MOTION, INERTIA, MASS

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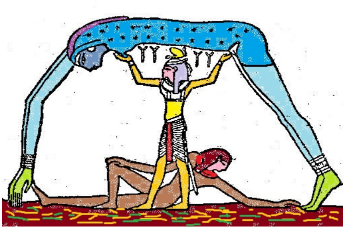

For the ancient Egyptians and Babylonians, the world was full of animate beings.

Water, air, earth, the Sun and Moon, the stars - everything was thought as deities endowed with their own soul and will

Shu, the god of air and vital breath, and *Tefnut*, the goddess of heat, begot *Geb*, the god of Earth, and *Nut*, the goddess of Sky, bearing stars. Figure shows ancient Egyptian representation of Earth (Geb) and Sky (Nut) supporteded by Air (Shu)..

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- Instead of asking "Who is it?" they posed the question "What is it?" Thus, for Thales of Miletus, everything was produced from water; for his student Anaximander, from the "apeiron", a hypothetical primitive element from which all four elements derived; for Anaximenes, from the air, and for Heraclitus, from fire.

Parmenides and Heraclitus

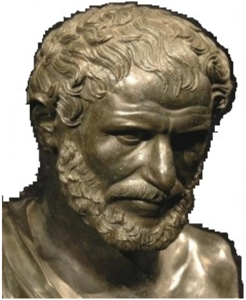
 

Figure: Ancient Greek philosophers Parmenides (left) and Heraclitus of Ephesus (right).

For Parmenides (540 - 470 BCE) the Universe was static, continuous and eternal

For Heraclitus (535-475 BCE) the Universe was in constant

Zeno of Elea and his paradoxes

Very little is known about Zeno's of Elea life, except that he lived between ca. $495 - 430$ **B.C.E. and founded a philosophical school in the Greek city of Elea in Magna Grecia (southern Italy now).**

He was Parmenides' pupil, and has visited Athens when Socrates was still a very young man. $Z\eta\nu\omega\nu$ o $E\lambda\epsilon\alpha\tau\varsigma$.

Zeno's Achilles and turtle paradox

Figure: Achilles and the Turtle endless run

Geometric progression

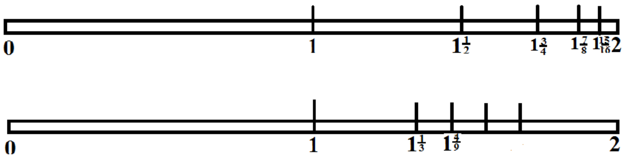


Figure: The sum of infinite geometric series is finite

For any complex number $q < 1$ one has

$$\sum_{k=0}^{N} q^k = \frac{1 - q^{N+1}}{1-q}, \rightarrow \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}.$$

Zeno's arrow paradox

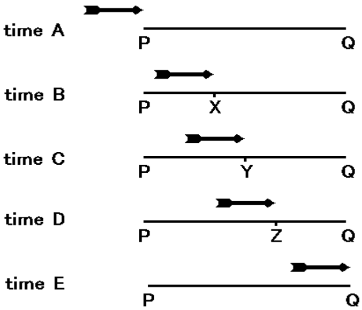


Figure 1

Zeno's Arrow Paradox - motion is impossible.

The Uncertainty principle $\Delta x \cdot \Delta p \geq \frac{1}{2}\hbar$

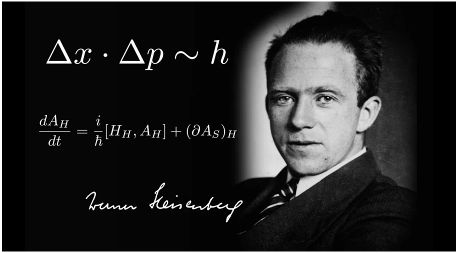
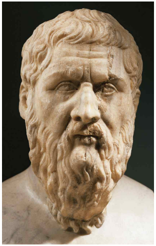
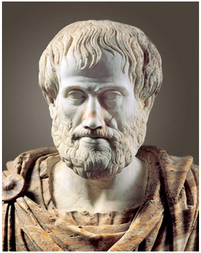


Figure: Werner Heisenberg, 1901 – 1976

Simultaneous measurement of position and velocity is impossible; therefore the total rest is impossible, not the motion. Heraclitus was right, Zeno was wrong.

Left: Plato, Center: Plato and Aristotle. Fragment of Raphael's fresco painted in $1509 - 1511$ **on the walls of the Sixtine chapels of Vatican. Elderly Plato points his finger to the sky, while Aristotle shows Earth with his hand. Right: Aristotle (384-322 B.C.).**

Aristotle's physics

- The material world is composed of *four elements*, which are: **fire, air, water and earth**.
  The elements are endowed with two types of opposite qualities: "lightness" and "heaviness", also "dry" or "wet".
  The light elements display a natural tendency to go up, and the heavy elements tend to go down.

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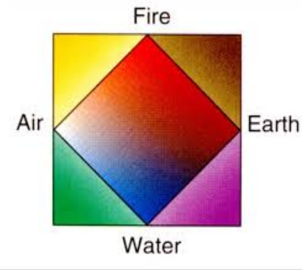
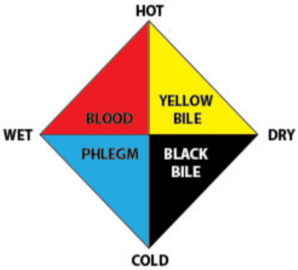
- The material world is composed of *four elements*, which are: **fire, air, water and earth**.
  The elements are endowed with two types of opposite qualities: "lightness" and "heaviness", also "dry" or "wet". The light elements display a natural tendency to go up, and the heavy elements tend to go down.
- Physical bodies are endowed with some internal force, or "spirit" that pulls or pushes them towards similar ones: the heavy metals and ores are found deep under ground, water naturally tends to join the ocean. If some amount of air is contained in water, upon heating it liberates itself and joins the atmosphere. The flames of fire go up attracted by the eternal fire above the skies.

The four elements + the fifth one

All physical bodies are just different combinations of four elements, in various proportions. The four elements are found in the "sublunar world", while the heavens are filled with the fifth element, (from which the word *quintessence* is derived).





The four elements of which all matter on Earth is composed. Parallelly, four fluids called "humors" circulate in human body, according to Hippocrates (460 – 370 B.C.E.)

