Discrete Newtonian Cosmology

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References

- Ellis & Gibbons (2013) - arxiv:1308.1852
- Ellis & Gibbons (2014) - arxiv:1409.0395
- Vieira, H.S., Bezerra V.B. (2014) - Revista Brasileira de Ensino de Física 36, Issue 3
- Historical papers found in Ellis (2013)

Newtonian Cosmology

Einstein without Einstein

- How to describe cosmological dynamics without using GR?
  - A fluid (usually dust, with $\omega = 0$)
  - A scaling function with nonzero derivative ($a(t)$, $\dot{a} \neq 0$)
  - Hubble-Lemaître law ($v = H(t)r$, $H(t) = \dfrac{\dot{a}(t)}{a(t)}$)
  - Classical hydrodinamical equations (Euler, Poisson and continuity)

NEWTONIAN UNIVERSES AND THE CURVATURE OF SPACE

By W. H. McCREA (*London*) and E. A. MILNE (*Oxford*)

[Received 7 March 1934]

THE present paper investigates the relationship between the universes of relativistic cosmology and the universes that can be constructed using only Newtonian dynamics, Newtonian gravitation, and Newtonian relativity. It is shown that the governing differential equations are identical in form in the two cases, and that the locally observable phenomena predicted on the two theories are indistinguishable. It is further shown that a space of positive curvature corresponds to a Newtonian universe in which every particle has a velocity less than the parabolic velocity of escape from the observer, a space of negative curvature to one in which every particle has a velocity greater than the parabolic velocity. Thus universes of positive, zero, and negative curvatures correspond to elliptic, parabolic, and hyperbolic Newtonian universes respectively. These results are applied to obtain a number of new results concerning Doppler effects and accelerations in relativistic universes. Finally an investigation due to Lemaître is briefly discussed.

Newtonian Cosmology

Einstein without Einstein

- What we CAN describe using Newtonian Cosmology??
  - Background Evolution at Horizon scales
  - Accelerated expansion and Cosmography
  - Nonrelativistic perturbations and structure formation at large enough scales
  - N-body simulations are just starting to include relativistic corrections

Newtonian Cosmology

Einstein without Einstein

- What we CAN'T describe using Newtonian Cosmology?
  - Background Evolution before matter domination ($z \approx 3000$)
  - Structure formation at astrophysical scales
  - Tensor modes (gravitational waves)
  - Pretty much everything else

