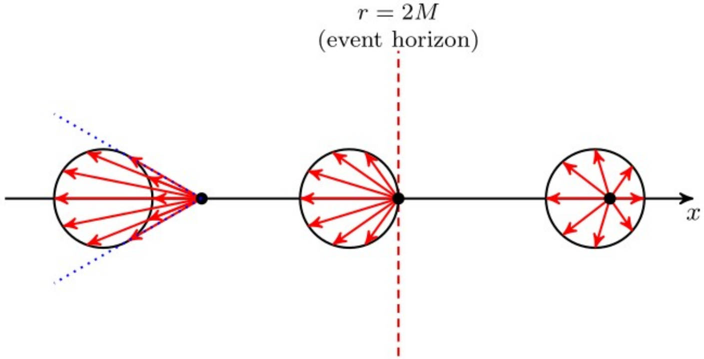
arXiv > gr-qc > arXiv:2501.09536

General Relativity and Quantum Cosmology

[Submitted on 16 Jan 2025]

Finslerian structure of black holes

Hengameh R. Dehkordi, Mauricio Richartz, Alberto Saa



Outline

- Finsler geometry (FG)
- Zermelo navigation problem and FG
- Painlevé-Gullstrand coordinates and the Finslerian structure of black holes
  - Spherical static cases (Schwarzschild)
  - Kerr black holes

Finsler geometry

Finsler geometry

Finsler geometry

Bernhard Riemann

Dissertation (Hab.) (1854)

Ueber

die Hypothesen, welche der Geometrie zu Grunde liegen.

Von

B. Riemann.

Aus dem Nachlass des Verfassers mitgetheilt durch R. Dedekind[1]).

Plan der Untersuchung.

Bekanntlich setzt die Geometrie sowohl den Begriff des Raumes, als die ersten Grundbegriffe für die Constructionen im Raume als etwas Gegebenes voraus. Sie giebt von ihnen nur Nominaldefinitionen, während die wesentlichen Bestimmungen in Form von Axiomen auftreten. Das Verhältniss dieser Voraussetzungen bleibt dabei im Dunkeln; man sieht weder ein, ob und in wie weit ihre Verbindung nothwendig, noch a priori, ob sie möglich ist.

Diese Dunkelheit wurde auch von Euklid bis auf Legendre, um den berühmtesten neueren Bearbeiter der Geometrie zu nennen, weder von den Mathematikern, noch von den Philosophen, welche sich damit beschäftigten, gehoben. Es hatte dies seinen Grund wohl darin, dass der allgemeine Begriff mehrfach ausgedehnter Grössen, unter welchem die Raumgrössen enthalten sind, ganz unbearbeitet blieb. Ich habe mir daher zunächst die Aufgabe gestellt, den Begriff einer mehrfach ausgedehnten Grösse aus allgemeinen Grössenbegriffen zu construiren. Es wird daraus hervorgehen, dass eine mehrfach ausgedehnte Grösse ver-

1) Diese Abhandlung ist am 10. Juni 1854 von dem Verfasser bei dem zum Zweck seiner Habilitation veranstalteten Colloquium mit der philosophischen Facultät zu Göttingen vorgelesen worden. Hieraus erklärt sich die Form der Darstellung

Finsler geometry

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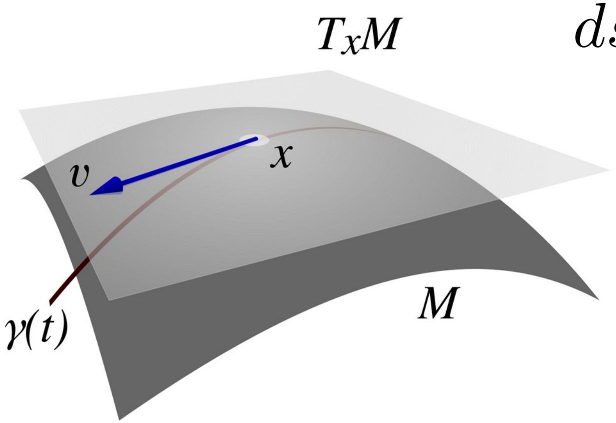

$$ds = F(x^1, \ldots, x^n; dx^1, \ldots, dx^2)$$

Finsler geometry

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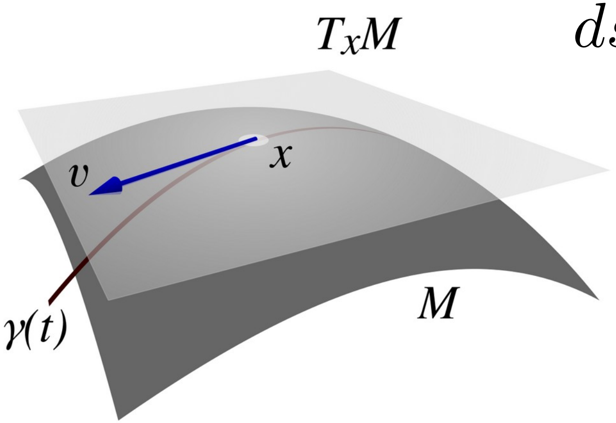

$$ds = F(x^1, \ldots, x^n; dx^1, \ldots, dx^2)$$

Quadratic restriction

$$F^2(x^k; dx^k) = g_{ij}(x) dx^i dx^k$$

Finsler geometry

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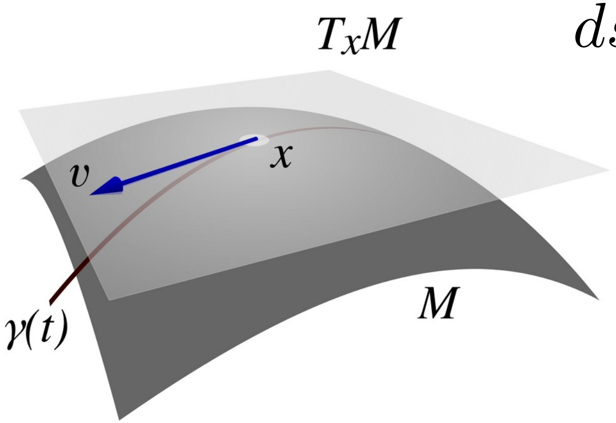

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Riemannian

Finsler geometry

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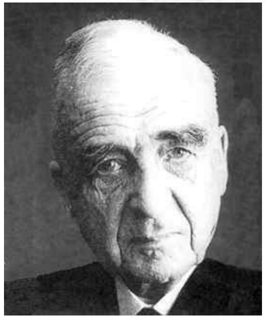
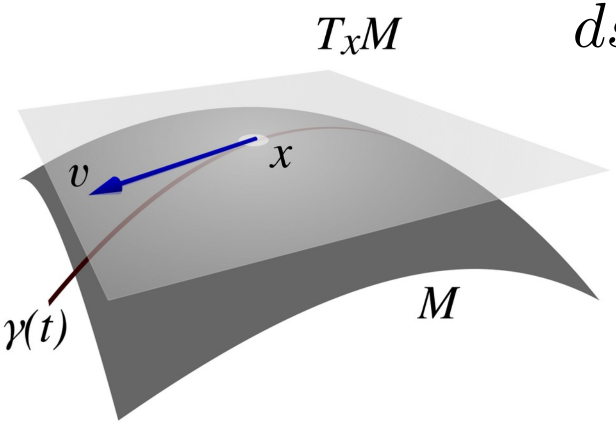


Paul Finsler



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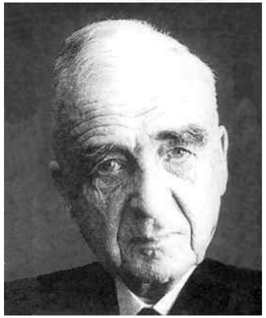

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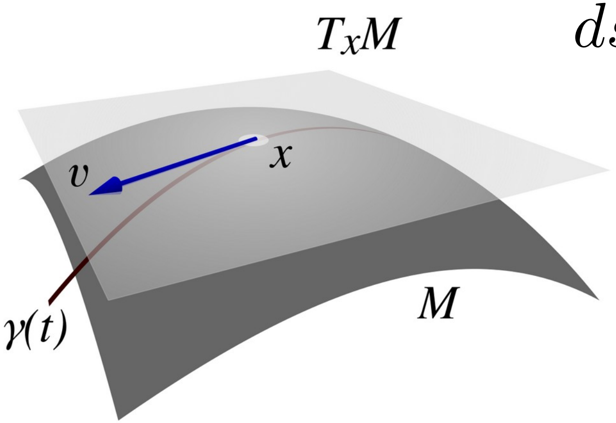

$$ds = F(x^1, \dots, x^n; dx^1, \dots, dx^2)$$

Without the quadratic restriction

$$F^2(x^k; dx^k) = g_{ij}(x) dx^i dx^k$$

Finslerian (1918)

Finsler geometry

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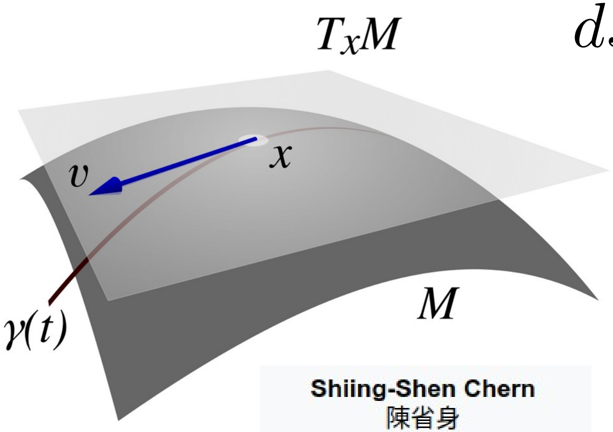
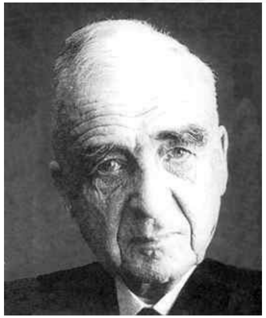
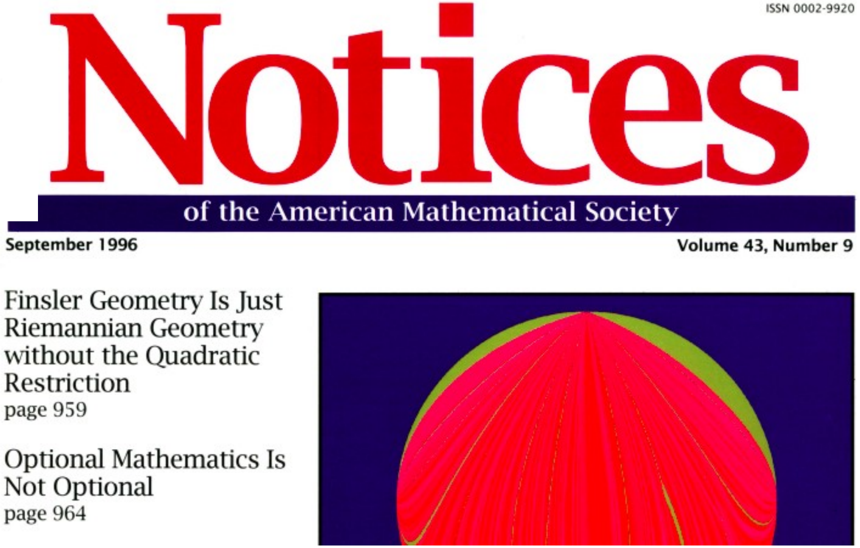




Paul Finsler

Shiing-Shen Chern
陳省身



ISSN 0002-9920

Notices of the American Mathematical Society

September 1996 Volume 43, Number 9

Finsler Geometry Is Just Riemannian Geometry without the Quadratic Restriction
page 959

Optional Mathematics Is Not Optional
page 964

Finsler geometry in Physics

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JANUARY 15, 1941 PHYSICAL REVIEW VOLUME 59

On an Asymmetrical Metric in the Four-Space of General Relativity

GUNNAR RANDERS
Yerkes Observatory of the University of Chicago, Chicago Illinois

(Received October 1, 1940)

Physical space-time allows of a metric of asymmetrical properties because of the unidirection of time. The simplest possible asymmetrical generalization of Riemannian metric is considered. The study of this metric is here mainly confined to the mathematical aspects. However, the physical consequences by application to space-time are obvious, and may be of interest by leading directly to a description of the electromagnetic field. A close formal connection with five-dimensional Riemannian geometry is shown, which might carry interest for the physical interpretation of five-dimensional relativity.

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$$ds = F(\mathbf{x}, d\mathbf{x}) = \sqrt{a_{\mu\nu}(x^\sigma)dx^\mu dx^\nu} + b_\mu(x^\sigma)dx^\mu$$

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PHYSICAL REVIEW D VOLUME 48, NUMBER 8 15 OCTOBER 1993

Relation between physical and gravitational geometry

Jacob D. Bekenstein*

Department of Physics, University of California at Santa Barbara, Santa Barbara, California 93106 and The Racah Institute of Physics, Hebrew University of Jerusalem, Givat Ram, Jerusalem 91904, Israel†

(Received 19 November 1992)

The appearance of two geometries in a single gravitational theory is familiar. Usually, as in the Brans-Dicke theory or in string theory, these are conformally related Riemannian geometries. Is this the most general relation between the two geometries allowed by physics? We study this question by supposing that the physical geometry on which matter dynamics takes place could be Finslerian rather than just Riemannian. An appeal to the weak equivalence principle and causality then leads us to the conclusion that the Finsler geometry has to reduce to a Riemann geometry whose metric, the physical metric, is related to the gravitational metric by a generalization of the conformal transformation involving a scalar field.

Zermelo navigation problem

Zermelo navigation problem

- Optimal control problem (1931)

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A (non-relativistic) particle moves with constant (unit) velocity V under the influence of a given time-independent drift (wind) vector field W.

Find the minimum-time trajectory between to fixed points A and B.