Aristotle's physics

- **1**. All bodies tend to approach their natural position with respect to the center of the Earth. For example, stones fall down, while sparks rush upwards from fire.
- **2**. The universal gravity acts either towards the center of the Earth, or outwards, depending on objects and their characteristic features. This is why fishes swim in water, and birds fly in the air.
- **3**. No motion can exist without natural cause. When external force different from gravity acts on a body, it starts to move along a straight line with constant velocity.
- **4**. Velocity of free fall is proportional to the density of the falling object and inversely proportional to the density of the surrounding medium. The same stone falls rapidly in air, and slowly in water.
- **5**. There is no vacuum in nature, because the bodies would fall down with infinite velocities.

Aristotle's physics

- **6**. The space is filled with matter. The outer space beyond the Earth's atmosphere is filled with aether, or the "fifth element" (*quinta essentia* in Latin), different from the four elements of which ordinary matter is composed.

- **7**. The Universe is finite and perfect. There is nothing beyond it, and the questioning the whereabouts of the Universe makes no sense.

- **8**. Matter can not be composed of atoms, because if such were the case, there would be vacuum between them.

- **9**. Also all celestial bodies, including the Sun, the Moon and the planets, are made of aether. They are radically different from material bodies we meet in our everyday life.

- **10**. The Cosmos is eternal and not subject to changes. The Sun, the Moon and the planets are perfect spheres which keep their form forever.

- **11**. All celestial bodies move with constant velocity along circular orbits.

The crystalline celestial spheres

Mathematician Eudoxus of Cnidus (IV century B.C.) conceived of a series of geometric models to explain the complex motions of the planets with respect to the Earth, considered immobile at the centre of the Universe. Each model employed three or four spheres concentric to the Earth, and uniformly revolving one inside the other.

Callippus of Cyzicus (IV century B.C.) made these models more faithful to the phenomena observed, increasing the number of spheres up to four or five per planet.

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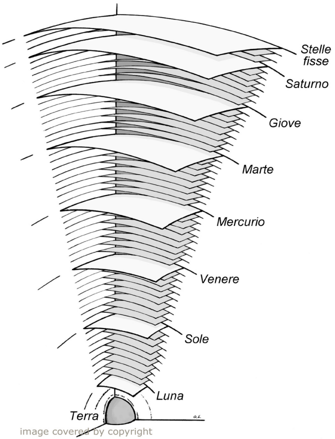
- All celestial spheres were formed of crystalline matter, innate, eternal, incorruptible, imponderable and perfectly transparent (De Caelo , II, 1) - ether or quintessence - quite different from the other four elements that made up the heavy and corruptible sub-lunar world: earth, water, air and fire.


Stelle
fisse
Saturno
Giove
Marte
Mercurio
Venere
Sole
Luna
Terra

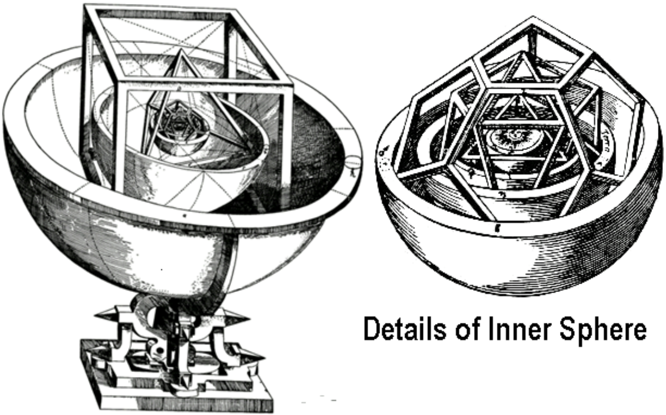


Figure: "Mysterium Cosmographicum" book (1596).

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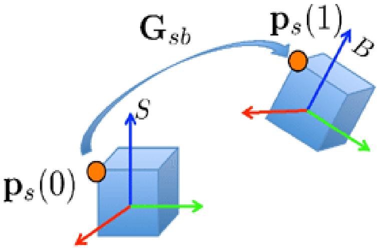
- In fact, motion, velocity, momentum and energy were quasi synonimous. The trace of such thinking persisted until XVIII century, when the energy $mv^2/2$ was called "VIS VIVA" ("living force")

Motion of an arrow after Aristotle

Motion of an arrow according to Aristotle: composition of rectilinear and circular motions.

Translations and rotations



Aristotle was partially right: any rigid motion can be decomposed into a superposition of a rotation with a translation.

Infinitesimal motions form a semi-simple product of two Lie groups, $SO(3)$ and T_3, three dimensional rotations and translations

- **In 1554 venetian mathematician Giovanni Battista Benedetti published a book "*Demonstratio*" refuting Aristotelean postulates concerning the free fall. The argument was simple: take two bodies A and B, with average densities ρ_A and ρ_B, respectively. If $\rho_A > \rho_B$, according to Aristotle the body A should be falling more rapidly than the body B.**

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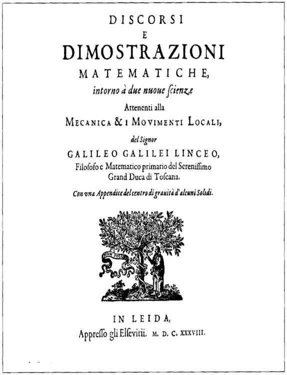
- Now let us glue the two bodies together so that they would fall as one. The average density of the combined body will be ρ_C, with $\rho_A > \rho_C > \rho_B$, and the agglomerate should fall down less rapidly then the body A, although more rapidly than the body B. However, this is not what is observed: the conglomerate of A and B falls down *at least* as rapidly as A itself, if not faster.

Galilei and his last book (1638)



DISCORSI
E
DIMOSTRAZIONI
MATEMATICHE,
intorno à due nuoue scienze
Attenenti alla
MECANICA & I MOVIMENTI LOCALI,
del Signor
GALILEO GALILEI LINCEO,
Filosofo e Matematico primario del Serenissimo
Grand Duca di Toscana.
Con una Appendice del centro di grauità d'alcuni Solidi.

IN LEIDA,
Appresso gli Elsevirii. M. D. C. XXXVIII.