Newtonian Cosmology

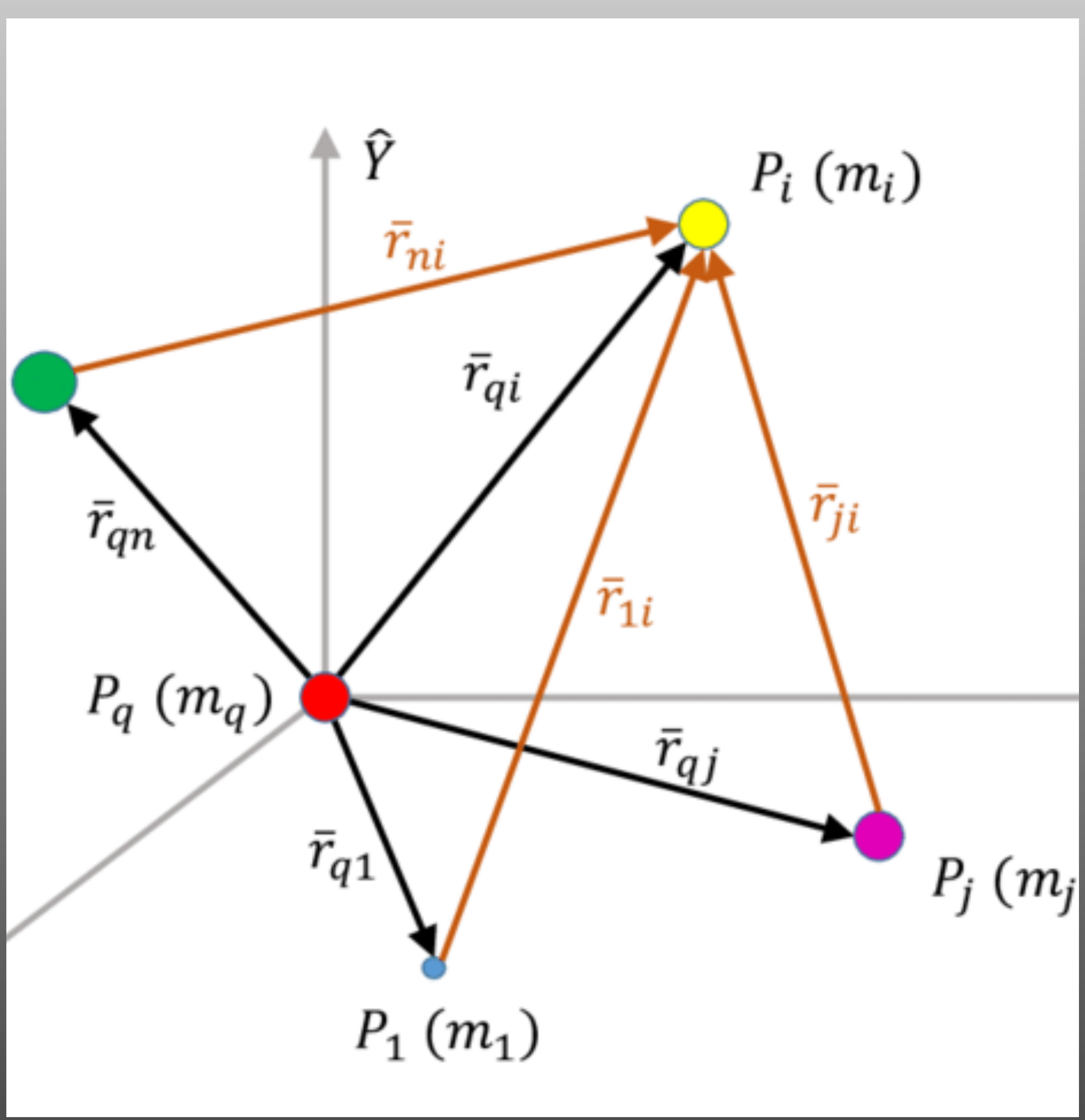
Einstein without Einstein

- Do we need the fluid description?
- Why not point particles, as in celestial mechanics?
- Can we describe the gravitational interactions at large scales as a Newtonian N-body problem?

Discrete Newtonian Cosmology

A discrete approach to Newtonian Cosmology

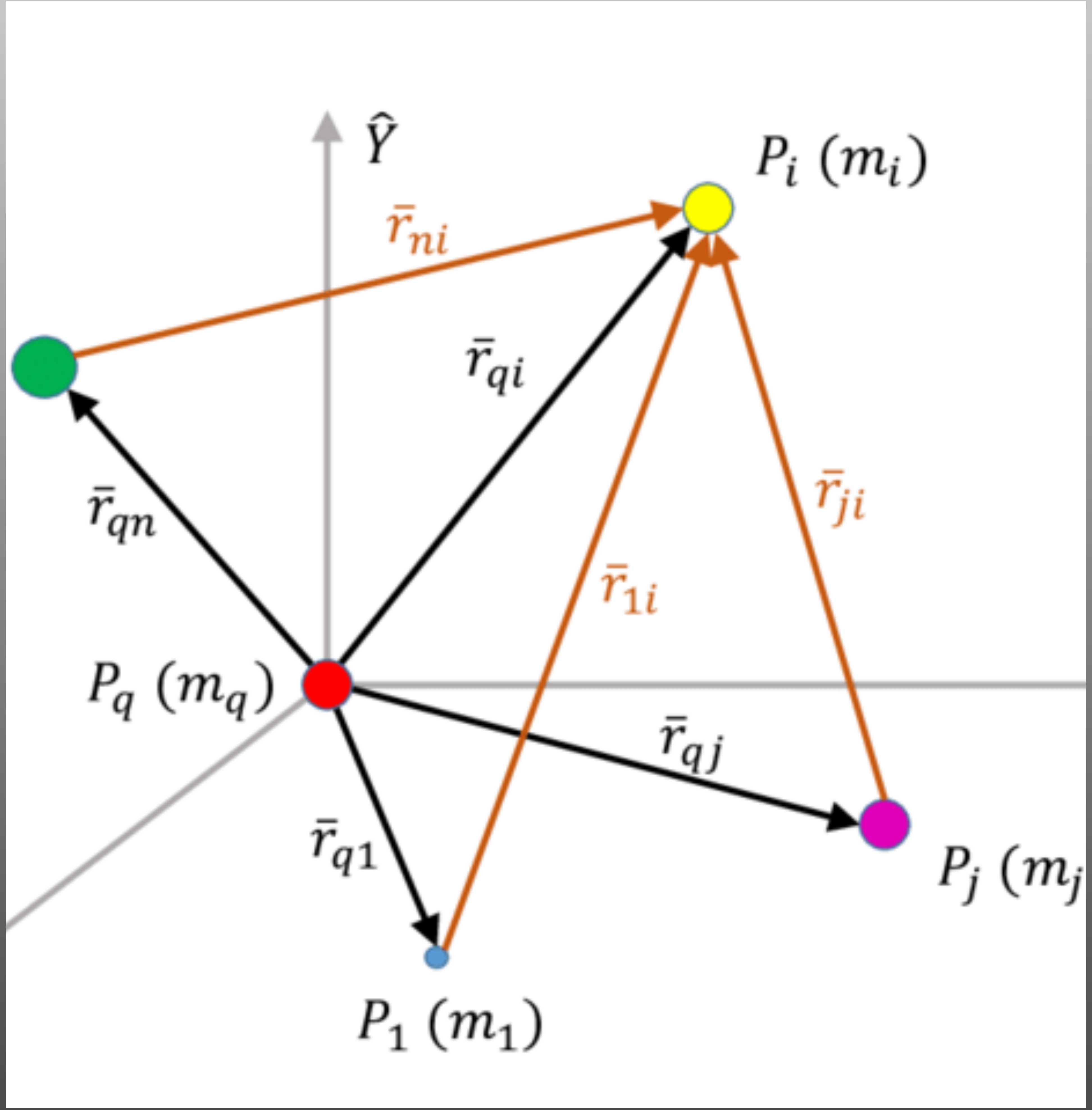
- Suppose we have a discrete set of mechanical point particles interacting gravitationally, instead of a fluid.
- The interactions are purely mechanical and do not suppose a continuum limit or an equation of state.
- No hydrodynamical equations.