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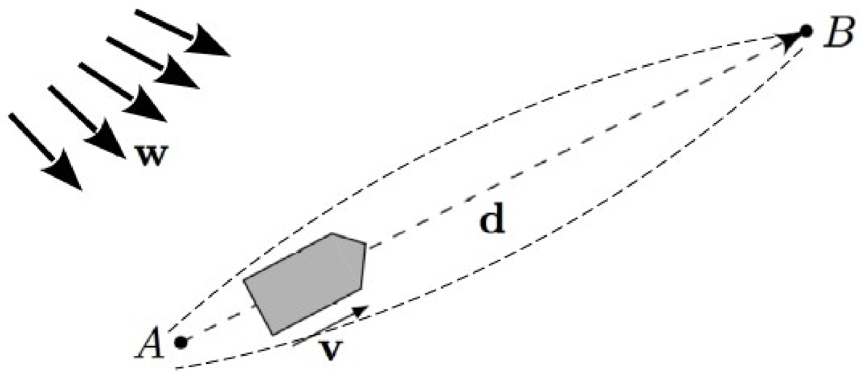

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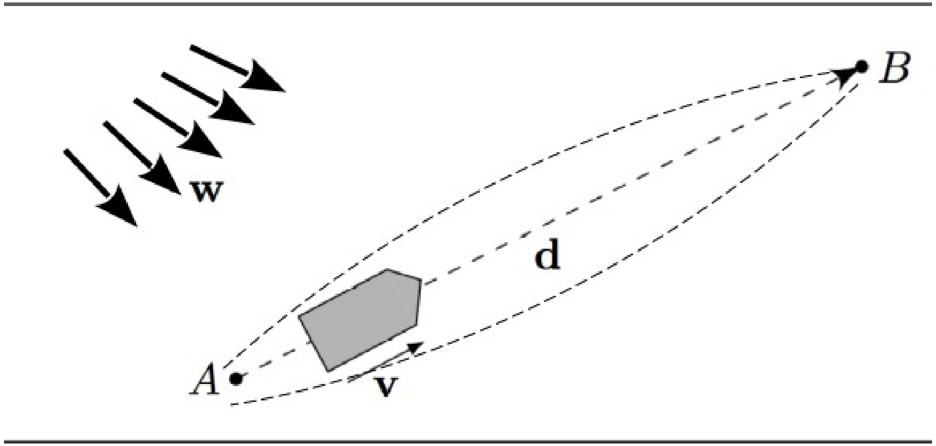

Ernst Zermelo

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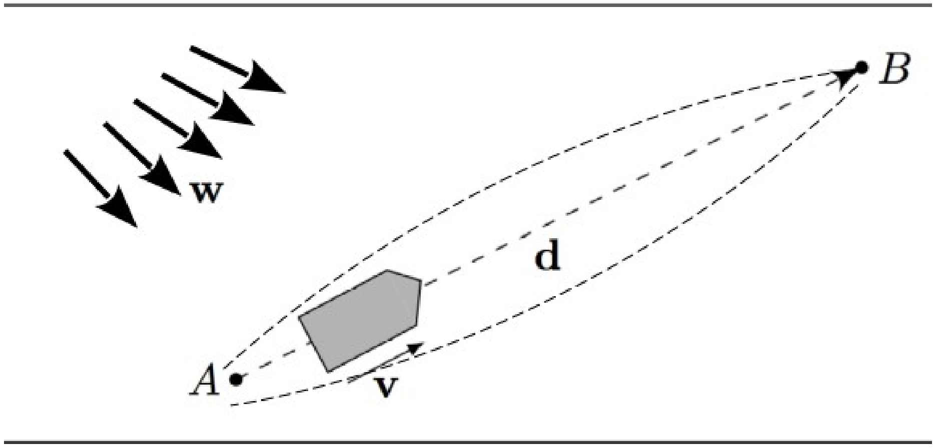

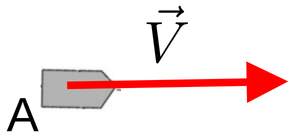

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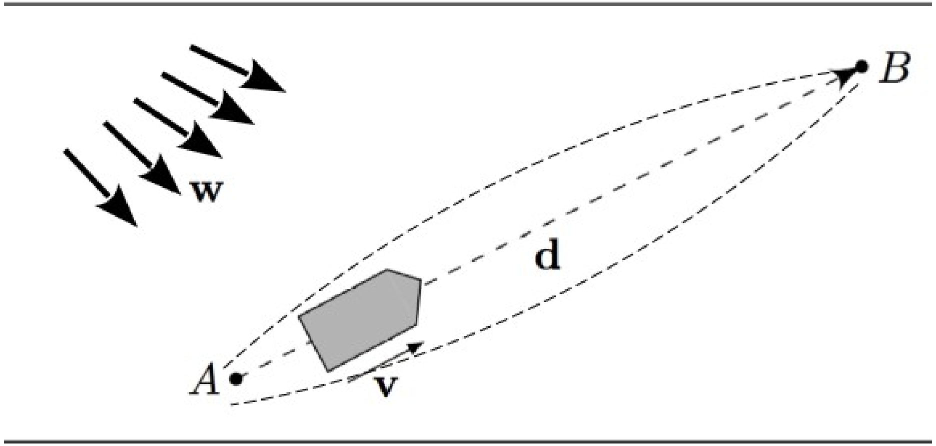

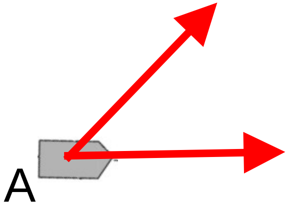

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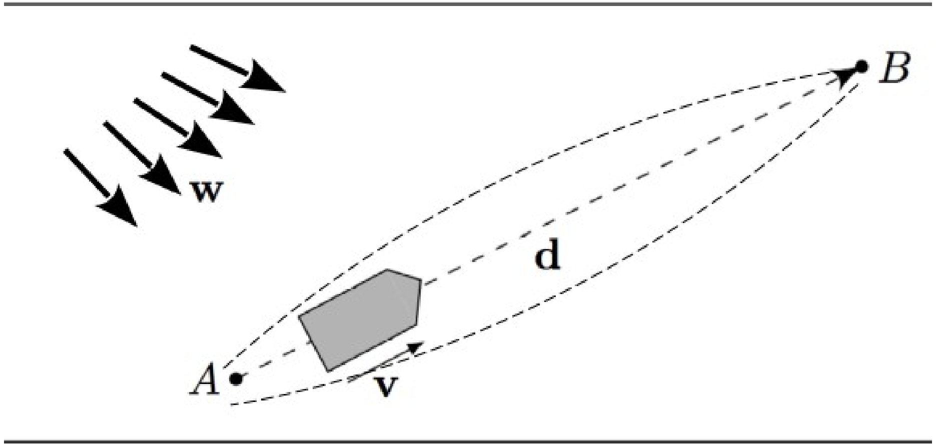

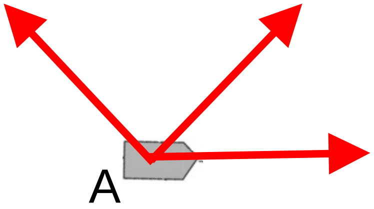

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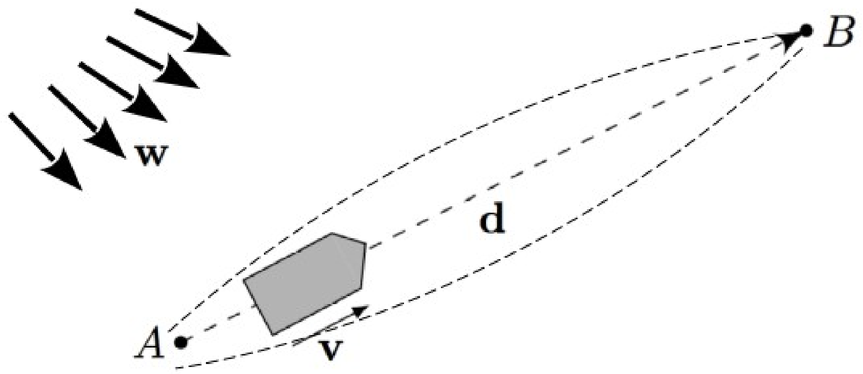

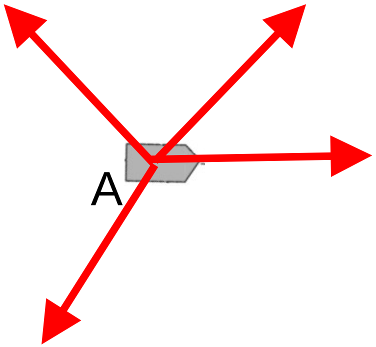

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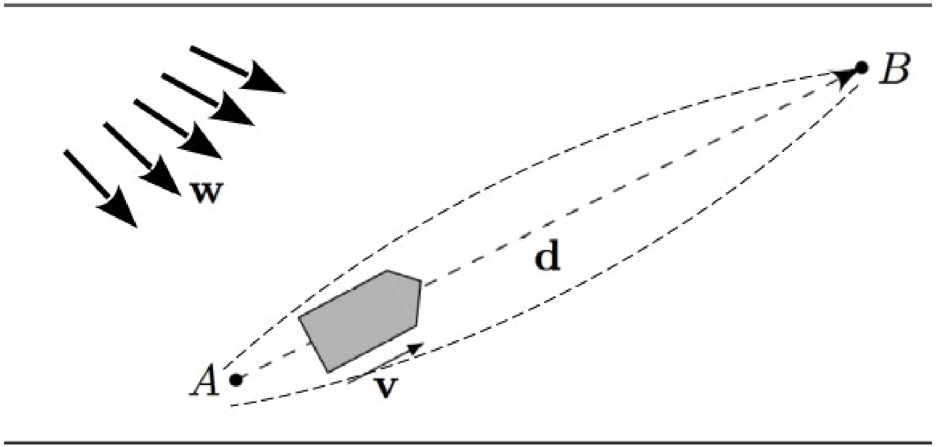

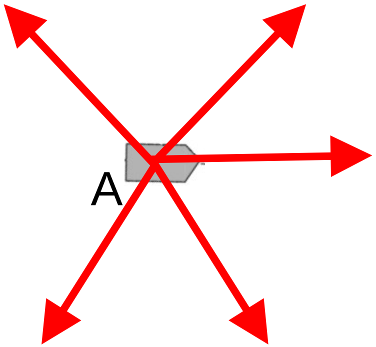

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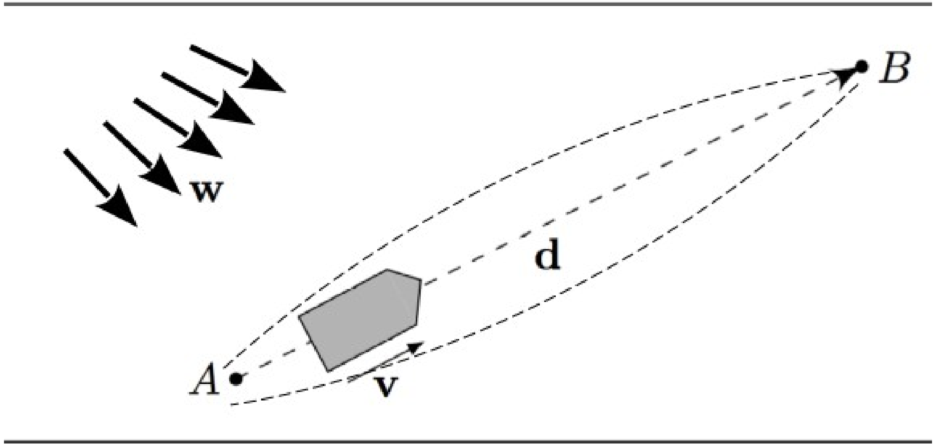

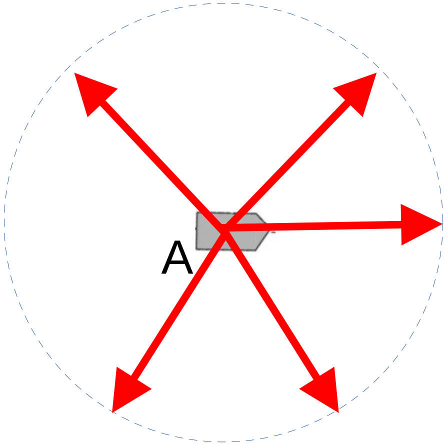

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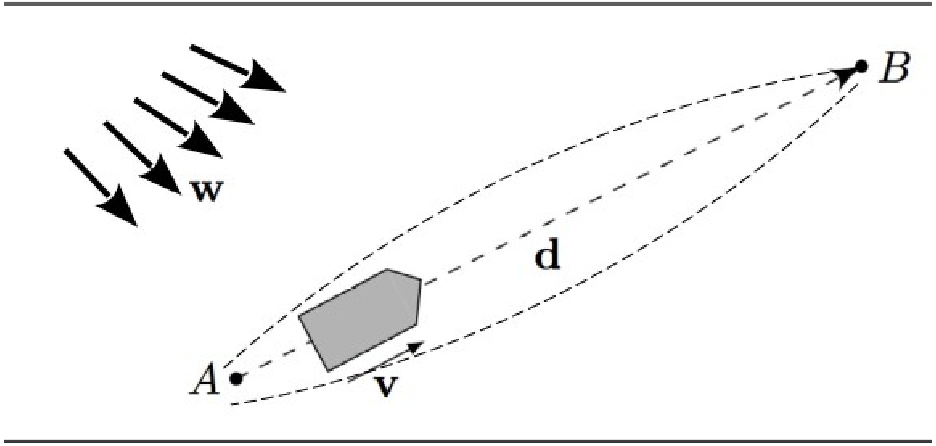

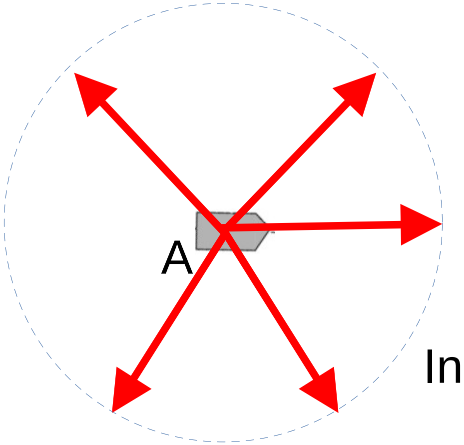

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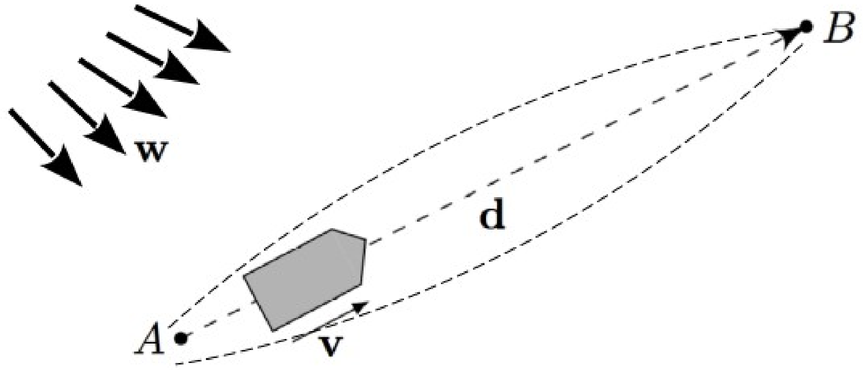

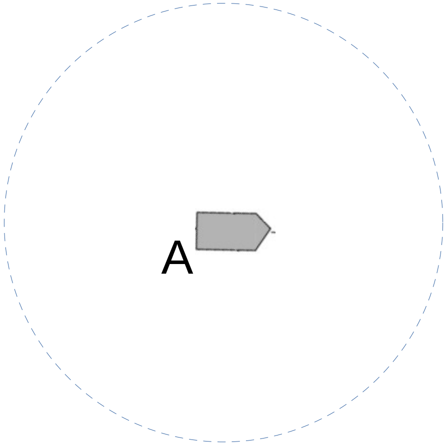

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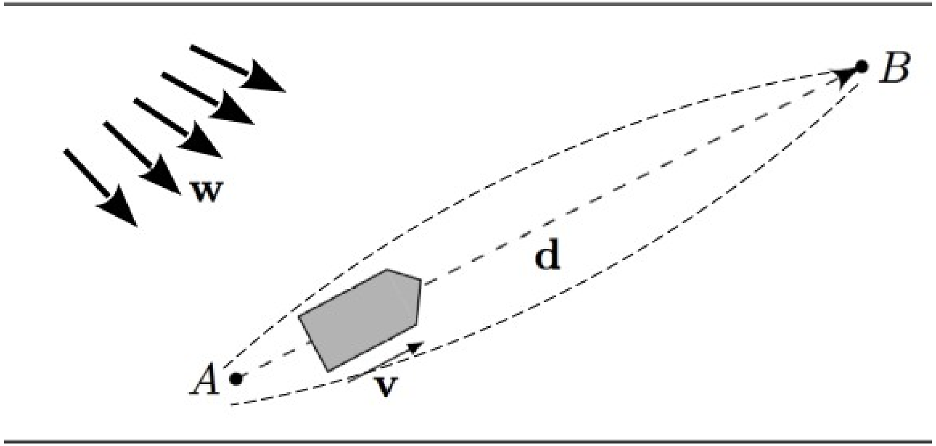