Galileo Galilei (1564-1642)


Between 1589 and 1592, Galileo Galilei is said to have dropped two spheres of different masses from the Leaning Tower of Pisa to demonstrate that their time of descent was independent of their mass. A similar experiment took place some years earlier (1585-1586) in Delft in the Netherlands, when Simon Stevin and Jan Cornets de Groot conducted the experiment from the top of the Nieuwe Kerk. The experiment is described in Simon Stevin's 1586 book De Beghinselen der Weeghconst (The Principles of Statics)

The experiment Galileo and Stevin were able to perform to show that masses fall down in the same manner independently of their weight and density provided both were heavy enough, was realized using the famous Tower of Pisa, which even at that time was leaning towards one side.

One of the consequences of universality of free fall discovered by Galileo was that a freely falling body is in the state of weightlessness. He illustrated this conclusion by considering a man carrying a quarter of beef on his shoulders.

The experiment is described in Simon Stevin's 1586 book De Beghinselen der Weeghconst (The Principles of Statics)
Let us take (as the highly educated Jan Cornets de Groot, the diligent researcher of the mysteries of Nature, and I have done) two balls of lead, the one ten times bigger and heavier than the other, and let them drop together from 30 feet high, and it will show, that the lightest ball is not ten times longer under way than the heaviest, but they fall together at the same time on the ground. (...) This proves that Aristotle is wrong.

The first Galileo's hypothesis: Zeno's paradox again!
Initially Galileo supposed that the velocity v of a freely falling body is proportional to the distance s it covered since the beginning of fall: $v = Cs$.
Here is Galileo's reasoning. Let the total distance covered by the freely falling mass be h, and let the final velocity when it hits the ground be $V = Ch$, supposing that the initial velocity was 0.

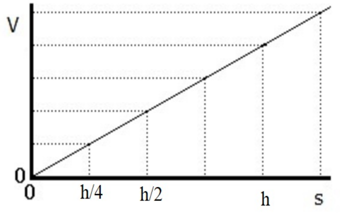


Figure: Linearly dependent velocity $V = Cs$ versus covered distance s.

Let us consider the last half of the total distance covered by falling mass, i.e. from the altitude $h/2$ till the ground level. According to linear law, the velocity was equal to $\frac{Ch}{2}$ when the object was half way to the ground, and Ch at the end. Therefore the average velocity between the positions $\frac{h}{2}$ and h is

$$< v >_1 = \frac{1}{2}\left(\frac{Ch}{2} + Ch\right) = \frac{3}{4}\,Ch.$$

According to the relation between distance Δs covered during time Δt and the average velocity $< w >$, $\frac{\Delta s}{\Delta t} =< w >$, we conclude that the time Δt_1 during which the body fell from the height $h/2$ down to the ground is equal to

$$\Delta t_1 = \frac{\Delta s}{< v >_1} = \frac{\frac{h}{2}}{\frac{3Ch}{4}} = \frac{2C}{3}$$

Consider now the part of the free fall starting from the altitude h above the ground contained between $\frac{3h}{4}$ and $\frac{h}{2}$, so that the distance between these points is equal to $h/4$. The average velocity on this part of the course is

$$< v >_2 = C\,\frac{1}{2}(\frac{h}{4} + \frac{h}{2}) = \frac{3C}{8}.$$

Dividing the length of the run $h/4$ by the corresponding average velocity $< v >_2$ we get the time Δt_2 necessary to travel the distance $h/4$ separating the points $3h/4$ and $h/2$ above the ground equal to the time Δt_1 that was necessary to travel the distance separating the points $h/2$; indeed,

$$\Delta t_2 = \frac{h/4}{\frac{3Ch}{8}} = \frac{2C}{3} = \Delta t_1.$$

- It is not necessary to repeat this exercise: we can easily conclude, as Galileo did in his time, that continuing to divide the run into smaller parts, each time dividing the previous distance by 2, one must conclude that the free fall from the height $y = H$ to the ground level $y = 0$ will take infinite time, being an infinite sum of equal contributions, each of them lasting $\Delta t_i = 2C/3$, for $i = 1, 2, 3, ...\infty$. In fact, this was another version of Zeno's paradox.

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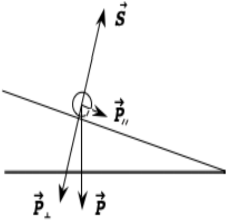
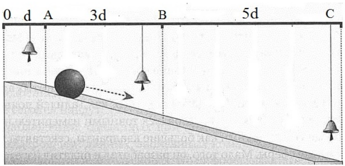
- In modern terms, it is enough to write the following integral: if $V = Cs$, then $dt = \frac{ds}{V(s)} = \frac{ds}{Cs}$;

$$t = \int_h^0 dt = -\int_h^0 \frac{ds}{Cs} = -(Ln0 - Lnh) = \infty.$$

((the minus sign is there because the distance s is decreasing from h till 0)

Galileo's laws of free fall





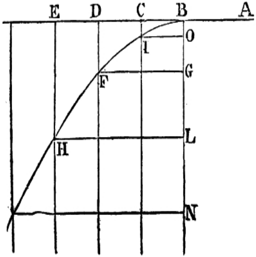
Decomposition of the force of gravity

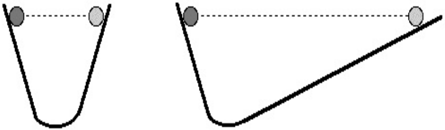
Galilei's experiment with bells.

$$V \simeq gt; \quad s(T) =< V >= \frac{V(0)+V(T)}{2} T = \frac{0+gT}{2} T = \frac{gT^2}{2}$$

(Fallacious reasoning, but right answer obtained nevertheless)

Galileian motion

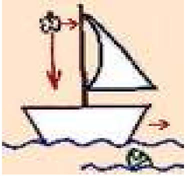




Galileo's sketch of parabolic motion in Earth's gravity

Zero acceleration = argument for the uniform motion

Galileian Relativity

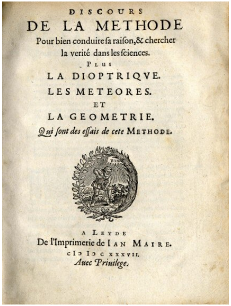


$$t' = t$$
$$x' = x - vt$$
$$y' = y$$
$$z' = z$$

Galileo's ship: the same laws of free fall

Galileo's transformation: the second time derivatives of coordinates x and x' are equal. Zero acceleration = zero force.