Discrete Newtonian Cosmology

A discrete approach to Newtonian Cosmology

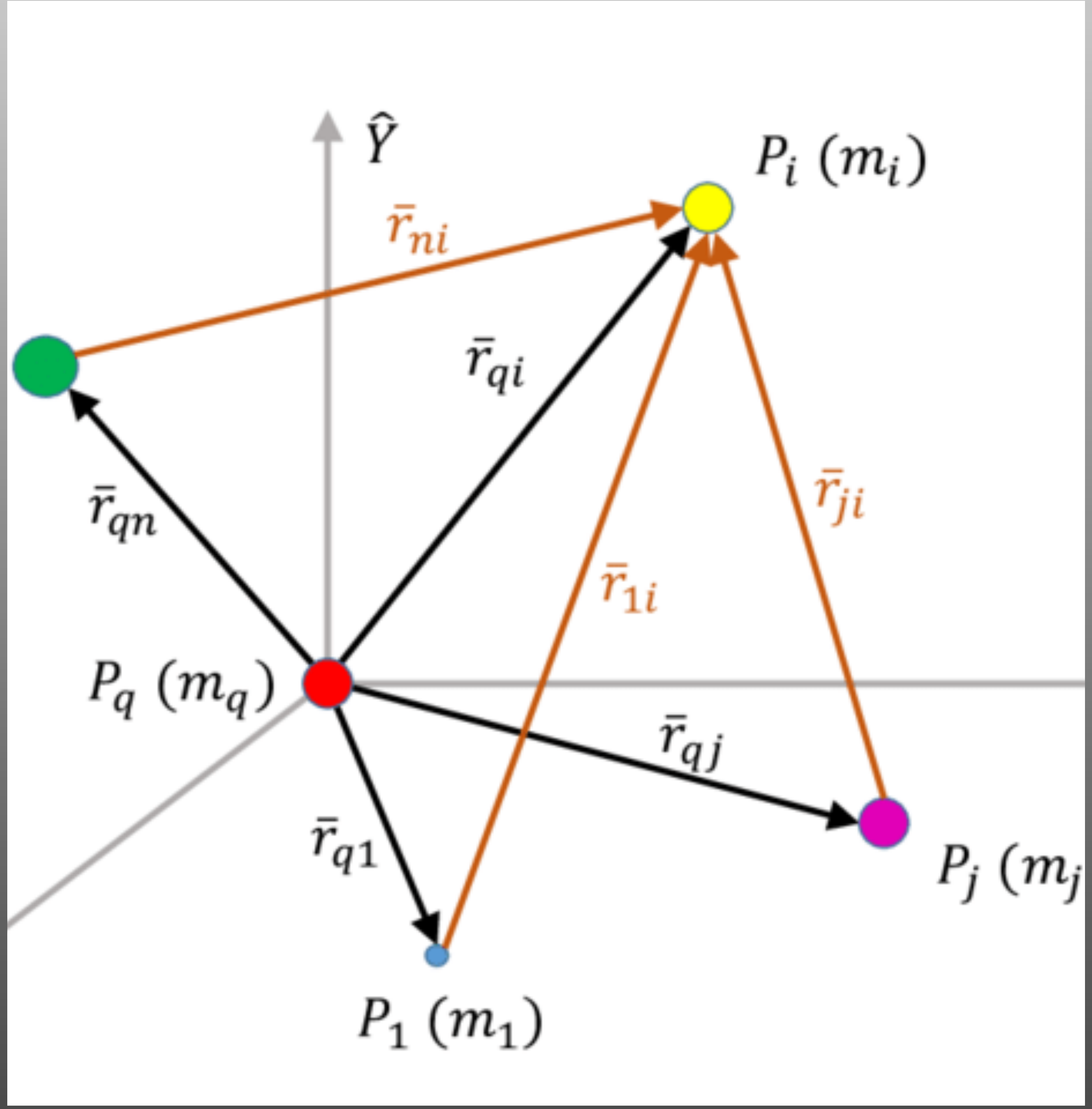
- What are the advantages?