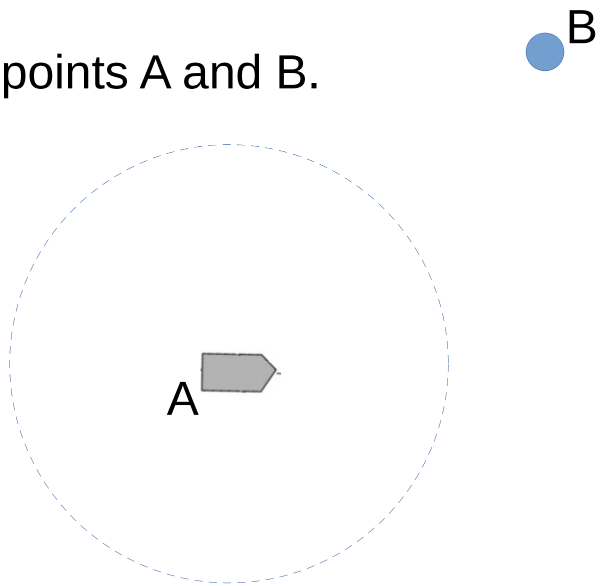

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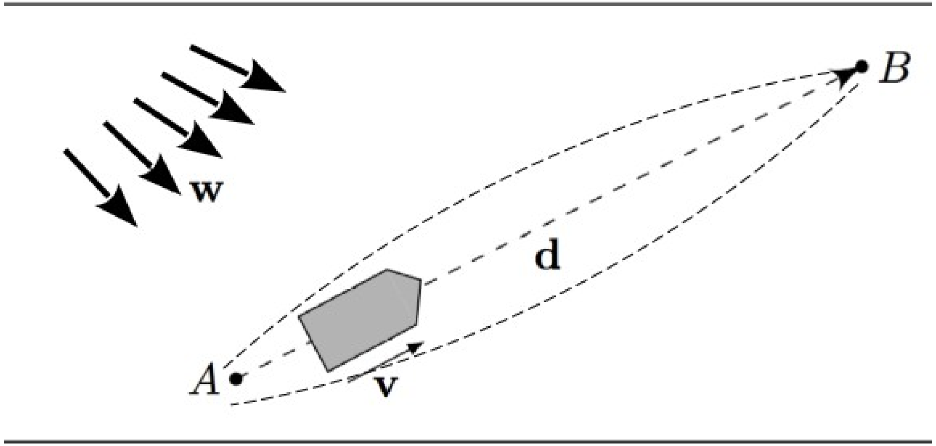

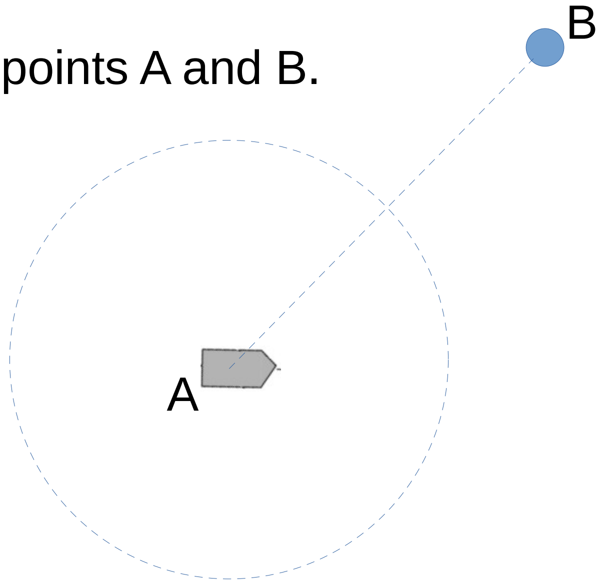

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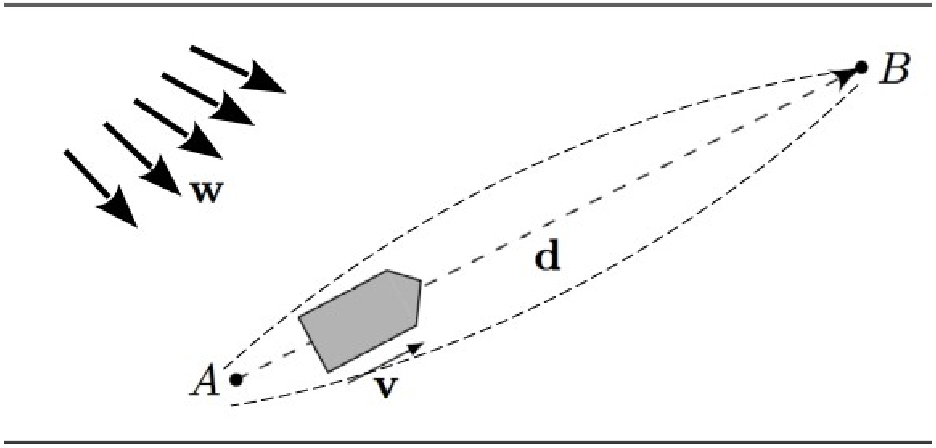

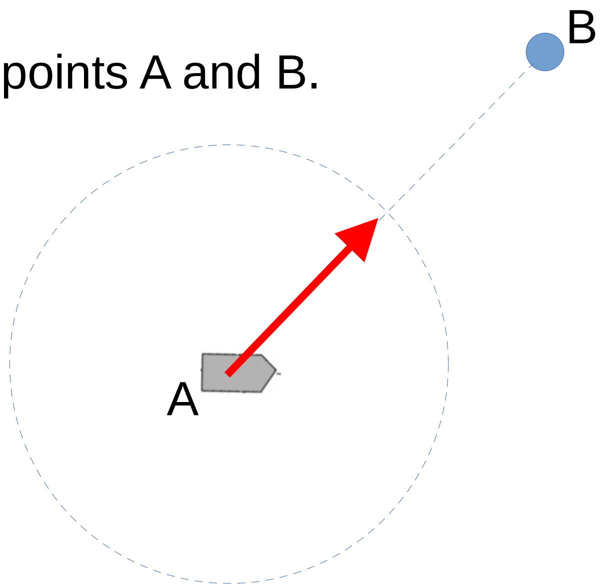

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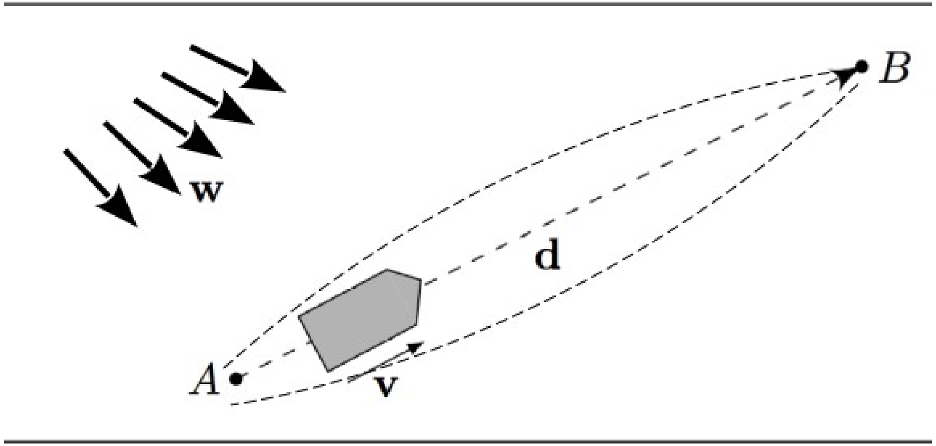

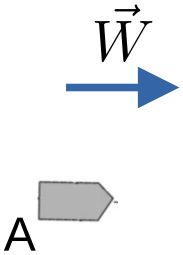

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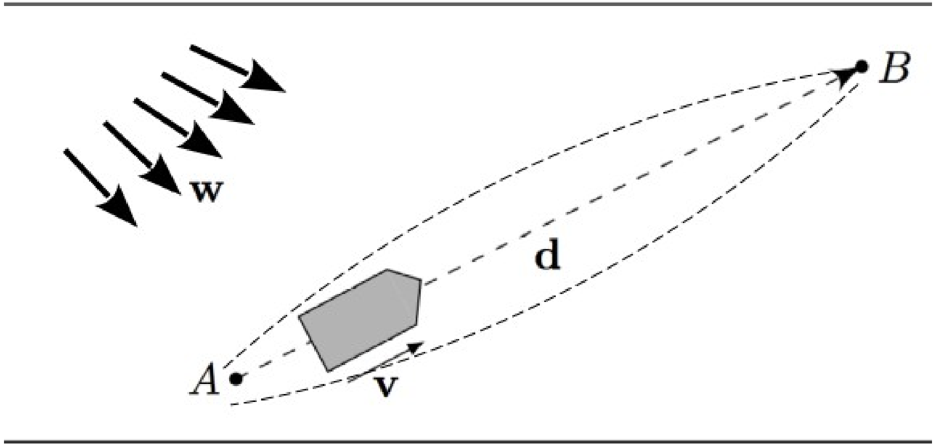

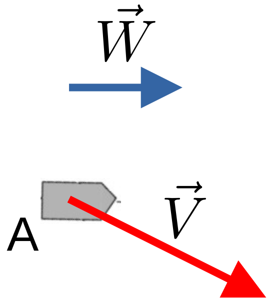

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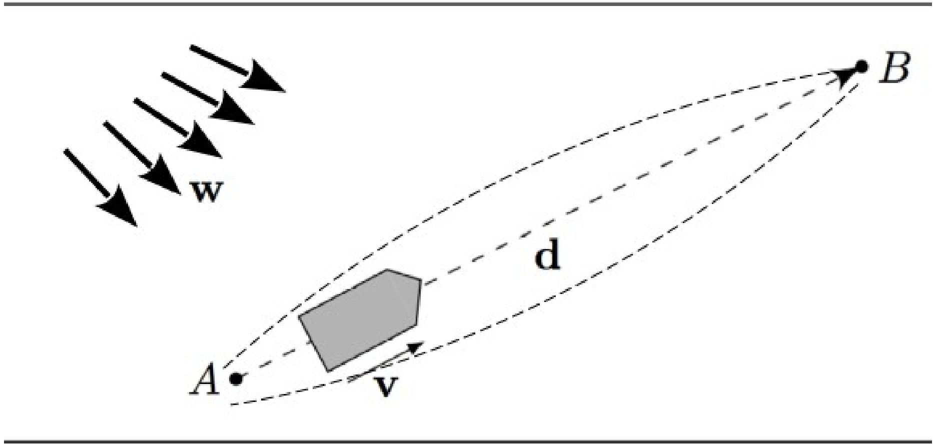

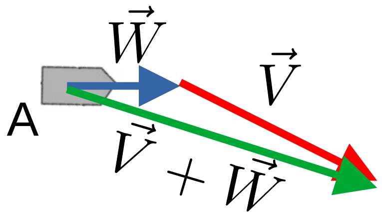

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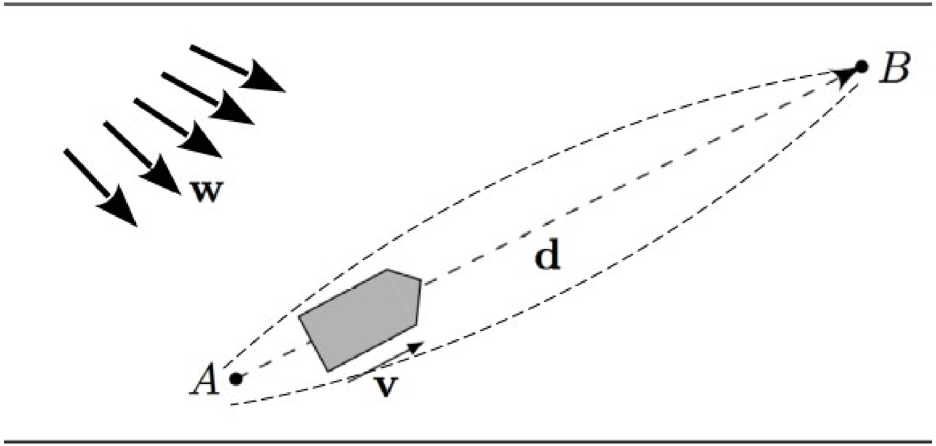

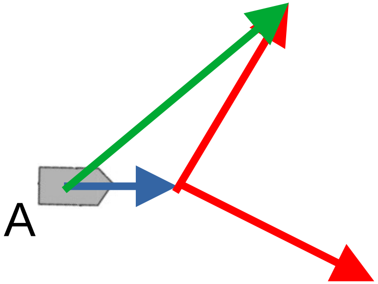

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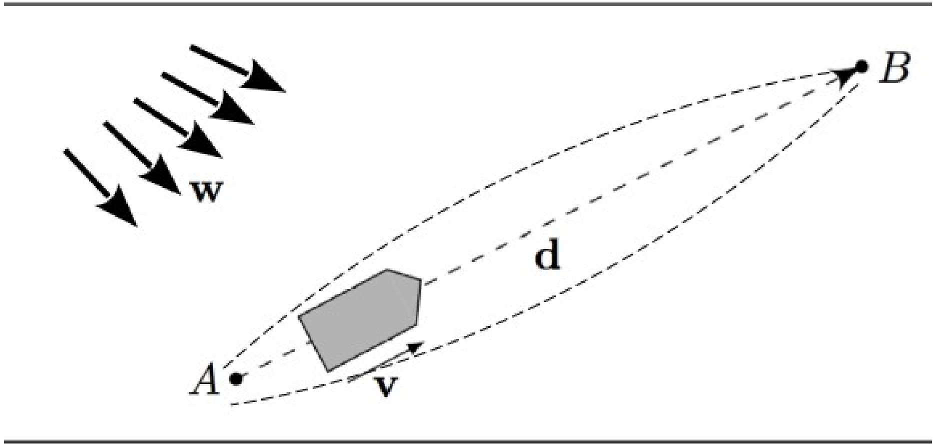

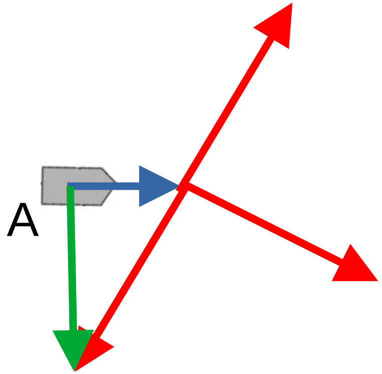

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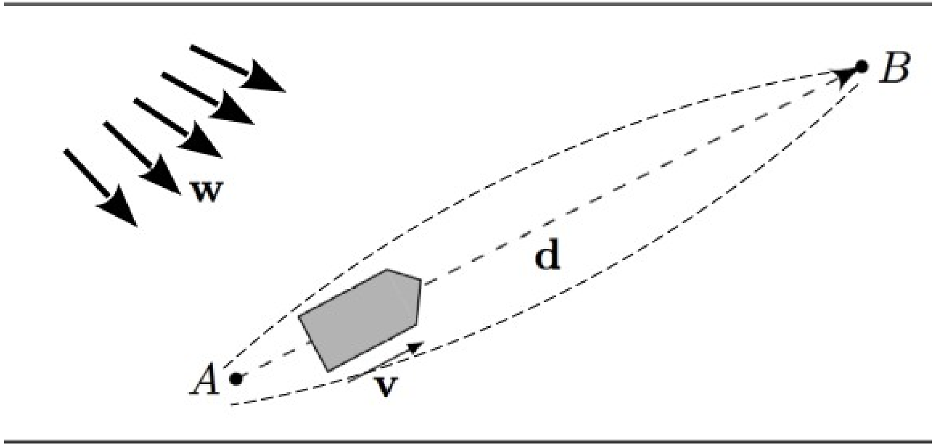