René Descartes and principle of inertia


DISCOURS
DE LA METHODE
Pour bien conduire sa raison, & chercher
la verité dans les sciences.
Plus
LA DIOPTRIQVE.
LES METEORES.
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LA GEOMETRIE.
Qui sont des essais de cete METHODE.
A LEYDE
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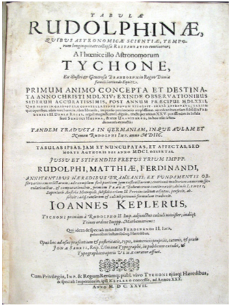
French philosopher René Descartes (1596-1650).

According to Descartes, no change of the state of motion is possible without external action. First hint towards a conservation law of quantity of motion

Johannes Kepler and planetary motion


TABULÆ
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A Phœnice illo Astronomorum
TYCHONE,
PRIMUM ANIMO CONCEPTA ET DESTINA-
TA ANNO CHRISTI MDLXIV: EXINDE OBSERVATIONIBUS
SIDERUM ACCURATISSIMIS, POST ANNUM PRÆCIPUE MDLXXII,
TANDEM TRADUCTA IN GERMANIAM, INQUE AULAM ET
TABULAS IPSAS, JAM ET NUNCUPATAS, ET AFFECTAS, SED
MORTE AUTHORIS SUI ANNO MDCI DESERTAS
JUSSU ET STIPENDIIS FRETUS TRIUM IMPPP.
RUDOLPHI, MATTHIÆ, FERDINANDI,
IOANNES KEPLERUS,
Cum Privilegiis, Imp. & Regum Rerumq; publ. vivo TYCHONI ejúsq; Hæredibus,
ANNO M. D C. XXVII.

Johannes Kepler (1576-1630) and Tycho Brahe's Martian Tables.
Analyzing astronomical data gathered by Tycho Brahe, Kepler discovered three laws of planetary motion

Kepler's laws of planetary motion

- • **First Law:** Planets move along elliptical orbits, the Sun being placed in one of the focal points.

  • **Second Law:** During the revolution, planets sweep equal areas in equal times.

  • **Third Law:** The squares of periods of revolution are proportional to the cubes of great axes of planet's elliptical orbits.

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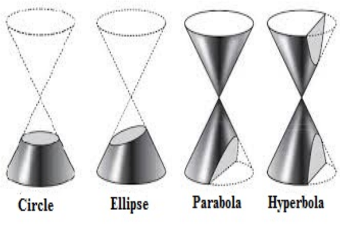
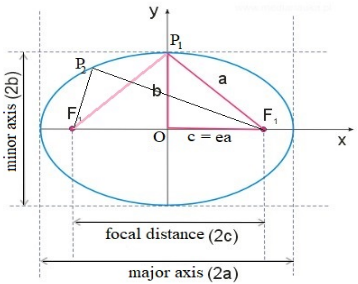
  • **Third Law:** The squares of periods of revolution are proportional to the cubes of great axes of planet's elliptical orbits.
- The first two laws are illustrated by the diagram below, in which a planet orbits around the Sun whose fixed position coincides with one of the foci. Equal areas swept by the planet during equal time intervals are represented by hatched sectors, thinner when the planet is close to aphelion, and wider close to perihelion.





In an analytic form, ellipse can be expressed either in a Cartesian, or in a polar system of coordinates:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad r(\varphi) = \frac{p}{1 - e\ \cos\varphi}. \tag{1}$$

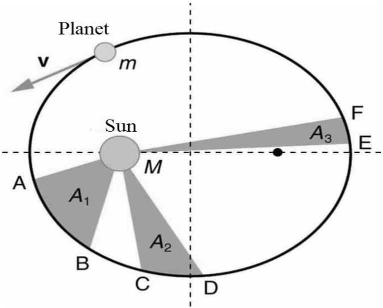


Figure: Illustration of Kepler's Second Law. The areas swept by a planet in equal times are equal.

In fact, the second law is the expression of conservation of planet's angular momentum (with Sun supposed to be motionless, at the centre of a Galilean reference frame)

- The proof was given by Newton in his *Principia*, using purely geometrical tools and the "calculus of fluxions" invented on this occasion. Of course Kepler did not have at his disposal Newton's mathematical apparatus and Newton's laws of motion and gravity; nevertheless his views on what causes planetary motion anticipated Newton's approach.

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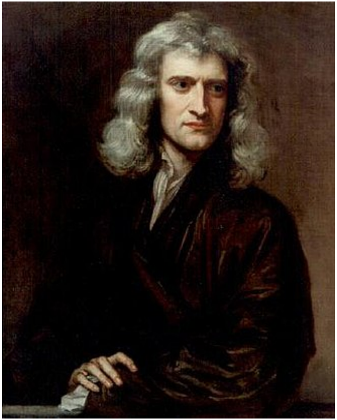
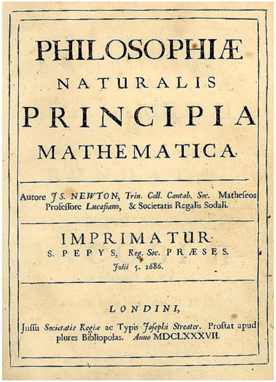
- Kepler wrote that the Sun, being in Universe's center, rules the planets and their motions through a mysterious force of attraction which he called "magnetism". He also foresaw that such a force should decrease with distance, although erroneously ascribed to it a simple inverse law $\simeq r^{-1}$

Isaac Newton

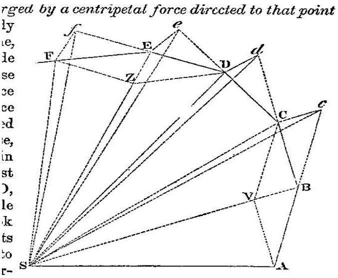


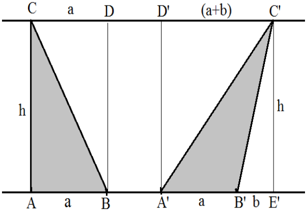

PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore J S. NEWTON, Trin. Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, Reg. Soc. PRÆSES.
Julii 5. 1686.