Discrete Newtonian Cosmology

A discrete approach to Newtonian Cosmology

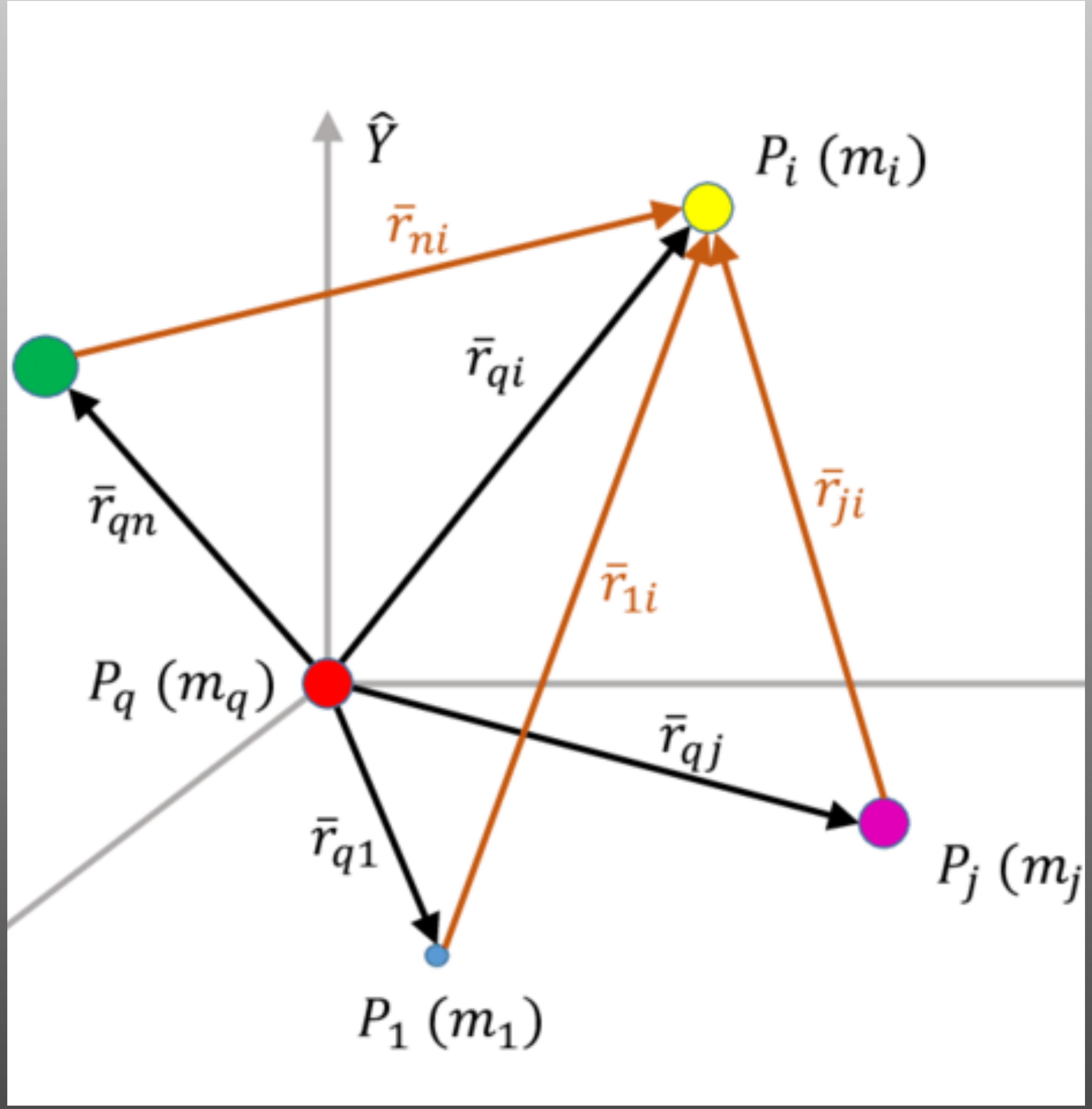
- What are the advantages?
  - No need for the fluid description, particles in the vacuum



Discrete Newtonian Cosmology

A discrete approach to Newtonian Cosmology

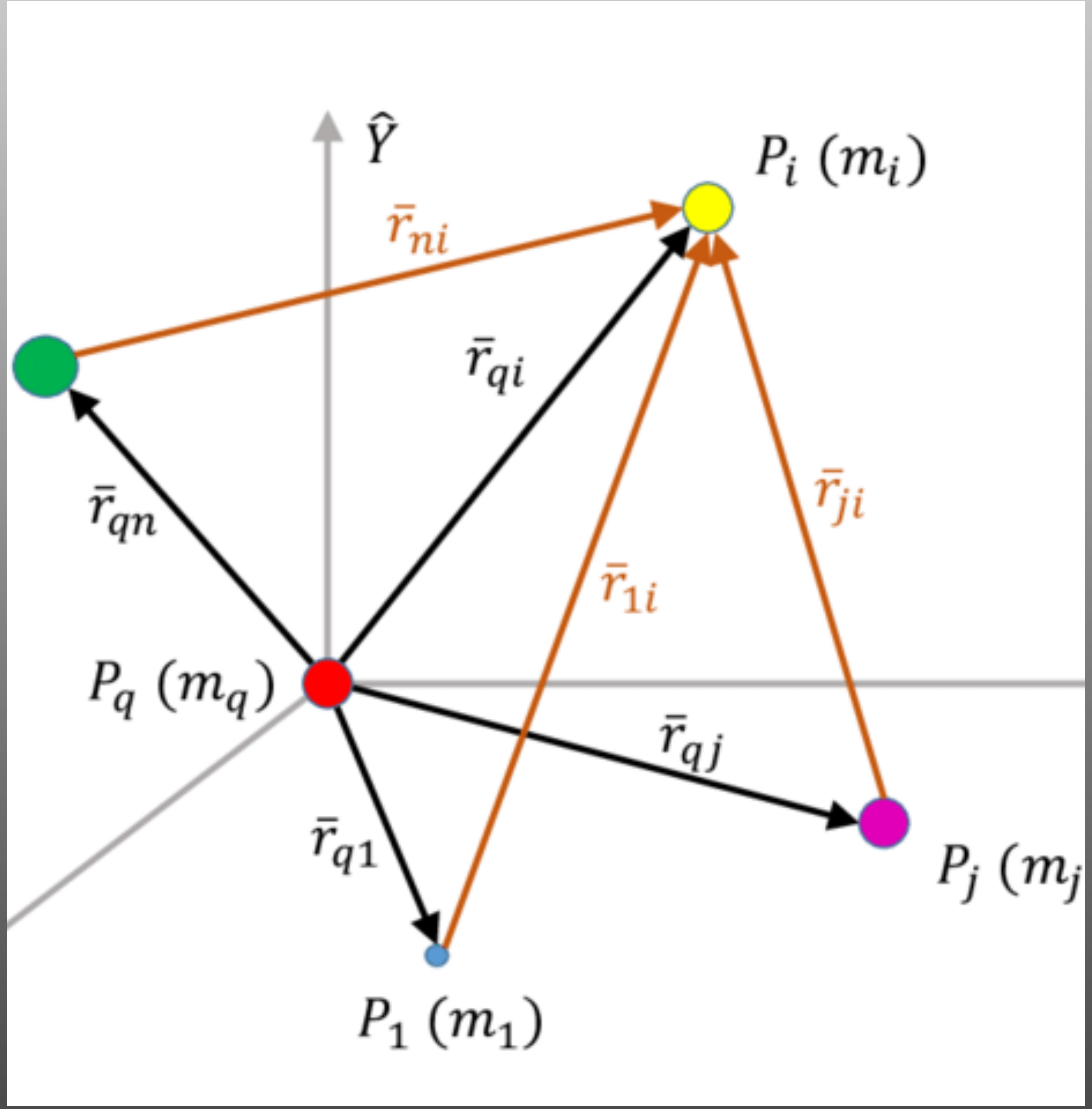
- What are the advantages?
  - No need for the fluid description, particles in the vacuum;
  - No infinities (the particles are finite and discrete