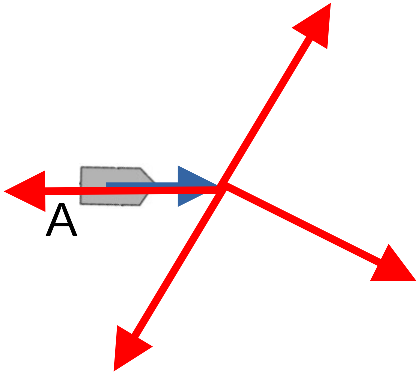

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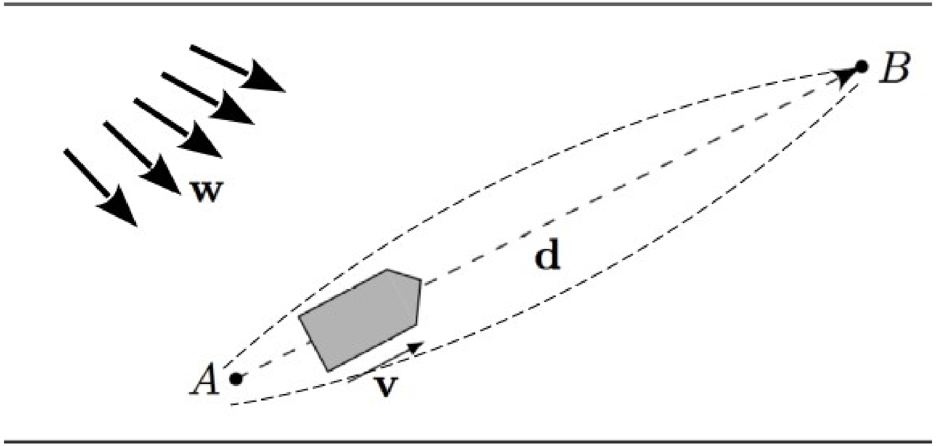

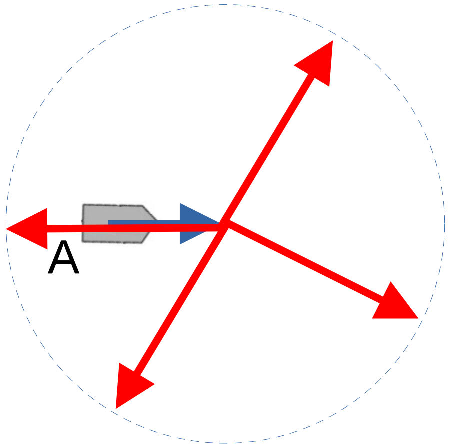

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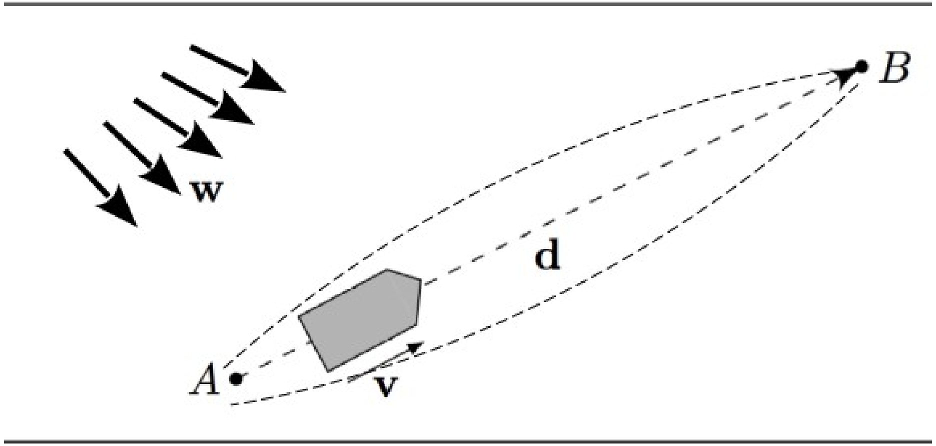

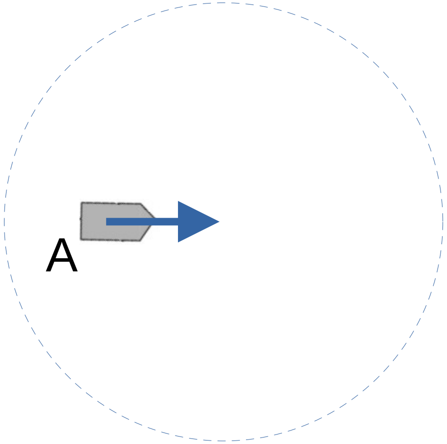

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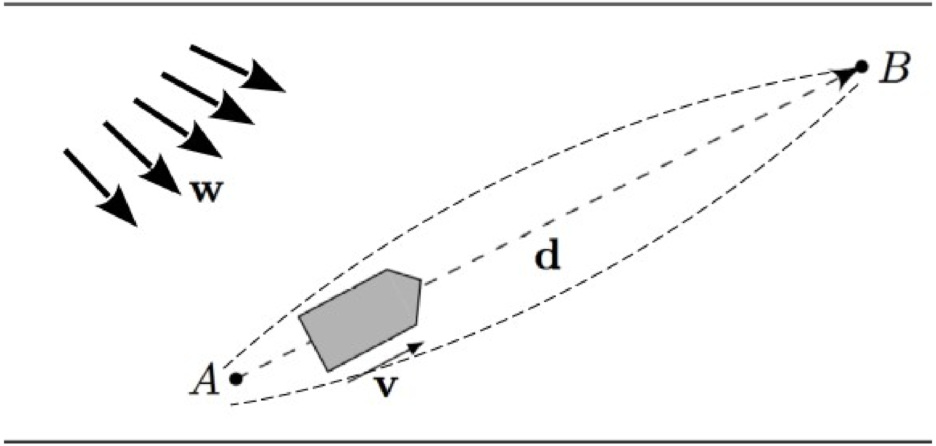

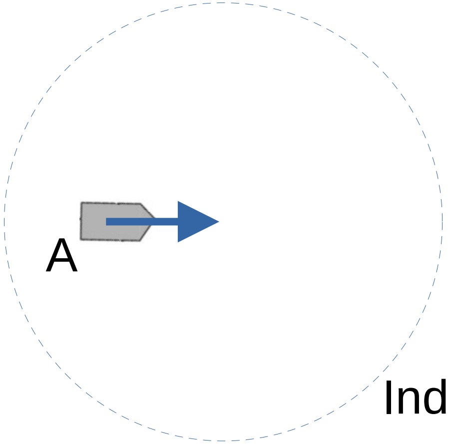

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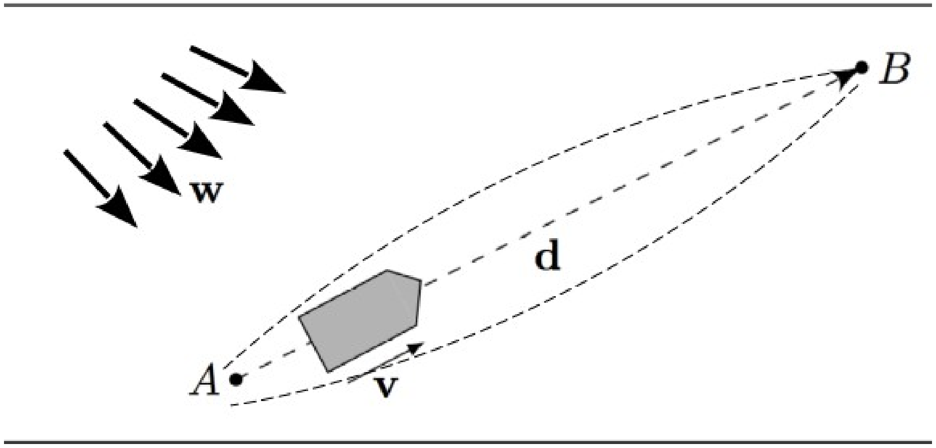

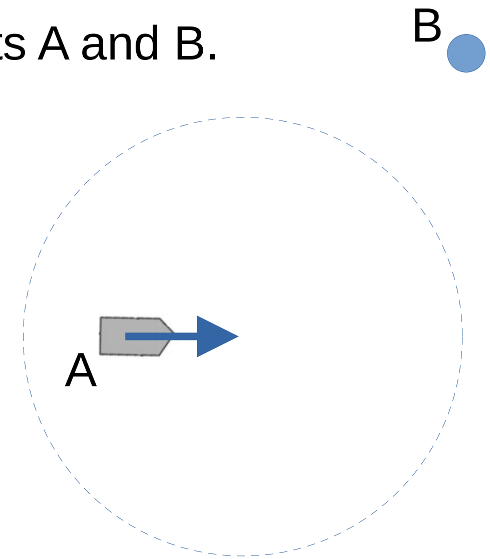

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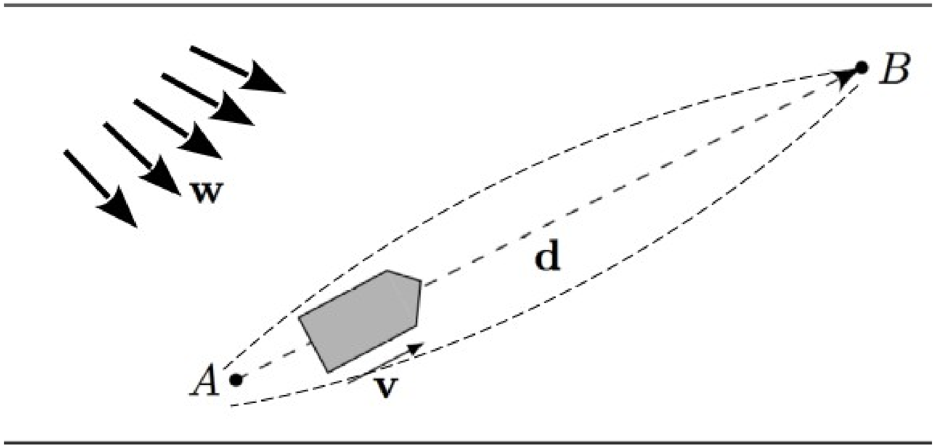

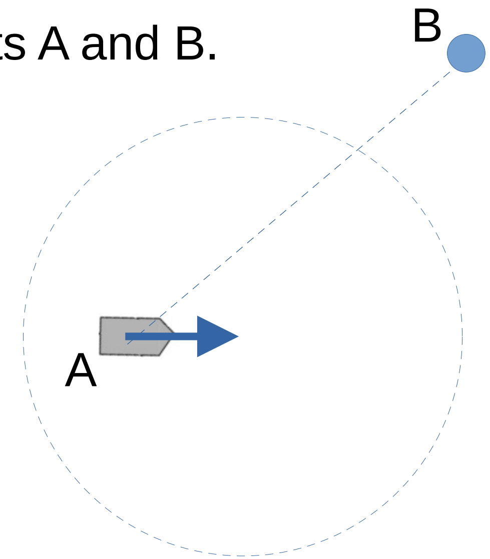

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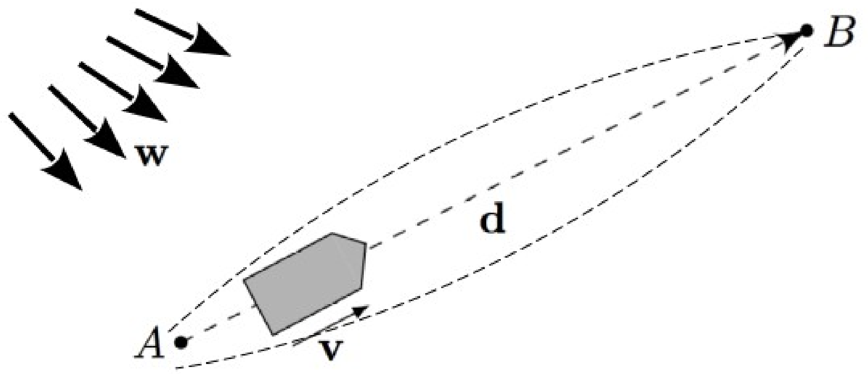

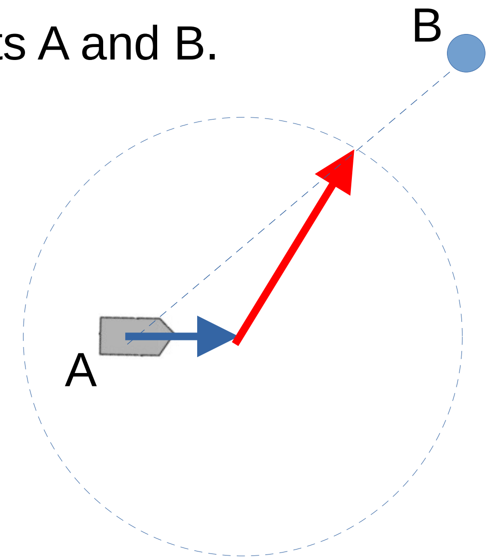

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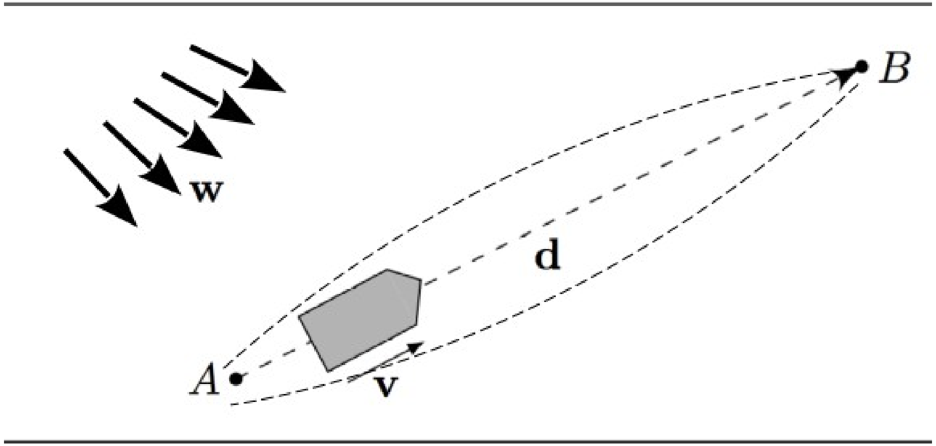

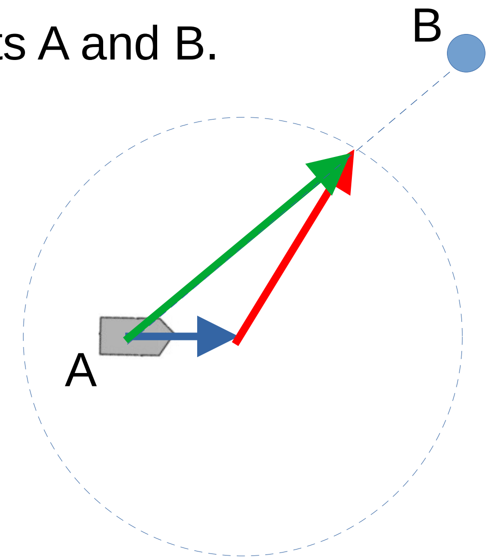

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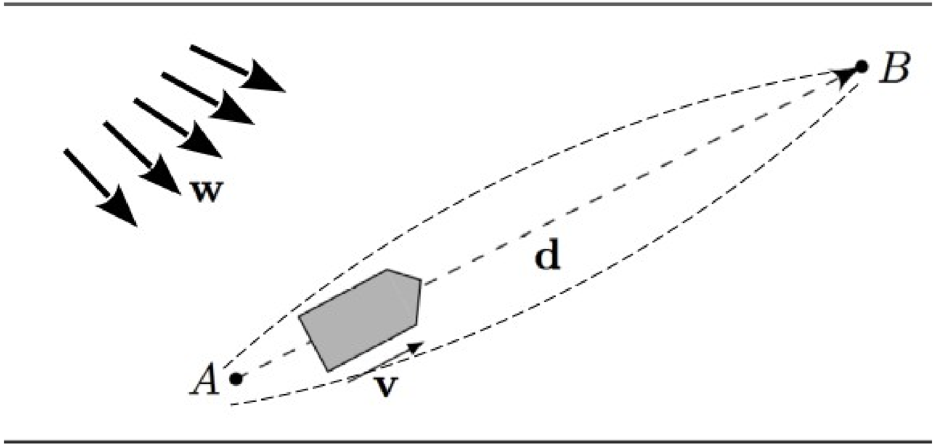

Geodesic (Fermat) formulation?