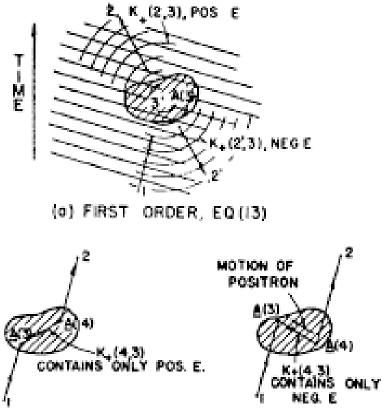
LONDINI,
Jussu Societatis Regiæ ac Typis Josephi Streater. Prostat apud plures Bibliopolas. Anno MDCLXXXVII.





This is how Newton proved Kepler's second law. A body attracted by a central force follows a rectilinear uniform motion until it receives a "kick" - the gravitational force from the central massive body. It changes its direction and velocity instantly, then the body continues rectilinear motion in new direction again, until the next "kick".

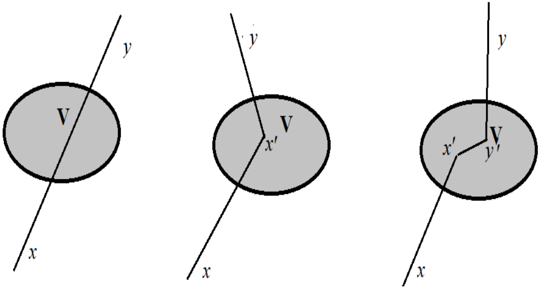
A similar idea of consecutive "kicks" separated by uniform rectilinear motion was used by Richard Feynman (1918 – 1988) in his novel aopproach to quantum mechanical description of scattering processes.



(a) FIRST ORDER, EQ (13)

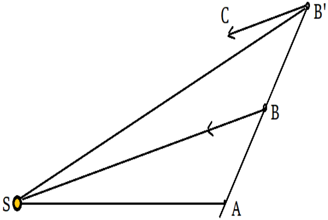
(b) VIRTUAL SCATTERING $t_4 > t_3$ (c) VIRTUAL PAIR $t_4 < t_3$

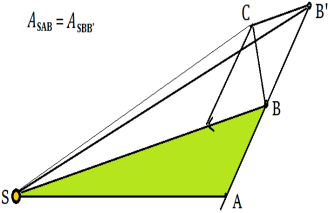
SECOND ORDER, EQ. (14)

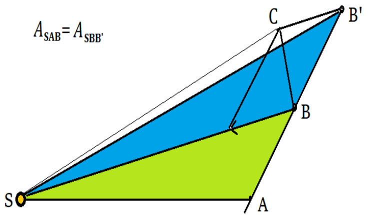


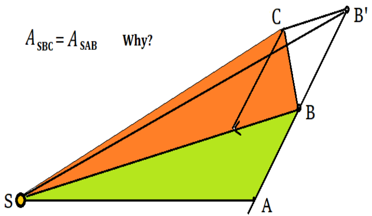
$$f(y) = \int G(x,y) \quad G(x,y) + \int G(x,x')V(x')G(x',y)dx' + \int G(x,x')V(x')G(x',y')V(y')G(y',y)dx'\,dy' + \ldots$$

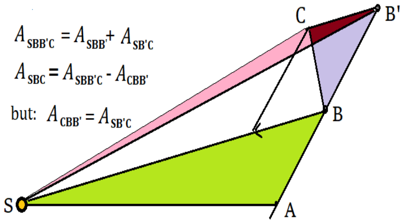
Scattering of particle-wave interacting with potential $V(x)$ limited in space is a sum of probabilities with zero, one, two,... N subsequent "kicks"

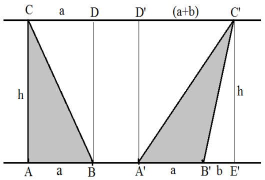

C
B'
B
S
A


$A_{SAB} = A_{SBB'}$
C
B'
B
S
A


$A_{SAB} = A_{SBB'}$
C
B'
B
S
A


$A_{SBC} = A_{SAB}$
Why?
C
B'
B
S
A


$A_{SBB'C} = A_{SBB'} + A_{SB'C}$
$A_{SBC} = A_{SBB'C} - A_{CBB'}$
but: $A_{CBB'} = A_{SB'C}$
C
B'
B
S
A


C
a
D
D'
(a+b)
C'
h
h
A
a
B
A'
a
B'
b
E'

The most amazing thing is that the proof of equal areas did not mention the dependence of the attracting force on distance - the only assumption was that the force is directed always towards S.
In modern terms, a radial attractive force directed towards a fixed central point O can be written as

$$\mathbf{F} = -\frac{f(r)\mathbf{r}}{r}$$

. The *angular momentum* of material body at a point $\mathbf{r}$ with velocity $\mathbf{V} = \frac{d\mathbf{r}}{dt}$ is

$$\mathbf{M} = \mathbf{r} \wedge \mathbf{V} = \mathbf{r} \wedge \frac{d\mathbf{r}}{dt}$$

We have then:

$$\frac{d\mathbf{M}}{dt} = \frac{d\mathbf{r}}{dt} \wedge \mathbf{V} + \mathbf{r} \wedge \frac{d\mathbf{V}}{dt} = \mathbf{V} \wedge \mathbf{V} + \mathbf{r} \wedge \mathbf{F} = 0 + \frac{f(r)}{r} \mathbf{r} \wedge \mathbf{r} = 0.$$

- **In his *Principia Mathematica* Newton formulated three fundamental laws of mechanics of material bodies. The motion with constant velocity, according to Galileo, could not be detected by the obserever without being able to see the surrounding world move in the opposite direction. Uniform motion can not be dinstinguished from rest. Only an accelerated motion is immediately detectable by inertial forces opposing the exterior force being the source of the acceleration.**

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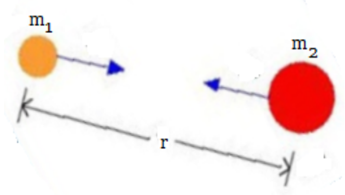
-

$$\Delta(m\mathbf{v}) = \mathbf{F}\,\Delta t. \tag{2}$$

Finally, the third law stipulated that if a body A acts on a body B with certain force **F**, then the body B acts on the body A with an opposite force $-\mathbf{F}$ (this law is often called the action-reaction law).