Discrete Newtonian Cosmology

A discrete approach to Newtonian Cosmology

- What are the advantages?
  - No need for the fluid description, particles in the vacuum;
  - No infinities (the particles are finite and discrete;
  - No need for Fourier analysis in dealing with scales



Discrete Newtonian Cosmology

A brief review of the classical mechanics of gravitation

1. $m_a \ddot{\mathbf{x}}_a = \mathbf{F}_{\text{ext}} + \sum_{a \neq b} \mathbf{F}_{ab}$

2. $F_a^{grav} = -\dfrac{Gm_b}{|\mathbf{x}_a - \mathbf{x}_b|^3}(\mathbf{x}_a - \mathbf{x}_b)$

3. If $N \to \infty$ or if the particles obey certain symmetries (e.g. spherical symmetry), then $\mathbf{F}_{\text{ext}} = 0$

4. $U(\mathbf{x}_a) = -\dfrac{Gm_b}{|\mathbf{x}_a - \mathbf{x}_b|}(\mathbf{x}_a - \mathbf{x}_b)$

Discrete Newtonian Cosmology

A brief review of the classical mechanics of gravitation

$$\implies \nabla_{\mathbf{x}_a} U = -\mathbf{F}_a$$

$\implies$ Time symmetry $(t \to t + t_0)$

$\implies$ Translation symmetry $(\mathbf{x}_a \to \mathbf{x}_a + \mathbf{x}_0)$

$\implies$ Rotational symmetry around origin

Discrete Newtonian Cosmology

A brief review of the classical mechanics of gravitation

The symmetries of equations **don't** imply the symmetries of the solutions:

- Homothetic solutions have a preferred barycenter (no translational invariance);
- With an enough number of particles the solution is approximately rotationally symmetric;
- Time symmetry only valid for static, stationary solutions.

Discrete Newtonian Cosmology

A brief review of the classical mechanics of gravitation

The symmetries of equations **do** imply the conservation of the classic invariants:

- $M = \sum_a m_a = M_0 > 0$
- $\mathbf{P} = \sum_a m_a \dot{\mathbf{x}}_a = \mathbf{P}_0$
- $\mathbf{L} = \sum_a m_a (\mathbf{x}_a \times \dot{\mathbf{x}}_a) = \mathbf{L}_0$

Discrete Newtonian Cosmology

A brief review of the classical mechanics of gravitation

...and the conservation of mechanical energy:

- $T = \frac{1}{2} \sum_a m_a (\dot{\mathbf{x}}_a)^2$
- $V = \sum_a V_a = - \sum_a \sum_{b \neq a} \frac{G m_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|}$

$$\implies E = T + V = E_0$$

Discrete Newtonian Cosmology

A brief review of the classical mechanics of gravitation

Moment of Inertia and Virial Theorem

- $I \equiv \frac{1}{2}\sum_a m_a x_a^2$

$\implies$ *Sundman's Theorem:*

$$(\mathbf{L}_0)^2 + (\dot{I})^2 \leq 4IT$$

There's no singularity in finite time and, furthermore, no collapse if

$$\mathbf{L_0}^2 \neq 0$$

Discrete Newtonian Cosmology

A brief review of the classical mechanics of gravitation

Moment of Inertia and Virial Theorem

- Notice that

  1. $U(\mathbf{x})$ is a central field potential $\Longrightarrow \mathbf{F}(\mathbf{x})$ is a homogeneous function of coordinates;

  2. The particles, being arbitrarily numerous but finite, occupy a finite amount of space; that is, the interactions take space in a finite amount of the coni figuration space.

- Then the *Virial Theorem* is valid for this system (Landau & Lifshitz)

Discrete Newtonian Cosmology

A brief review of the classical mechanics of gravitation

Moment of Inertia and Virial Theorem

Virial Theorem:

$$V = \ddot{I} - 2T, \text{ and } < \ddot{I} > = 0 \implies < V > = -2T$$

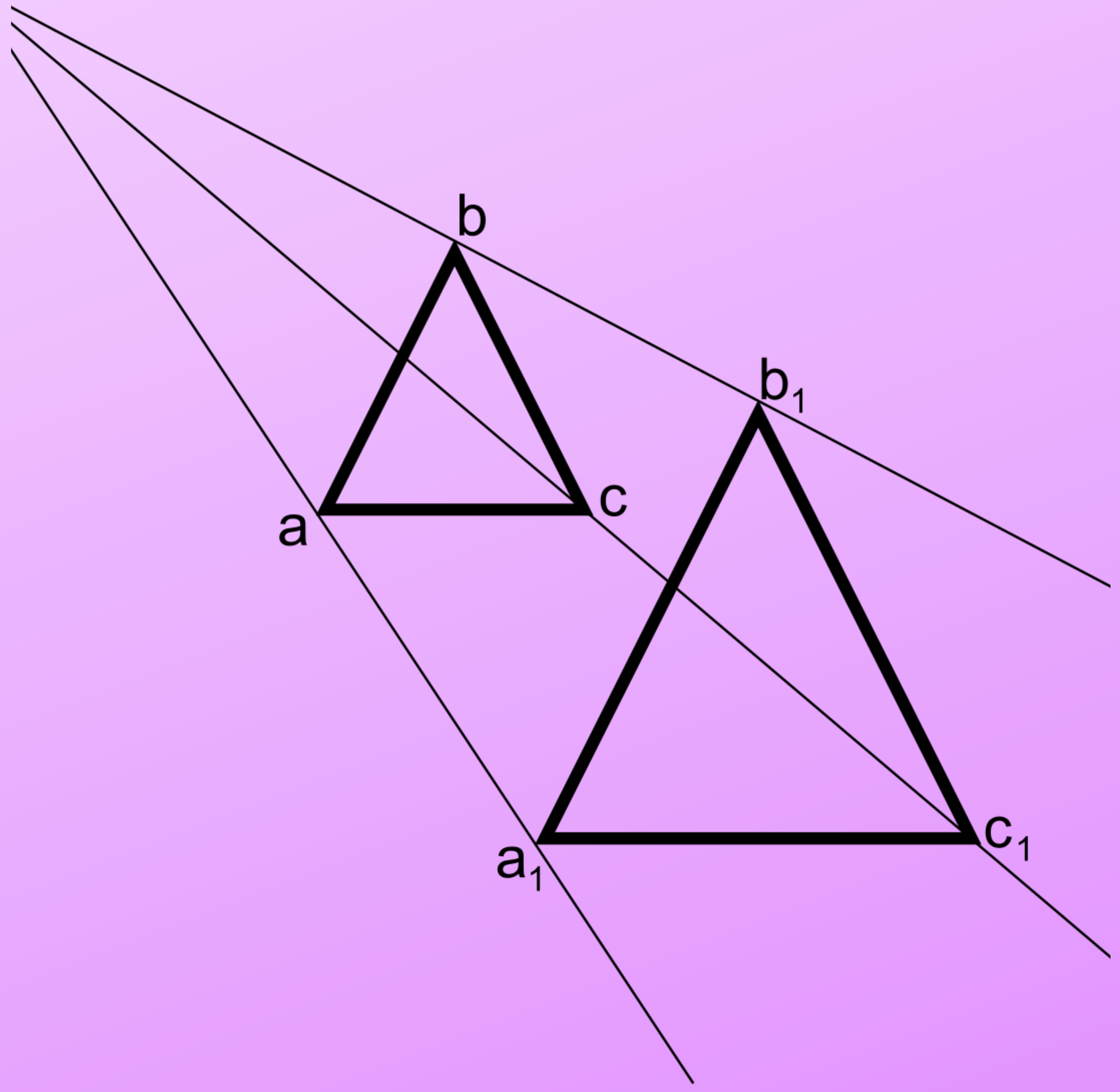
One can make an analogy with the Raychaudhuri equation and the singularity theorems.

Discrete Newtonian Cosmology

The Cosmology

How to create the Universe (with Newtonian mechanics):

- An homothetic relation on the coordinates
- A central configuration
- A large enough number of particles



Intermission

Homework:

Using the general coordinates and the Hubble law

$$q(t) := a(t), \dot{q} = H(t)q,$$

together with the Lagrangian

$$L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 + \frac{GMm}{q},$$

derive the Friedmann equations from the Euler-Lagrange equations and the conservation of the Hamiltonian function , where M is the total mass of a sphere of homogeneous density ρ.