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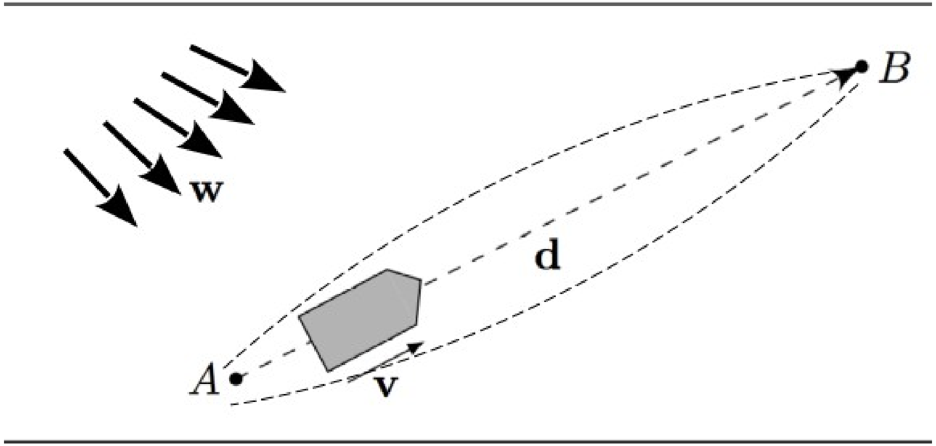

Geodesic (Fermat) formulation?

Yes!

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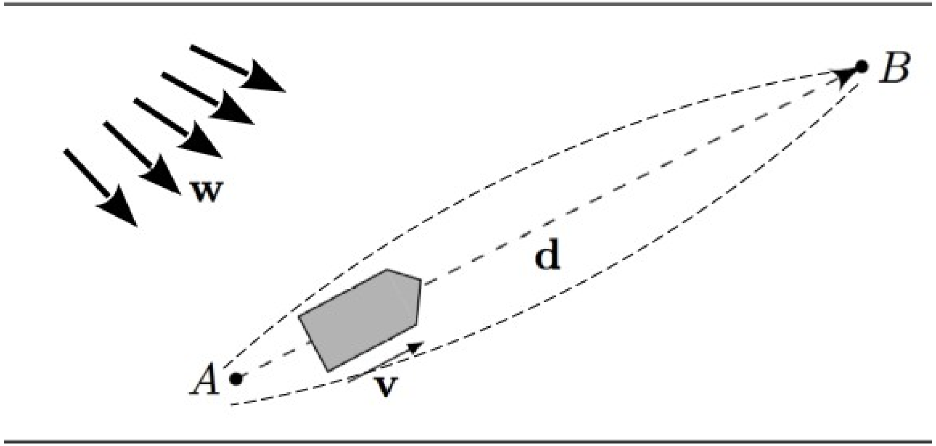

Geodesic (Fermat) formulation?

Yes!

Weak drift
$||W|| < 1$

Critical drift
$||W|| = 1$

Strong drift
$||W|| > 1$

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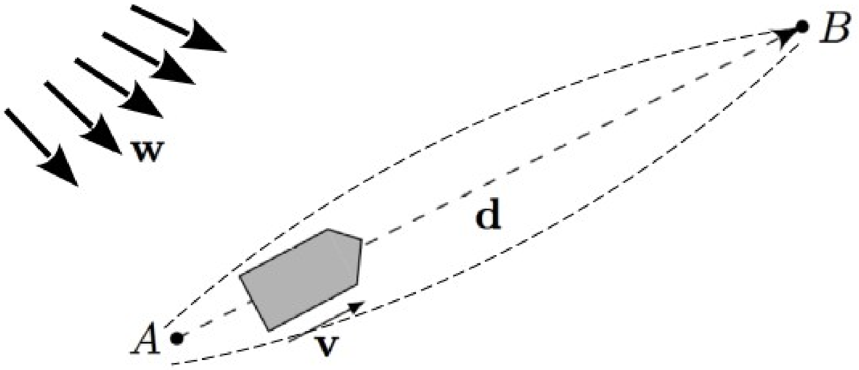

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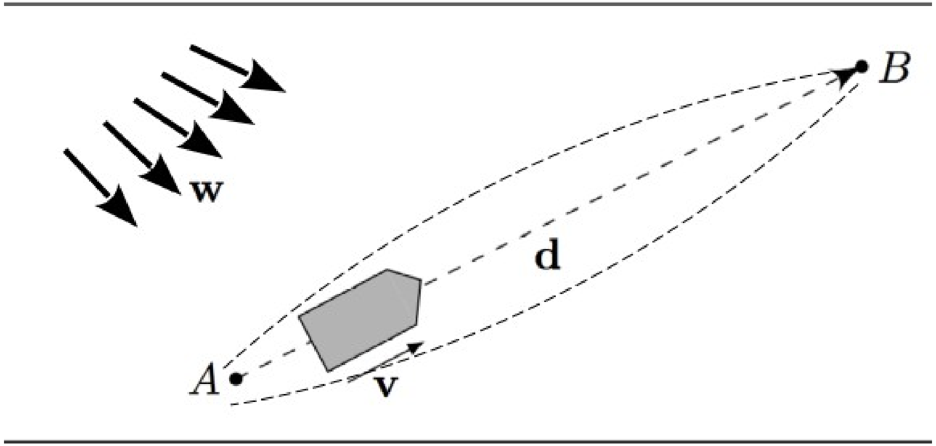

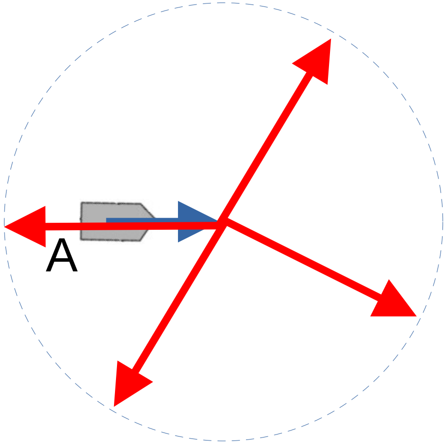

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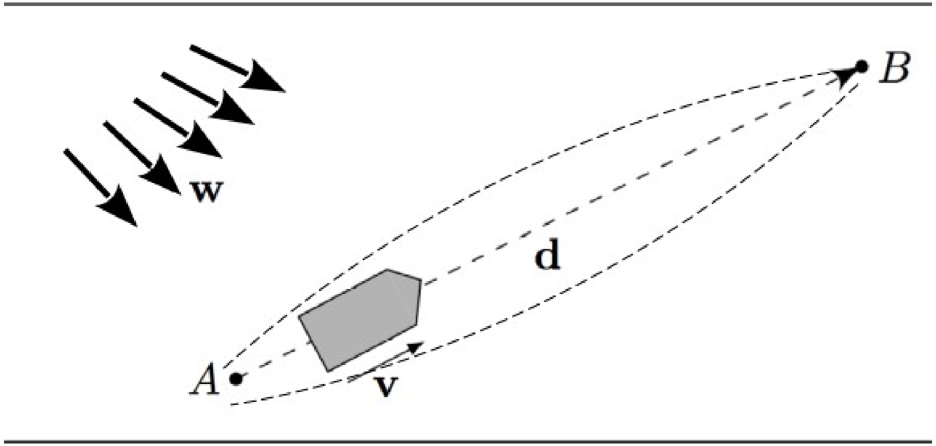

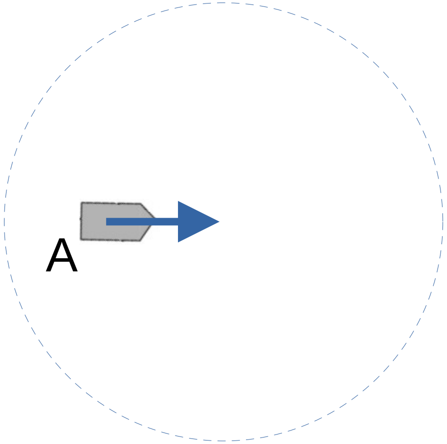

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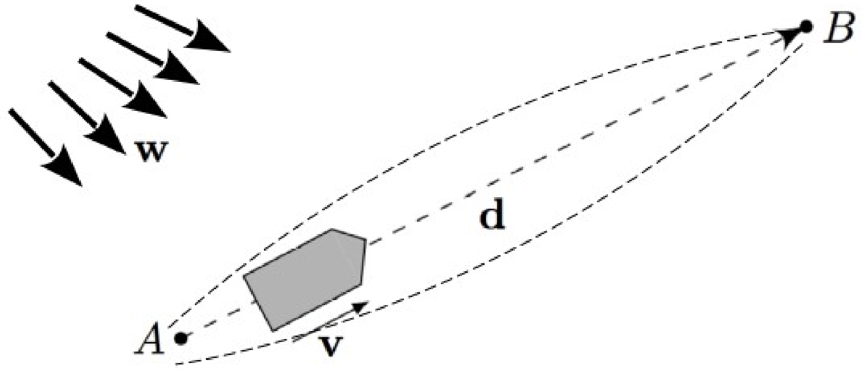

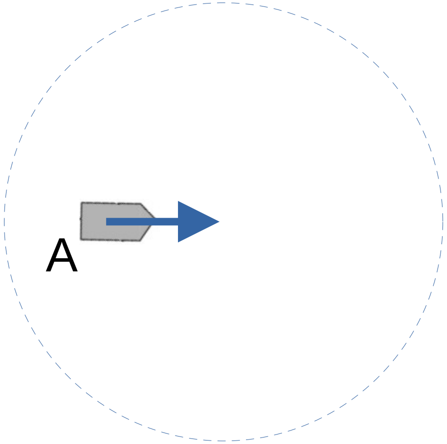

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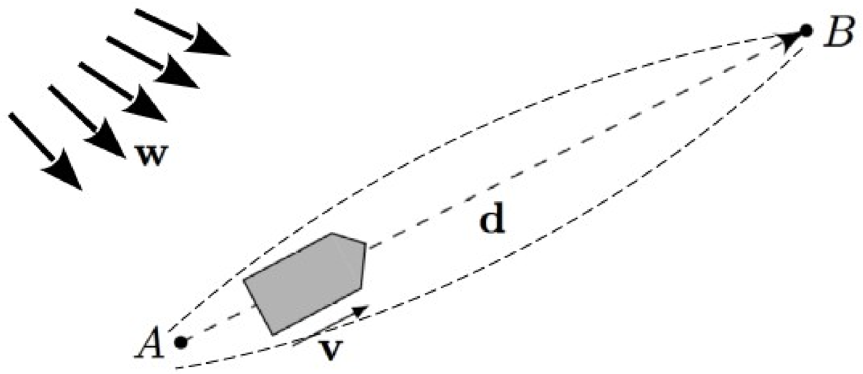

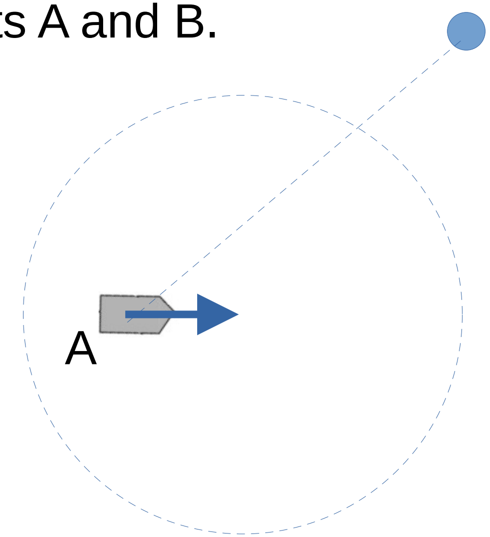

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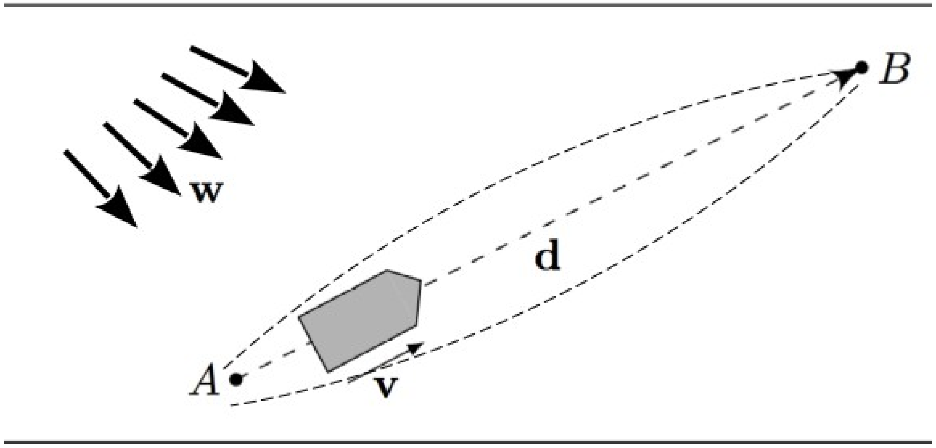

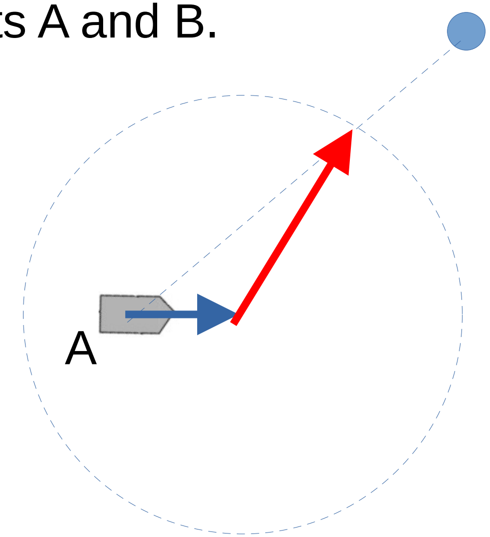

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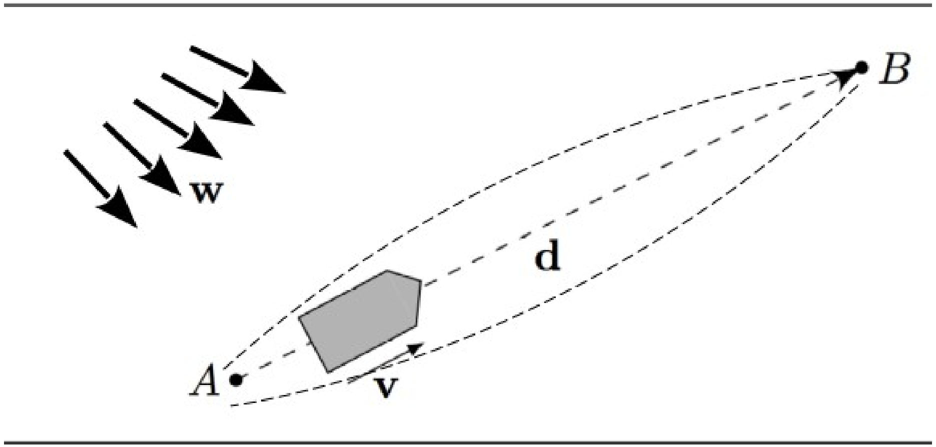