Universal gravity



$$F_g = G\frac{M\,m}{d^2}$$

$$G = 6.67 \times 10^{-11}\ N\ m^2 / kg^2$$

Newton found the proof that elliptic orbits in a central potential are possible only with the inverse-square law

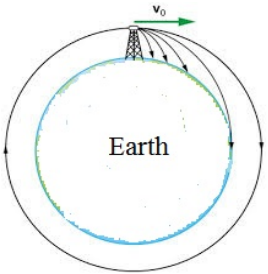
Newton's Law of universal gravitational attraction can be formulated as follows: any massive body attracts another massive body with force that is parallel to the line joining their centers of mass, proportional to the product of both masses, and decreasing as the inverse square of distance separating them. According to Newton's third law,

$$\mathbf{F}_{12} = -\frac{GMm\mathbf{r}_{12}}{\mid \mathbf{r}_{12} \mid^3} = \mathbf{F}_{21} \tag{3}$$

In the case when the massive spherically symmetric body M is placed at the center of coordinate system, the gravitational force it produces acting on a small mass m at a distance $\mathbf{r}$ can be written in simple form:

$$\mathbf{F} = -\frac{GmM\mathbf{r}}{\mid \mathbf{r} \mid^3} \tag{4}$$

Let us consider a shot with horizontal initial velocity, just little above the surface of our planet, as displayed in (8) below:



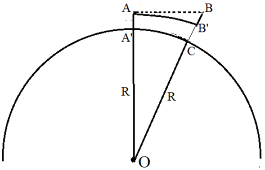


Figure: A horizontal shot with sufficient initial velocity may end up by creating an artificial satellite. On the right: the scheme explaining the calculus of first cosmic velocity (satellization near Earth's surface).

Let us evaluate the velocity at which an object will never fall down, continuing its way at the same initial height above the surface. Using the additivity of velocities, the horizontal component of the initial velocity **V** is conserved, while the vertical component is subjected to the law of the free fall. The downward acceleration at Earth's surface has the value $g = 9.81\ m/sec^2$; according to the law of free fall, during the first second the falling body travels $h = gt^2/2 = 4.905$ **m** towards the Earth's center.

The picture on the right of (8) shows (in an exaggerated manner) that in order to remain at the same distance from the Earth, the body must travel horizontally the distance denoted by x.
All distances in the figure are exaggerated; in fact, the difference between the curved segment AB' and the straight segment AB is negligible, compared to Earth's radius $R = 6\,380 km$. Similarly, the height AA' of the initial point A above the surface should be invisible on the drawing, so small it is with respect to Earth's radius, as well as h.

Applying Pythagoras' theorem to the triangle AOB, and supposing that the initial position A is so close to the surface that we can pose $OA = R$, we get the following identity:

$$(R+h)^2 = R^2 + x^2, \quad \text{or} \quad R^2 + 2Rh + h^2 = R^2 + x^2, \qquad (5)$$

from which we get a simple fomula neglecting the term $h^2 << Rh$:

$$x^2 \simeq 2Rh, \quad \text{so that} \quad v_{sat} = \sqrt{2Rh} = \sqrt{gR}. \qquad (6)$$

if we insert $h = gt^2/2$ with $t = 1$ second, so that the dimension of $\sqrt{gR}$ is that of velocity - here denoted by v_{sat}, the *satellization velocity* on the surface of Earth. Inserting in (6) $g = 9.81 m/sec^2$ and $R = 6\,380$ km, we obtain the satellization velocity equal to 7.91 kilometers per second.

Let us to evaluate the period of revolution of such an artificial satellite: the full rotation around the big circle is equivalent to traveling the distance of 40000 km; Earth's circumference. Dividing this distance by the satellization velocity, we get

$$T_{min} = \frac{40000}{7.91} = 5057 \text{ seconds} = 84 \text{ minutes} = 1 \text{ hour } 84 \text{ minutes}. \quad (7)$$

This period on a circular orbit with radius equal to 6 380 km will be useful when we shall compare it with another Earth's satellite, with greater orbital radius and with longer period, accordingly to Kepler's third law

If mechanical laws apply equally well to heavenly bodies as to material objects near the surface of the Earth, then the motion of the Moon should be analyzed in the same terms as the imaginary artificial satellite close to Earth's surface. This means that the equation (6) can be used, too, but with different unknown quantity to be determined: the centripetal acceleration a due to Earth's at a distance $D_M = 384\,000$ km. Moon's orbital velocity on its trajectory around the Earth was already well known: it is enough to divide the circumference of lunar orbit D_L by the sidereal month expressed in seconds. The result is

$$< v_L >= \frac{2\pi \times 384\,000 \mathrm{km}}{27.32 \times 24 \times 3600 \text{ seconds}} = 1.021 \ \mathrm{km/sec}. \qquad (8)$$

Inserting this value along with the average radius of Moon's orbit D_L into the formula (6), where gravitational acceleration on Earth's surface is replaced by the unknown value a of gravitational acceleration on Moon's orbit, we get

$$< v_L > = \sqrt{a D_L} \quad \rightarrow a = \frac{< v_L >^2}{D_L} = 2.714 \times 10^{-6} \mathrm{km \cdot sec}^{-2} = 2.714 \mathrm{mm}/ \tag{9}$$

an extremely low value, which means that the gravitational pull of the central mass (here the Earth) is considerably weakened with growing distance. But what is the exact mathematical expression describing the phenomenon? Well, let us compare the two accelerations first:

$$\frac{a}{g} = \frac{2.714 \times 10^{-3} \mathrm{msec}^{-2}}{9.81 \mathrm{msec}^{-2}} = 2.766 \cdot 10^{-4}. \tag{10}$$

If the gravitational pull was inversely proportional to the distance from the attracting central mass, this ratio would be equal to the inverse ratio between the distance to the Moon and the Earth's radius; but that ratio is equal to $(D_L/R)^{-1} = R/D_L = (6380\mathrm{km})/(384000\mathrm{km}) = 1.66 \cdot 10^{-2}$, which is by two orders of magnitude greater than the value obtained in (10). But the square of this number gives a quasi-perfect fit:

$$\left(\frac{D_L}{R}\right)^2 = \frac{R^2}{D_L^2} = \frac{(6380\mathrm{km})^2}{(384000\mathrm{km})^2} = 2.761 \cdot 10^{-4} \qquad (11)$$

The coincidence is remarkable, taking into account that Newton was using a simplified version of Moon's orbit, which in fact is not circular, but elliptic with eccentricity more than 5%; but even so, the accuracy of the law so obtained is of the order of less than 1%.