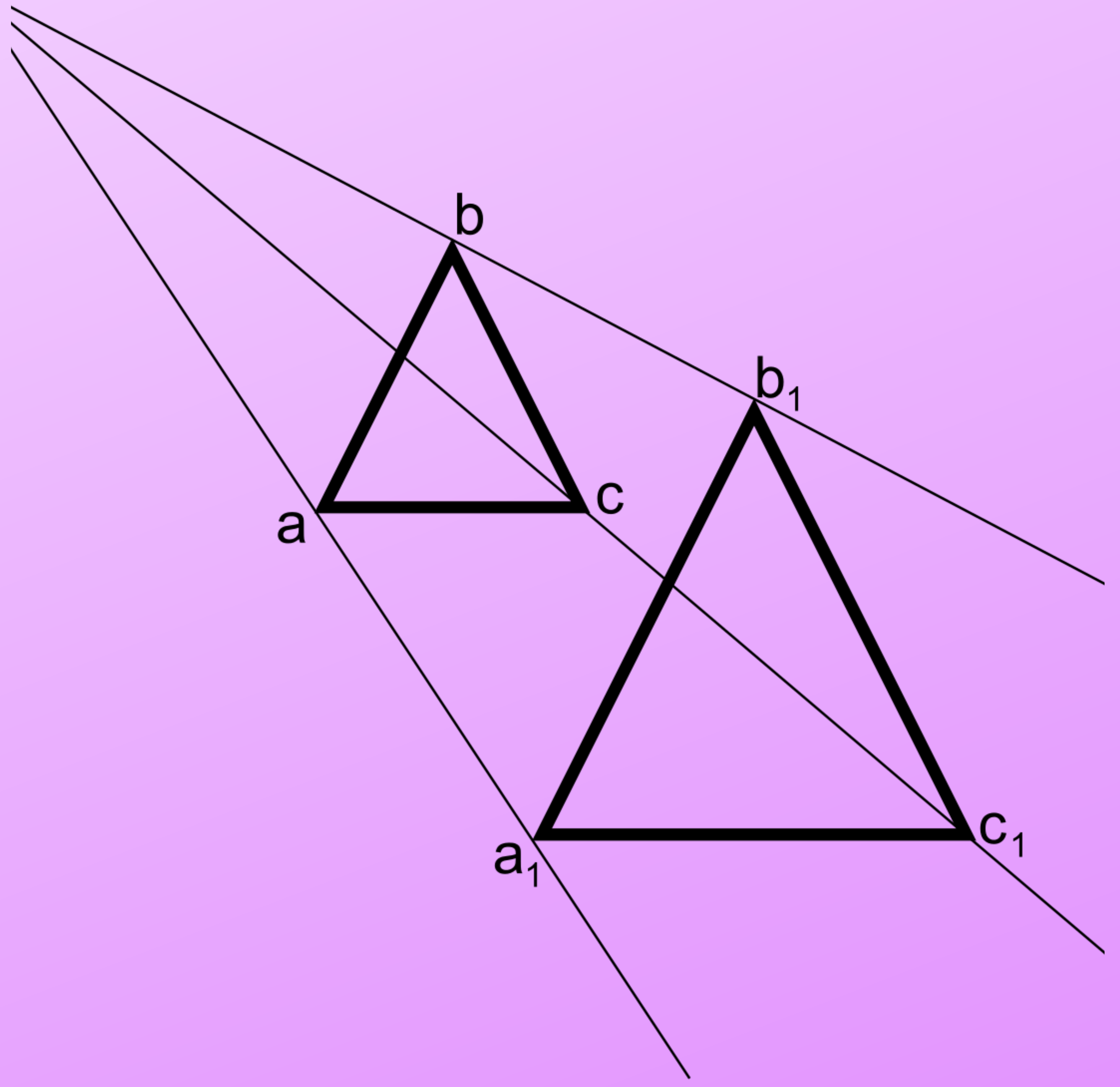
Discrete Newtonian Cosmology

The Cosmology

We postulate an homothetic scaling in relation to the origin, where the physical coordinates are given by

$$\mathbf{x}_a = S(t)\mathbf{r}_a,\ \dot{\mathbf{r}}_a = 0,$$

with $\mathbf{r}_a$ the comoving coordinates.