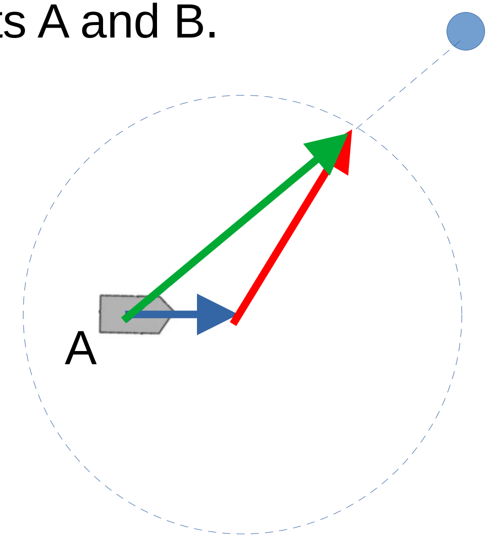

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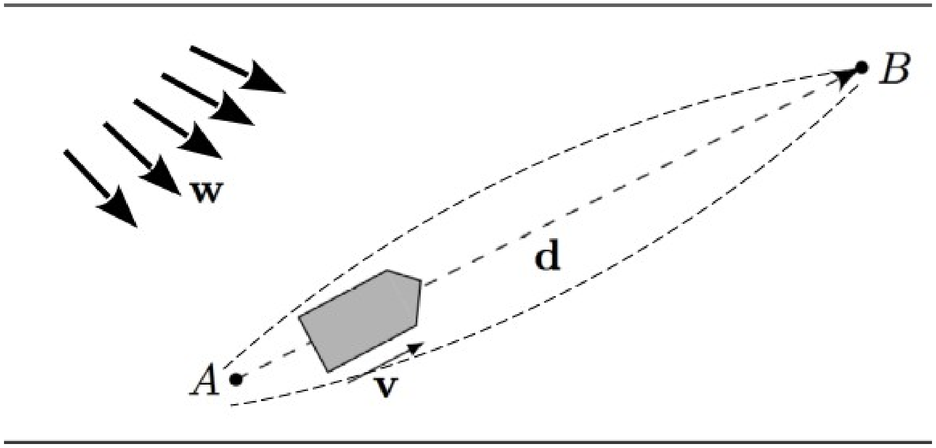

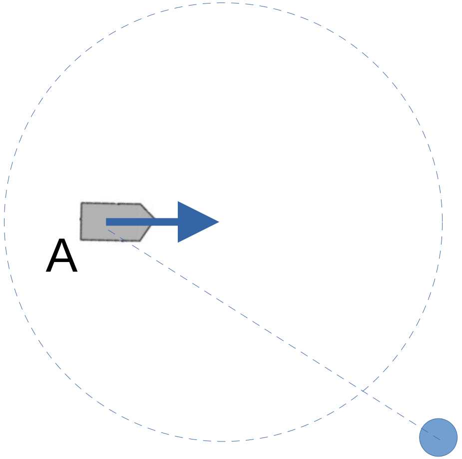

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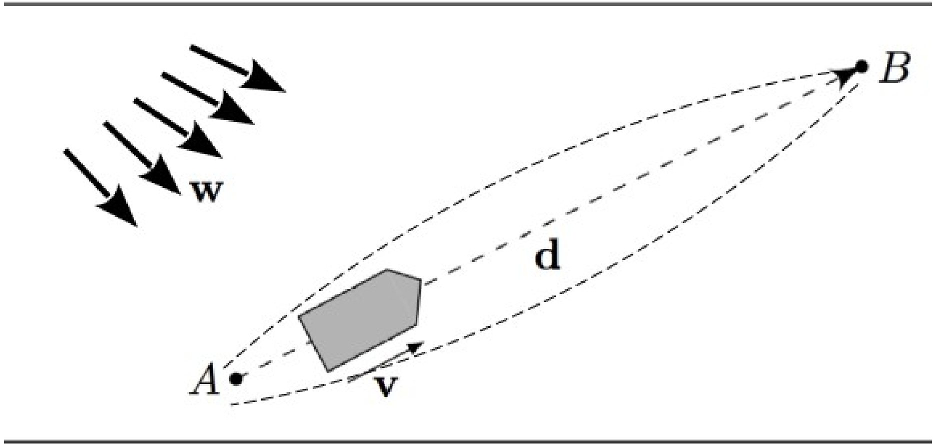

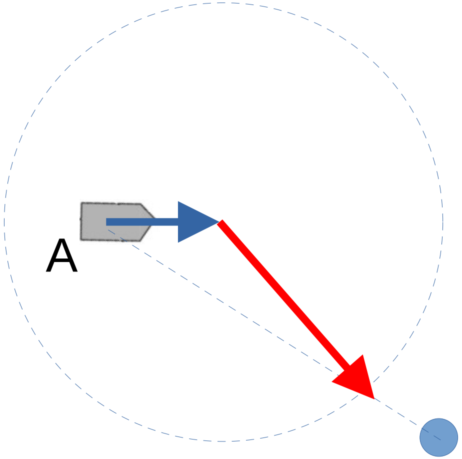

Weak drift
$||W|| < 1$

Zermelo navigation problem

Ernst Zermelo

- Optimal control problem (1931)

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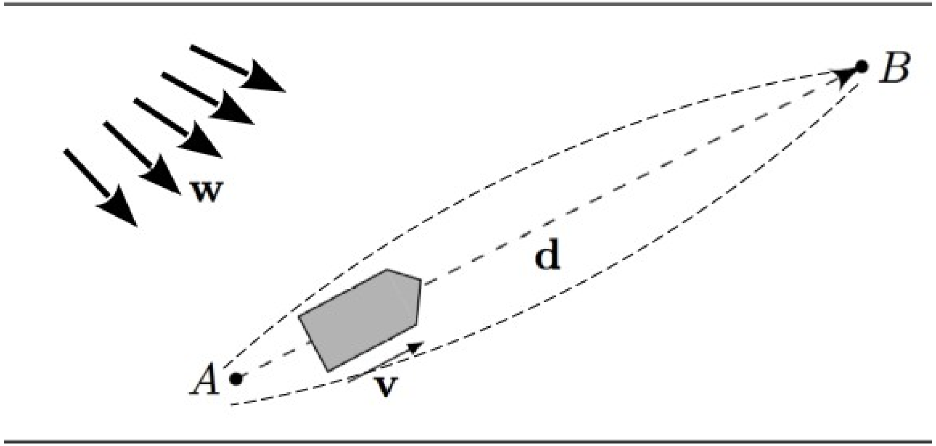

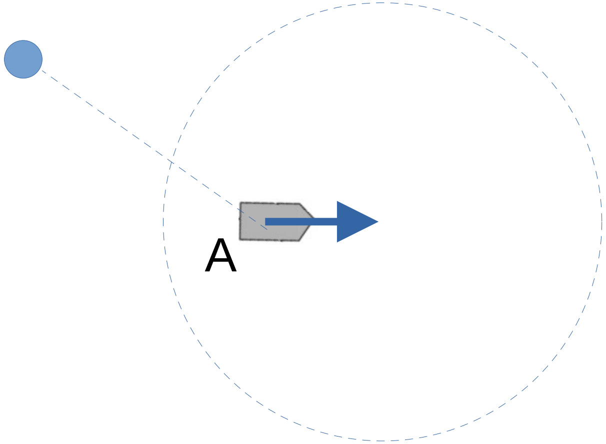

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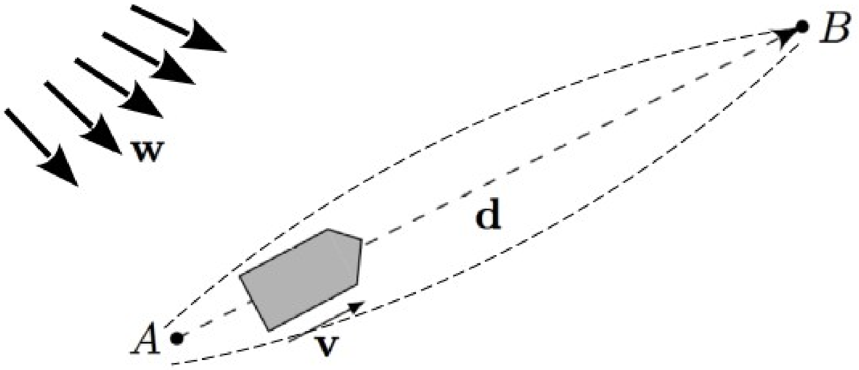

Critical drift

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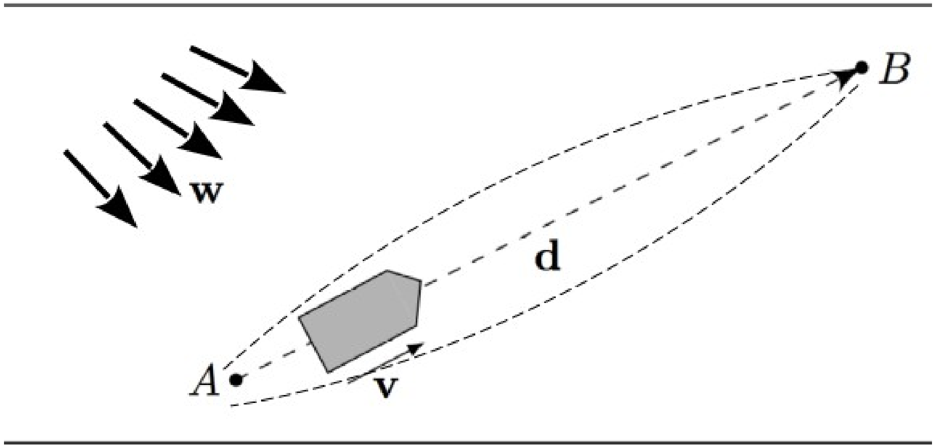

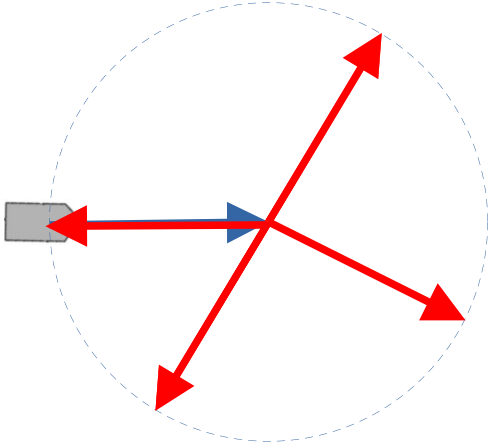

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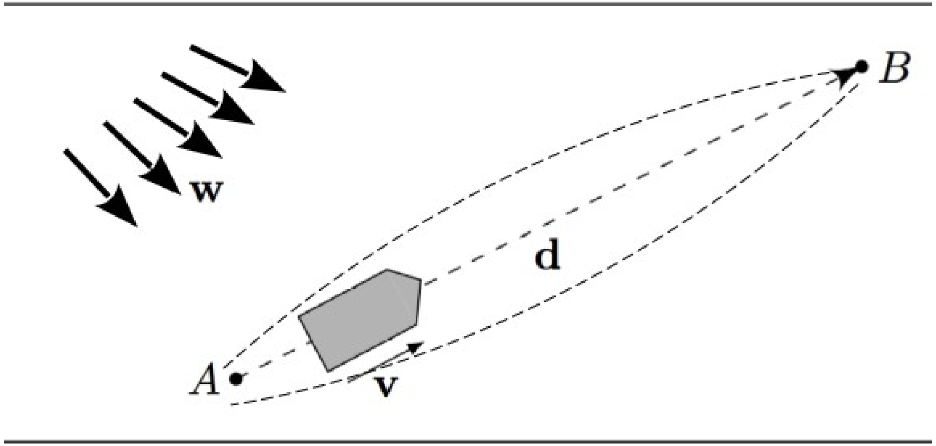

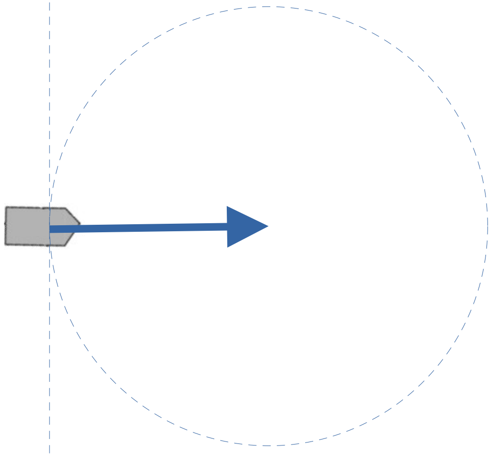

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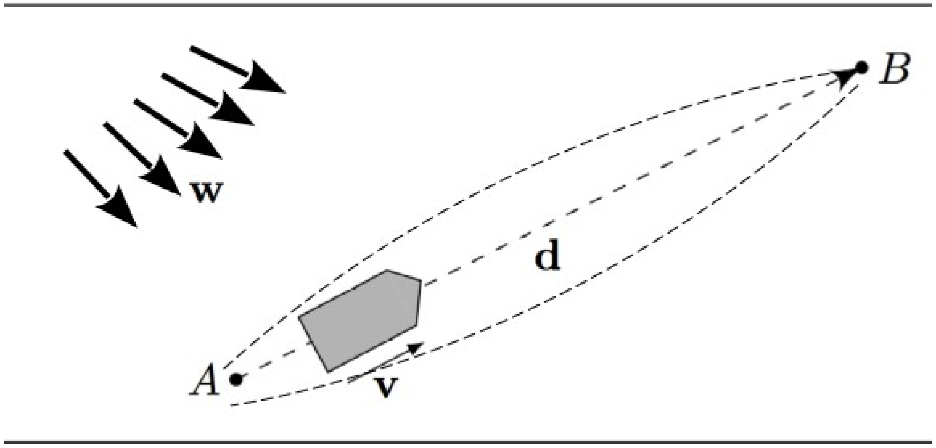

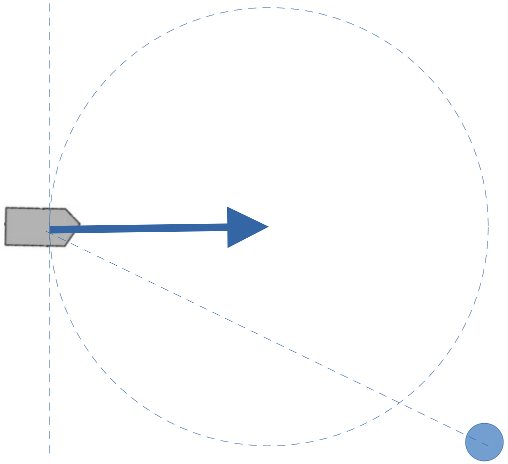

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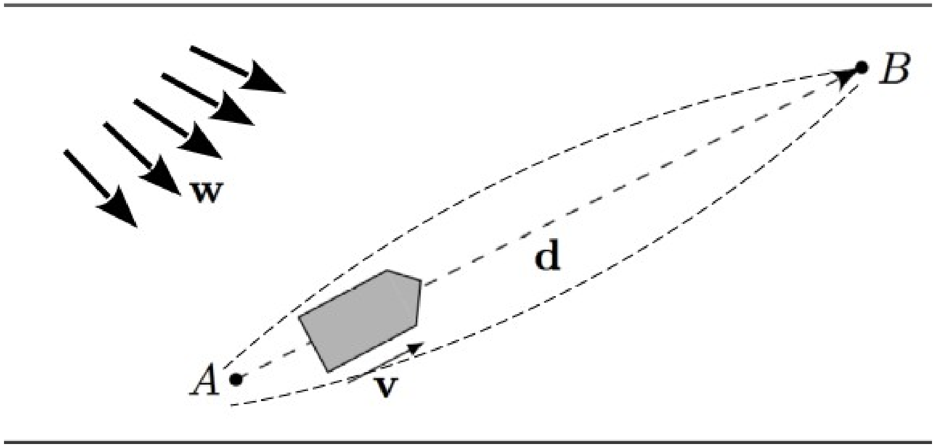