Kepler's Third Law states that for satellites revolving around the same central body the ratio between the squares of their rotation periods is equal to the ratio between the cubes of corresponding orbits' radii (at this moment, we are considering circular orbits only). Let us compare:

$$\left(\frac{T_{Moon}}{T_{sat}}\right)^2 = \left(\frac{39\,341}{84}\right)^2 = (468.35)^2 = 219347;$$

$$\left(\frac{D_{Moon}}{R_T}\right)^3 = \left(\frac{384\,400}{6380}\right)^3 = (60.251)^3 = 218721. \qquad (12)$$

Again, the coincidence is excellent: the ratio between the two numbers is 1.0029, the error less than one-third of 1%. However, Kepler's third law is not restricted to circular orbits: in general case closed orbits are ellipses, and circle's radius is replaced by the semi-great axis a.

The analytic definition of elliptical orbit combined with Kepler's second law lead directly to the inverse square law of gravitational force. Here is one of the shortest mathematical demonstrations. Let us take the time derivative of $r(t)$ describing an elliptical orbit:

$$\dot{r} = \frac{d}{dt}\left[\frac{p}{1 + e\cos\varphi(t)}\right] = \frac{p\; e\dot{\varphi}\sin\varphi}{(1 + e\cos\varphi)^2}. \tag{13}$$

Substituting $(1 + e\cos\varphi)^{-2} = p^{-2}r^2$, we get $\dot{r} = ep^{-1}r^2\dot{\varphi}\;\sin\varphi$ and using the second Kepler's law $r^2\dot{\varphi} = l =$ Constant, we can write the radial velocity as:

$$\dot{r} = \frac{el}{p}\sin\varphi. \tag{14}$$

With constants p, l and e the only time dependent function in this expression is $\varphi(t)$, which makes the time derivation of $\dot{r}$ particularly easy:

$$\frac{d}{dt}(\dot{r}) = \ddot{r} = \frac{el}{p}\,\dot{\varphi}\cos\varphi. \tag{15}$$

Now, $e\ \cos\varphi$ can be expressed as a function of r using the constitutive equation of the ellipse:

$$e\cos\varphi = \frac{p}{r} - 1, \tag{16}$$

which we insert now in (15):

$$\ddot{r} = \frac{el}{p}\,\dot{\varphi}\cos\varphi = \frac{l}{p}\,\dot{\varphi}\,\left(\frac{p}{r} - 1\right) \tag{17}$$

Recalling that $l = r^2\dot{\varphi}$ we can substitute l in the first term, and express $\dot{\varphi}$ as function of r in the second, $\dot{\varphi} = l\ r^{-2}$ to obtain

$$\ddot{r} = r\dot{\varphi}^2 - \frac{l^2}{p}\,\frac{1}{r^2}, \tag{18}$$

where we recognize two essential terms on the right side of the equation for radial acceleration: the centrifugal acceleration $r\dot{\varphi}^2$ and acceleration directed towards the focus (i.e. the center of the coordinate system), proportional to the inverse square of radius-vector's length r.

There is an extra bonus in formula (18): it is valid for *any* value of eccentricity e. This means that the inverse square law for central attracting force causing centripetal acceleration keeps it validity not only for closed elliptical orbits, but also the parabolic ($e = 1$) and hyperbolic ($e > 1$) ones. The converse is also true, i.e. the motion of a point-like mass under the action of central force proportional to the inverse square of distance follows necessarily one of the conic curves.

The Kepler's Third Law

Let us rewrite the formula for radial acceleration:

$$\ddot{r} = r\dot{\varphi}^2 - \frac{l^2}{p}\,\frac{1}{r^2}, \qquad (19)$$

For a circular motion $r = R =$ Constant, therefore both $\frac{dr}{dt}$ and $\frac{d^2r}{dt^2}$ vanish; and we get

$$R\left(\frac{d\varphi}{dt}\right)^2 - \frac{MG}{R^2} = 0 \qquad (20)$$

The constant angular velocity is $\omega = \frac{d\varphi}{dt}$, bt $\omega = \frac{2\pi}{T}$, where T is the period of the orbital motion. Therefore we get

$$\frac{4\pi^2}{T^2} = \frac{MG}{R^3}$$

Even after *Principia* were published, Newton continued to have profound doubts concerning action at a distance, as evidenced by these words from a letter to Bentley written around 1692:
"*It is inconceivable that inanimate Matter should, without the Mediation of something else, which is not material, operate upon, and affect other matter without mutual Contact [...] Gravity must be caused by an Agent acting constantly according to certain laws*".
It took almost two centuries until the concept of *field* was introduced by Maxwell and Faraday. Nevertheless they introduced the *aether*, hypothetical luminoferous substance as medium through which light propagates. How such substance, no matter how thin but permeating whole space, would not stop the motion of planets and stars, remained unexplained.

Another shortcoming of which Newton was equally aware concerned the concept of absolute space and absolute time. That the time flows at the same rate everywhere in the Universe seemed obvious and conformal with common sense, as Newton expressed it in "Principia":

"*I do not define time, space, place, and motion, as they are well known to all. Absolute space by its own nature, without reference to anything external, always remains similar and unmovable.*"

However the existence of the unique coordinate system attached to the empty space independent of any material bodies and their mutual positions was much less obvious, as follows from another passage:
"*It is indeed a matter of great difficulty to discover and effectually to distinguish the true motions of particular bodies from the apparent, because the parts of that immovable space in which these motions are performed do by no means come under the observations of our senses.*"
Nevertheless, Newton found arguments that seemed irrefutably prove that although motion with constant speed cannot have absolute meaning as rightly Galileo has shown, the accelerated motion is detectable even without referring to other bodies existing in the Universe.

Another example of physical system able to detect the absolute acceleration was proposed by Newton in the same *Scholium*. Take two massive balls attached to a short rope at its ends. If the balls are at rest and at a distance smaller than rope's length, the rope does not feel any tension; but let the balls turn around their center of mass, and the rope will be immediately under tension, the stronger the higher the angular velocity of rotation.