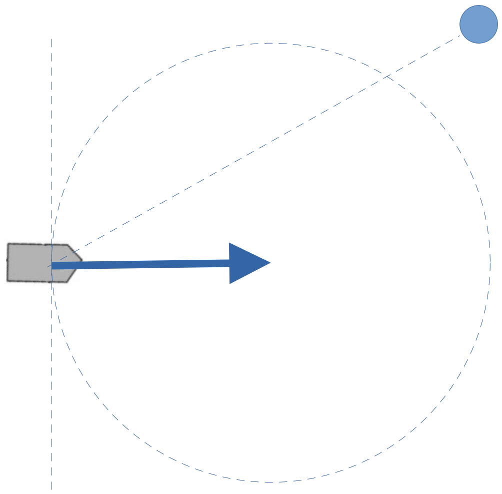

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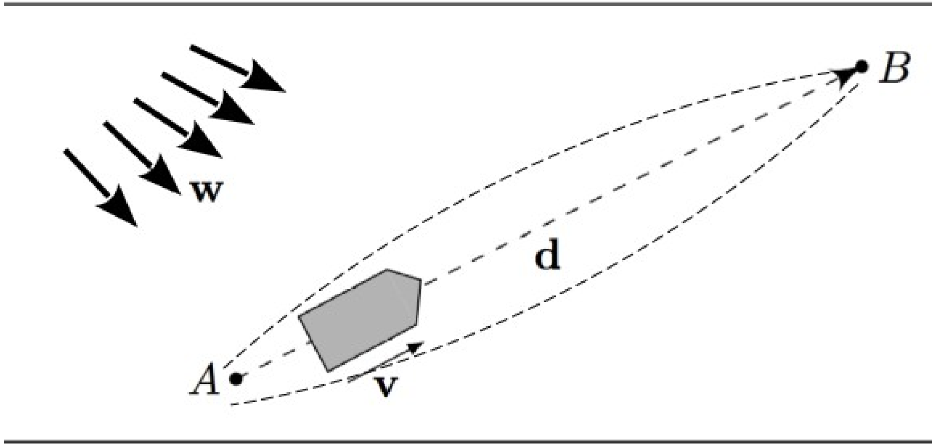

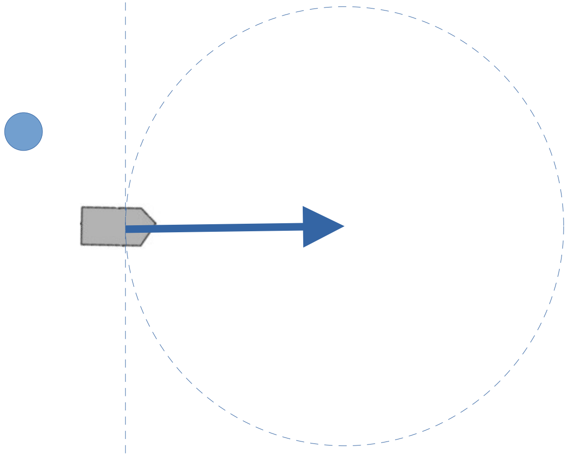

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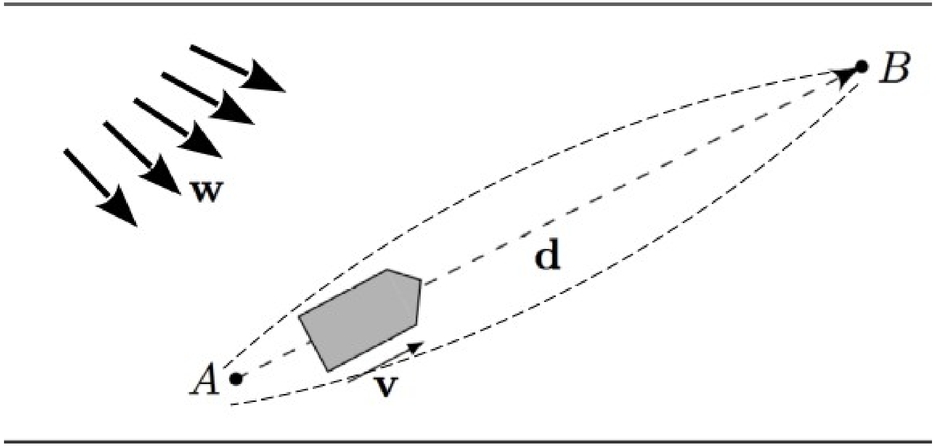

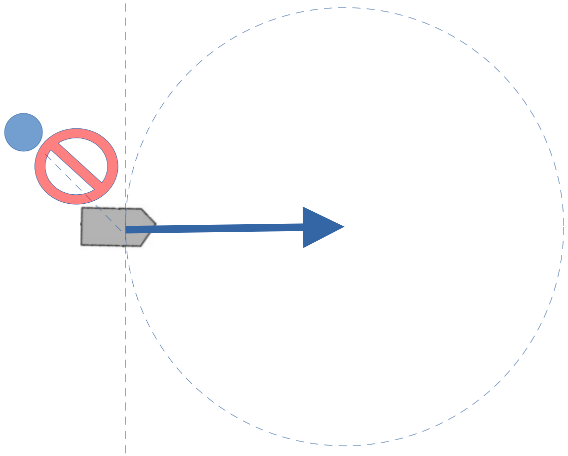

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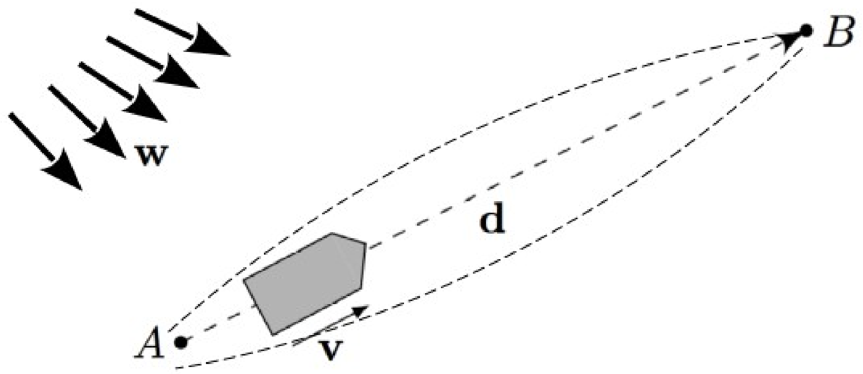

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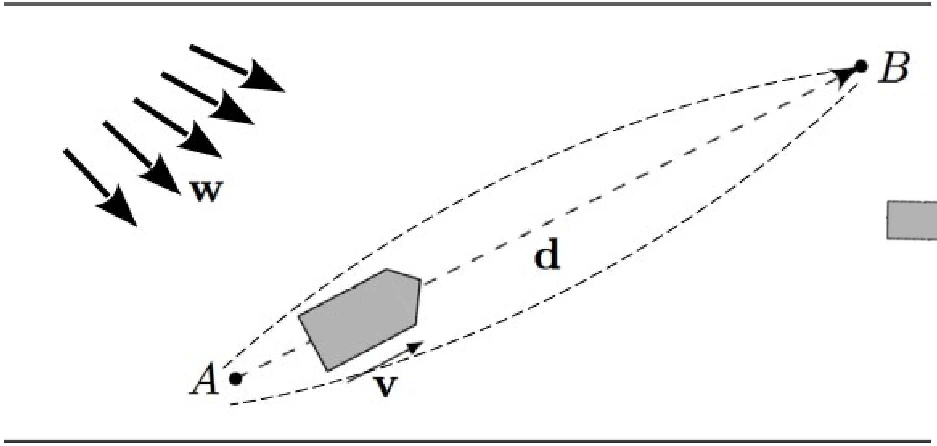

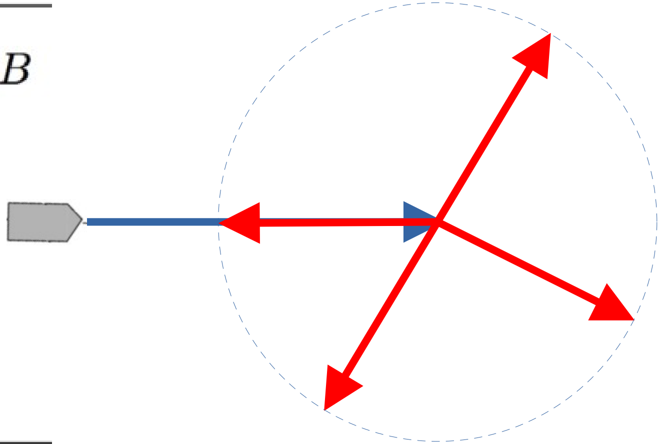

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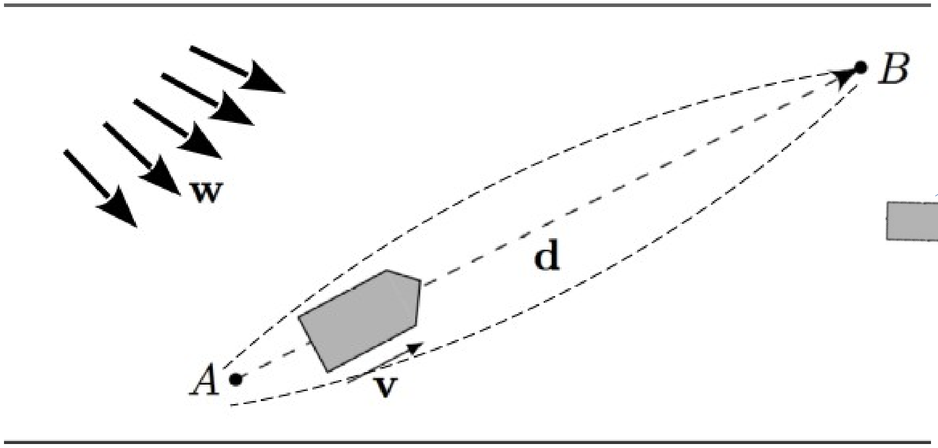

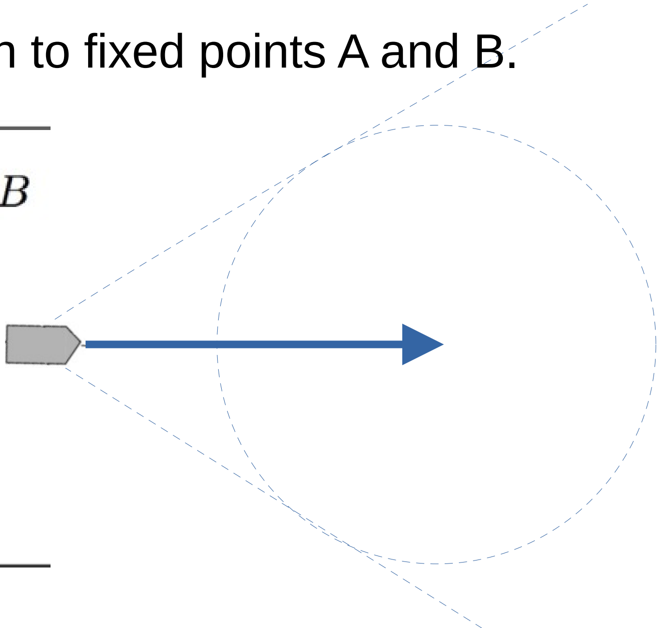

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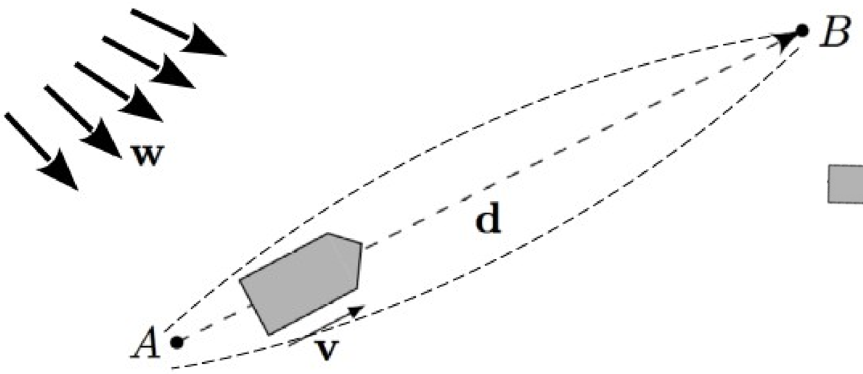

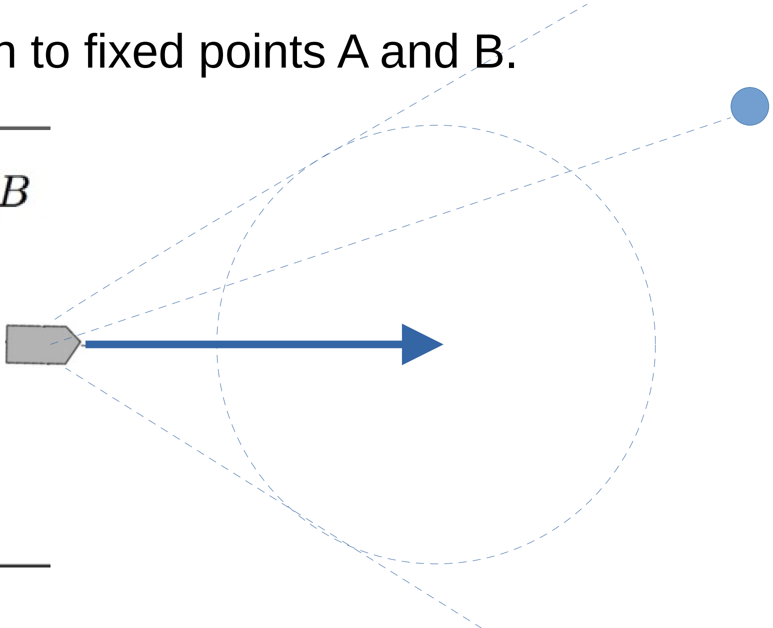

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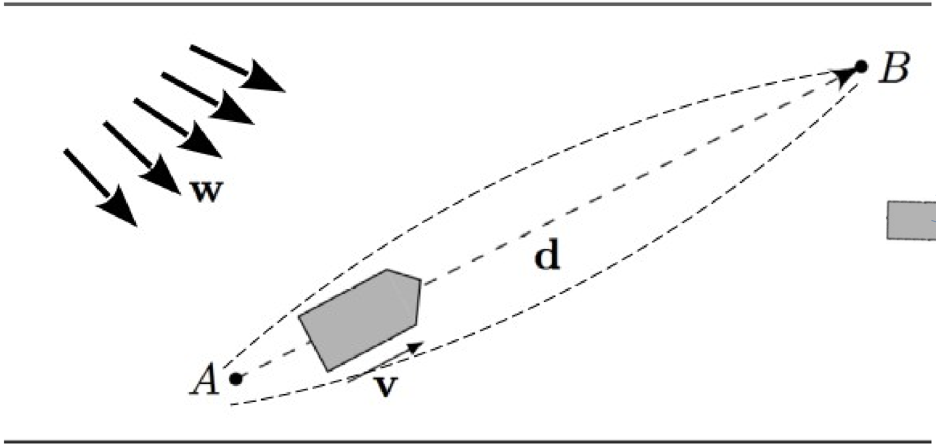

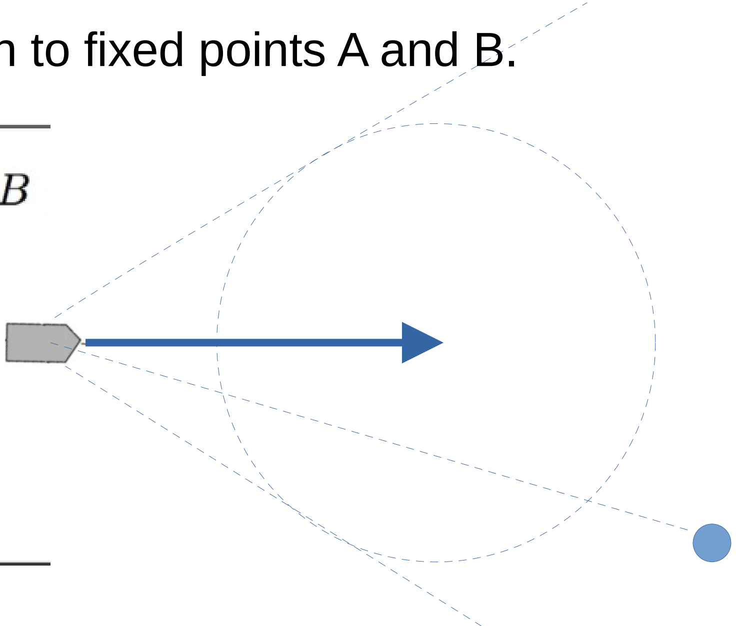

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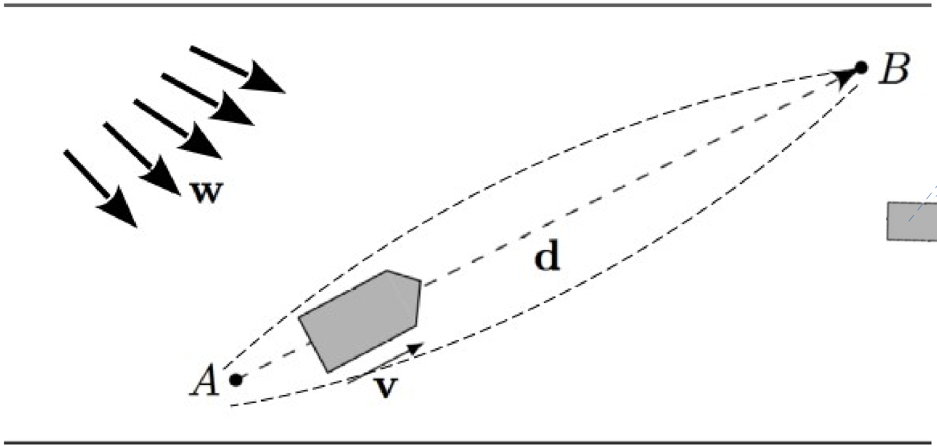

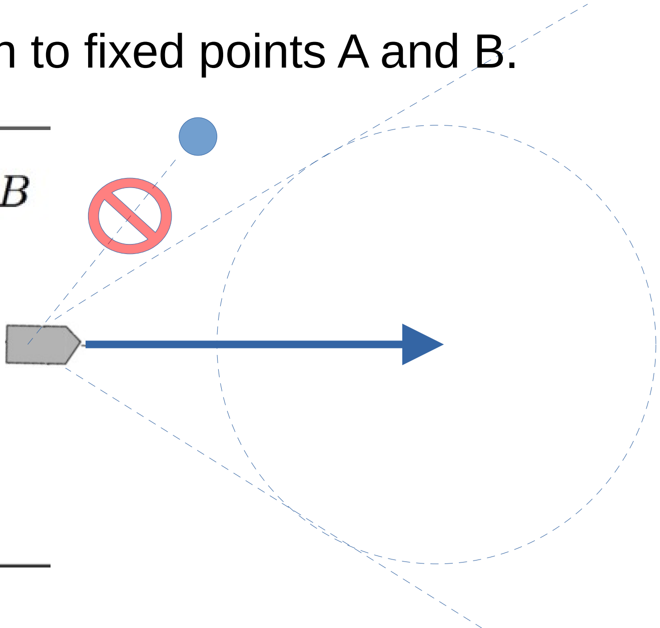

Strong drift

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Zermelo navigation problem and FG

Zermelo navigation problem and FG

Formulation as a geodesic (Fermat) problem

Zermelo navigation problem and FG

Formulation as a geodesic (Fermat) problem

Consider a non-relativistic particle moving with constant (unit) speed in an N-dimensional Riemannian manifold $(\mathcal{M}, h_{ij})$ subject to a time-independent vector field drift (wind) W^i such that $||W||^2 = h_{ij}W^iW^j$.

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YES!

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Formulation as a geodesic (Fermat) problem

Weak drift

$$||W|| < 1$$

Zermelo navigation problem and FG

Formulation as a geodesic (Fermat) problem

Weak drift

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Randers metric(!)

$$ds = F(\mathbf{x}, d\mathbf{x}) = \sqrt{a_{ij}dx^i dx^j} + b_j dx^j,$$

$$a_{ij} = \frac{1}{\lambda^2}\left(\lambda h_{ij} + W_i W_j\right) \quad \text{and} \quad b_i = -\frac{W_i}{\lambda},$$

where $W_i = h_{ij}W^j$ and $\lambda = 1 - ||W||^2$.

Zermelo navigation problem and FG

Formulation as a geodesic (Fermat) problem

Critical drift

$$\boxed{||W|| = 1}$$

Zermelo navigation problem and FG

Formulation as a geodesic (Fermat) problem

Critical drift

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Kropina metric

$$F(\mathbf{x}, d\mathbf{x}) = \frac{h_{ij}dx^i dx^j}{2W_j dx^j},$$

defined on the half-tangent space

$$A = \{d\mathbf{x} \in T_{\mathbf{x}}\mathcal{M} \; : \; W_j dx^j > 0\},$$

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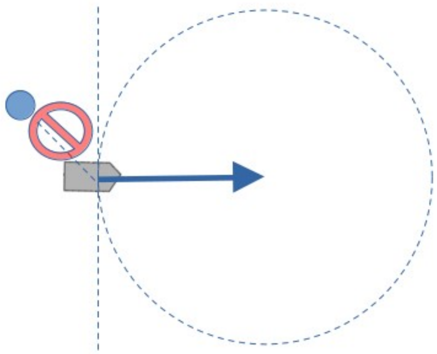

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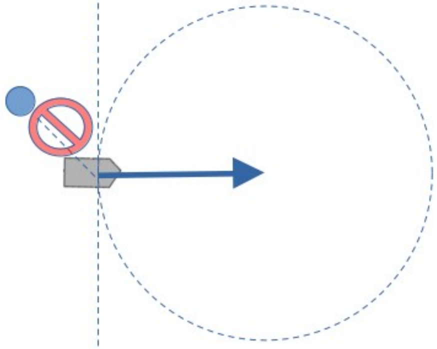

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Limit $\lambda \to 0$ of the Randers metric

$$ds = F(\mathbf{x}, d\mathbf{x}) = \sqrt{a_{ij}dx^i dx^j} + b_j dx^j,$$

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Zermelo navigation problem and FG

Formulation as a geodesic (Fermat) problem

Strong drift
$$\boxed{||W|| > 1}$$

Zermelo navigation problem and FG

Formulation as a geodesic (Fermat) problem

Strong drift
$||W|| > 1$

Lorentz-Finsler metric

$$F(\mathbf{x}, d\mathbf{x}) = -\frac{1}{\lambda}\left(W_i dx^i \mp \sqrt{(W_j dx^j)^2 + \lambda h_{ij} dx^i dx^j}\right),$$

defined on the conic region of the tangent space $T_{\mathbf{x}}\mathcal{M}$ given by

$$A = \Big\{\mathbf{x} \in T_{\mathbf{x}}\mathcal{M} :$$
$$\lambda h_{ij} dx^i dx^j + (W_j dx^j)^2 > 0,\; W_j dx^j > 0\Big\}.$$

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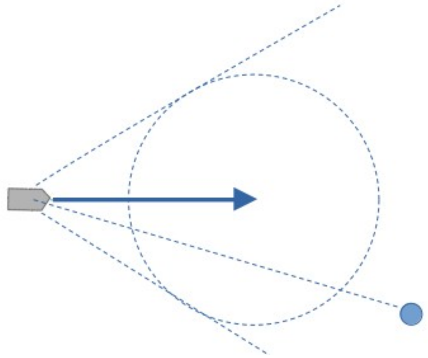

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Formulation as a geodesic (Fermat) problem

Strong drift

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Obtained by considering

$\lambda < 0$ in the Randers metric

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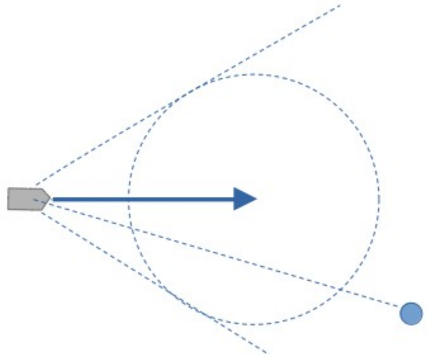

Painlevé-Gullstrand coordinates

Painlevé-Gullstrand coordinates

Stationary spacetimes

$$
\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\
&= -dt^2 + h_{ij} \left(dx^i - W^i dt\right) \left(dx^j - W^j dt\right)
\end{aligned}
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Null geodesics

$$ds = 0$$

Painlevé-Gullstrand coordinates

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Null geodesics

$$ds = 0 \quad \Rightarrow \quad dt = F(\mathbf{x}, d\mathbf{x})$$

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Schwarzschild black hole

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Painlevé-Gullstrand coordinates

Schwarzschild black hole

$$h_{ij} = \delta_{ij}$$

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Given the spherical symmetry of the problem, we can consider, without loss of generality, the Randers (1), Kropina (4), and the first Finsler-Lorentz (6) metrics along the x axis. A unified description is given by

$$F(\mathbf{x}, d\mathbf{x}) = \frac{1}{\lambda}\left(\sqrt{dx^2 + \lambda(dy^2 + dz^2)} + \sqrt{\frac{2M}{r}}dx\right), \tag{9}$$

with $\lambda = 1 - 2M/r$. The three metrics correspond, respectively, to $r > 2M$ ($\lambda > 0$), $r = 2M$ ($\lambda = 0$), and $r < 2M$ ($\lambda < 0$). Hence, a unique Finslerian Lagrangian is defined over the entire range of r, which implies that geodesics can smoothly cross the submanifold $r = 2M$ provided the half-space (5) and conic (7) constraints are satisfied in the respective regions.

The Finslerian indicatrix (8) associated with (9) reduces to the simple expression

$$\left(dx + \sqrt{\frac{2M}{r}}\right)^2 + dy^2 + dz^2 = 1, \tag{10}$$

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$$W^i = -\sqrt{\frac{2M}{r}}\hat{r}$$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$
$$= -dt^2 + h_{ij}\left(dx^i - W^i dt\right)\left(dx^j - W^j dt\right)$$

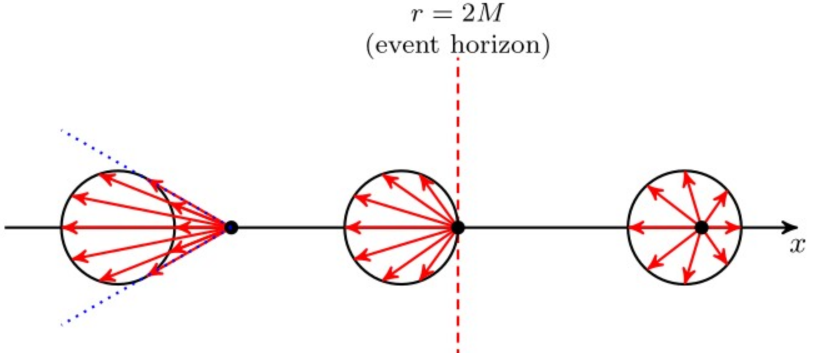
Given the spherical symmetry of the problem, we can consider, without loss of generality, the Randers (1), Kropina (4), and the first Finsler-Lorentz (6) metrics along the x axis. A unified description is given by

$$F(\mathbf{x}, d\mathbf{x}) = \frac{1}{\lambda}\left(\sqrt{dx^2 + \lambda(dy^2 + dz^2)} + \sqrt{\frac{2M}{r}}dx\right), \tag{9}$$

with $\lambda = 1 - 2M/r$. The three metrics correspond, respectively, to $r > 2M$ ($\lambda > 0$), $r = 2M$ ($\lambda = 0$), and $r < 2M$ ($\lambda < 0$). Hence, a unique Finslerian Lagrangian is defined over the entire range of r, which implies that geodesics can smoothly cross the submanifold $r = 2M$ provided the half-space (5) and conic (7) constraints are satisfied in the respective regions.

The Finslerian indicatrix (8) associated with (9) reduces to the simple expression

$$\left(dx + \sqrt{\frac{2M}{r}}\right)^2 + dy^2 + dz^2 = 1, \tag{10}$$



Painlevé-Gullstrand coordinates

Kerr black hole

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\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\
&= -dt^2 + h_{ij}\left(dx^i - W^i dt\right)\left(dx^j - W^j dt\right)
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$$h_{ij}dx^i dx^j = \frac{\rho^2}{\Sigma}dr^2 + \rho^2 d\theta^2 + \Sigma\sin^2\theta\,(d\phi + \delta d\theta)^2\,,$$ (Natario's coordinates)

and drift W^i such that

$$W_i dx^i = \frac{\rho^2}{\Sigma}v dr + \Sigma\sin^2\theta\Omega d\phi,$$

with

$$v = -\frac{\sqrt{2Mr(r^2+a^2)}}{\rho^2}, \quad \Omega = \frac{2Mra}{\rho^2\Sigma},$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Sigma = r^2 + a^2 + \frac{2Mra^2}{\rho^2}\sin^2\theta, \qquad \delta = -a^2\sin(2\theta)\int_r^\infty \frac{v\Omega}{\Sigma}dr.$$

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Indicatrix

$$\frac{r^2+a^2}{\Sigma}\left(dx + \sqrt{\frac{2M}{r}}\right)^2 + \frac{\Sigma}{r^2+a^2}\left(dy - \frac{2Ma\sqrt{r^2+a^2}}{r\Sigma}\right)^2 = 1,$$

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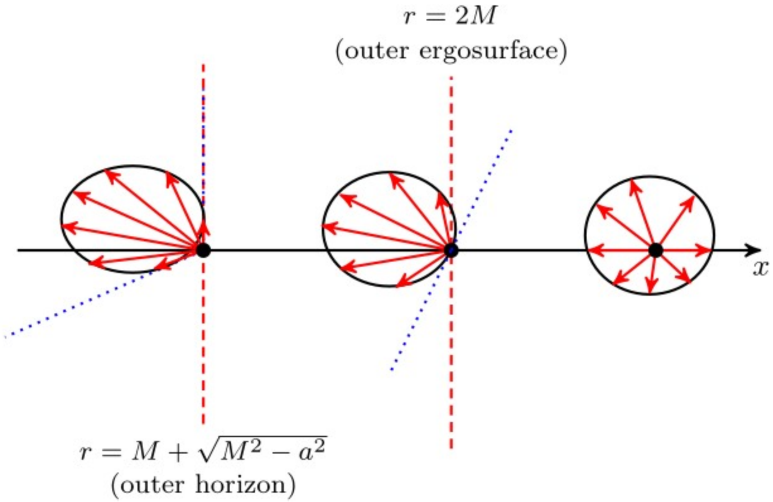

The “river model” of black holes

The river model of black holes

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We present a lesser known way to conceptualize stationary black holes, which we call the river model. In this model, space flows like a river through a flat background, while objects move through the river according to the rules of special relativity. In a spherical black hole, the river of space falls into the black hole at the Newtonian escape velocity, hitting the speed of light at the horizon. Inside the horizon, the river flows inward faster than light, carrying everything with it. The river model also works for rotating (Kerr–Newman) black holes, though with a surprising twist. As in the spherical case, the river of space can be regarded as moving through a flat background. However, the river does not spiral inward, but falls inward with no azimuthal swirl. The river has at each point not only a velocity but also a rotation or twist. That is, the river has a Lorentz structure, characterized by six numbers (velocity and rotation). As an object moves through the river, it changes its velocity and rotation in response to tidal changes in the velocity and twist of the river along its path. An explicit expression is given for the river field, a six-component bivector field that encodes the velocity and twist of the river at each point and encapsulates all the properties of a stationary rotating black hole.
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- Finding null geodesics of stationary spacetimes admitting Painlevé-Gullstrand coordinates is a Finsler geometry problem.
- Finslerian description offers an enhanced understanding of the causal structure of black holes, presenting the behavior of null geodesics on the ergosurfaces, horizons, and their interior regions in novel geometric terms.