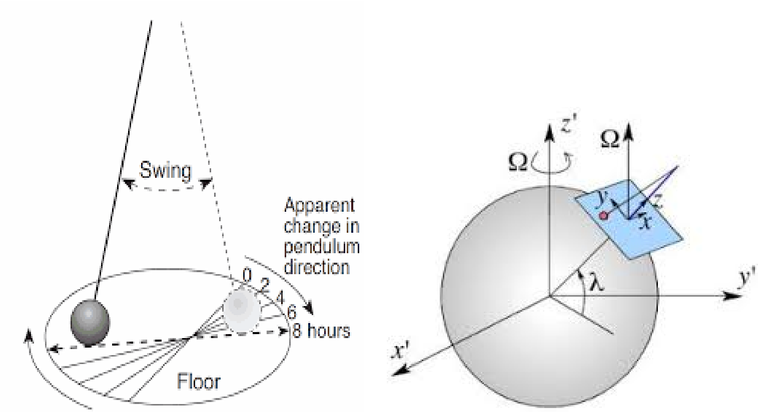


Figure: Left: Foucault's pendulum, Right: Foucault's pendulum on the Earth rotatiing with respect to the distant stars.

Figure: Left: A toy gyroscope, Right: An aircraft gyroscope.

This issue was severely criticized by George Berkeley in his book *De Motu* ("On Motion"). If the water-filled bucket were the unique material body in the Universe, it would be not obvious at all why the liquid should rise towards the walls if there is no material body with respect to which rotation takes place.

The same argument was used against the two balls's rotation tending the rope between them: if nothing else existed in the Universe, how to know whether the rope with the balls is rotating or not?

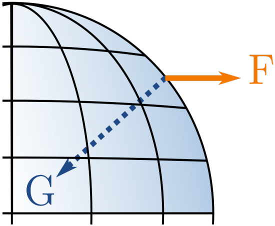
According to Berkley's analysis, not only rectilinear uniform motion can not be detected, but also the accelerated motion and its consequence in the form of inertial force would not exist if there were no other massive bodies in the Universe.

So, who is responsible for the inertial forces? - apparently, it cannot be anything else but all masses filling the Universe, the distant stars who play the role of absolute space - answered a century later German physicist **Ernst Mach** ($1838 - 1916$). As simple common experience shows, two phenomena always coincide: the inertial centrifugal forces and rotation around one's own vertical axis.

This is shown in the Figure (11) below. A man holding two heavy dumbbells starts to turn rapidly; immediately the dumbbells are pushed away from the axis of rotation, and at the same time the man sees the starry sky rotate in opposite direction above his head. Mach's conclusion was that if there were no distant stars, there would be no inertia either.

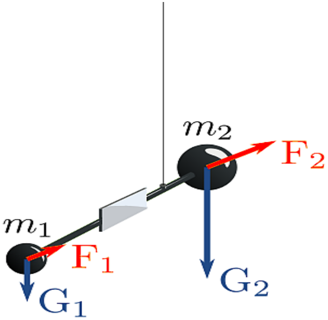
Figure: Left: Newton's rotating bucket, Right: Inertial forces caused by acceleration and simultaneous apparent rotation of distant stars.

Another troublesome fact is the strict equality (up to a choice of units) of *gravitational mass* that appears in Newton's law of universal gravity, and the *inertial mass* appearing in Newton's second law of mechanics, $\mathbf{F} = m\,\mathbf{a}$. Why do we need two different masses, if they are always equal? One is enough, and it should be the gravitational mass. If we adopt this point of view, the inertial mass is a manifestation of gravitational forces, albeit different from the fundamental inverse square law.



Loránd Eötvös, his torsion balance and the forces axcting on it on the rotating Earth.

Gravity and Inertia



Eotvos' torsion balance.

If Mach's hypothesis is true and the inertial mass of a body should not depend on masses that are close to it, but mostly on masses that are very far away - literally, the most remote stars and galaxies. Supposing a uniform distribution of masses in the Universe with mean density ρ, a spherical layer of radius R and of thickness ΔR contains the amount of mass $\rho \times (4\pi R^2 \Delta R)$. The gravitational influence of each particular mass belonging to this layer is proportional to R^{-2}, therefore the total influence of each such layer is the same, equal to the product $\rho \times (4\pi R^2 \Delta R) \times R^{-2} = 4\pi\rho\Delta R$. But according to Newtonian theory of gravitation, they should cancel if Universe is homogeneous on a big scale and all directions are equivalent.

The situation would be different if a very weak transversal component of gravitation existed, too, dependent on relative acceleration between masses, and behaving like r^{-1}. Then the influence of successive layers of matter behaves as $\Delta r \times r^2 \times r^{-1} \simeq r$, thus making the ultimate faraway layers of apparently fixed galaxies responsible for the force aligned along relative acceleration of a body, similar to inertial force opposing the accelerated motion. The General Relativity theory developed by Einstein in 1915 contains the possibility of transversal components of gravity field, but it remains unclear how it can produce inertial mass via interaction with faraway matter.

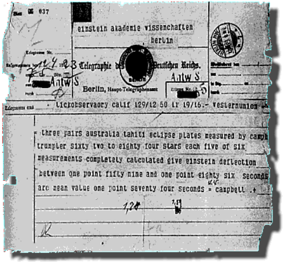
Nobody better than Newton himself expressed the feeling that his theory of gravity does not go deep enough in explaining the mystery of universal attraction. Here is what he wrote in "General Scholium" closing the *Principia*;
"But hitherto I have not been able to discover the cause of those properties of gravity from phaenomena, and I frame no hypotheses; for whatever is not deduced from phaenomena is to be called a hypothesis; and hypotheses, whether metaphysical or physical, have no place in natural philosophy".
The statement about framing no hypotheses, sounding in Latin "Hypotheses non fingo", has become one of the best known Newton's quotations.

The d'Alembert principle

Jean le Rond d'Alembert (1717-1783)

$$\mathbf{F} = ma \to \mathbf{F} - ma = 0. \quad (21)$$

Figure: Einstein in 1922 during his visit in Kyoto, Japan. The General Relativity was already tested by Eddington's observations in 1919.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arc mean value one point seventy four seconds = campbell .+

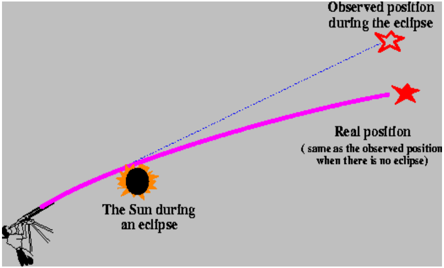

Figure: The telegram sent from South Africa by Eddington to Einstein in Berlin in 1919.