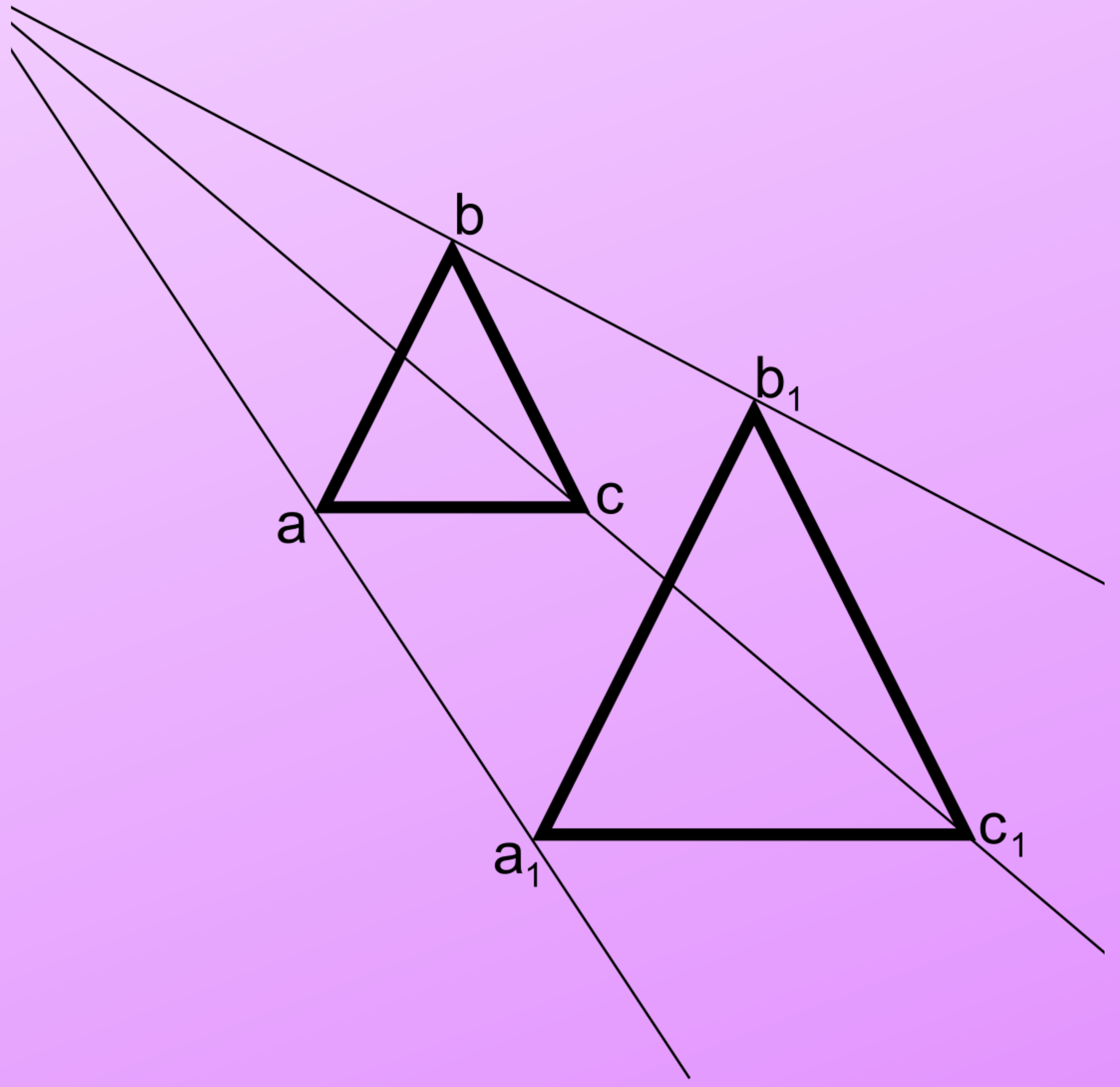
Discrete Newtonian Cosmology

The Cosmology

$$\Longrightarrow \mathbf{v}_a \equiv \dot{\mathbf{x}}_a = \dot{a}(t)\mathbf{r}_a = H(t)\mathbf{x}_a$$

We obtain straightforwardly the Hubble-Lemaître law.

- Notice that the only inertial motion is of the particle at the origin; the only other possibility for inertial motion is if $\dot{H}(t) = H_0 = \text{cte}$



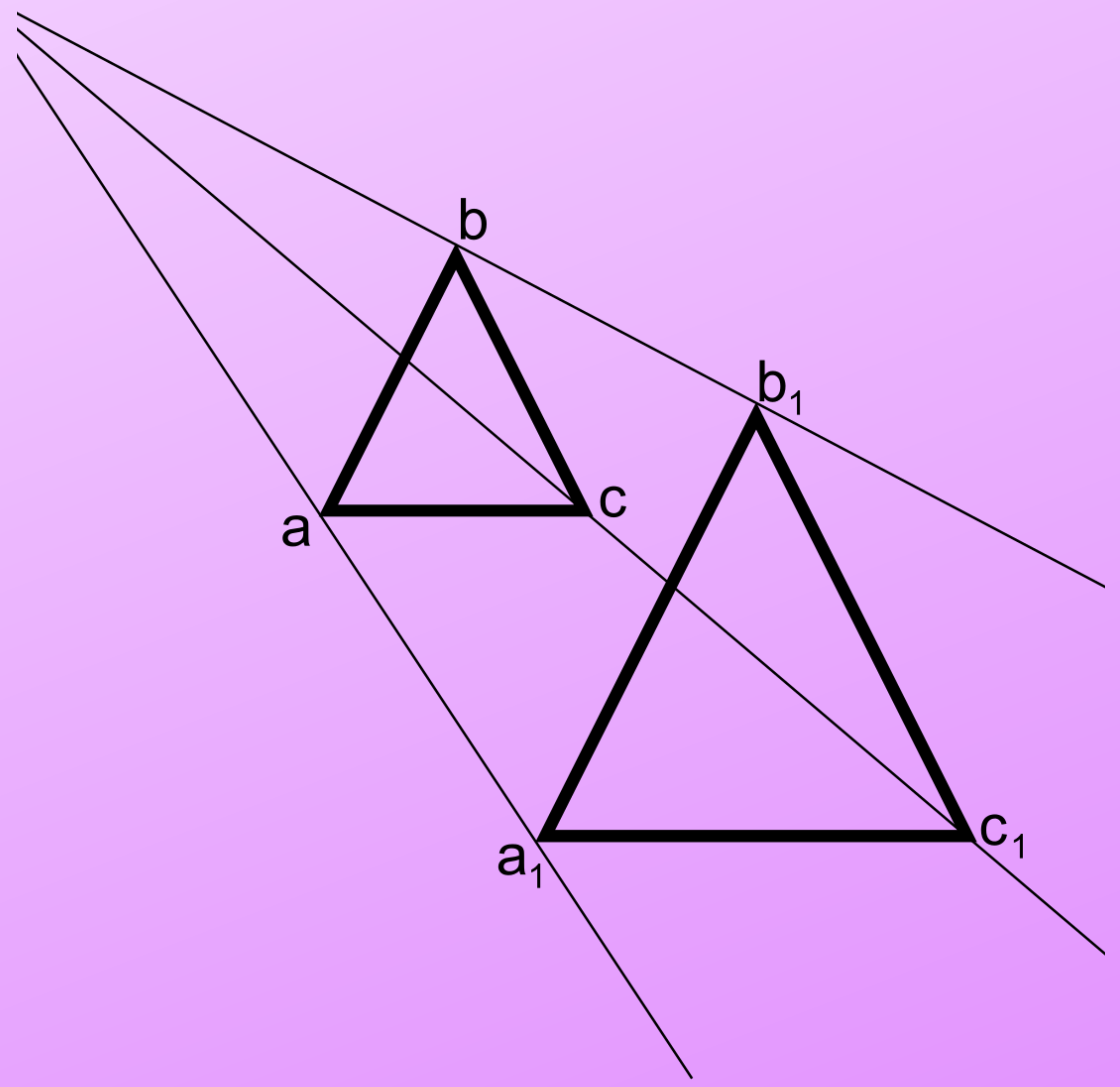
Discrete Newtonian Cosmology

The Cosmology

$$\Longrightarrow \;\; \mathbf{v}_a \equiv \dot{\mathbf{x}}_a = \dot{a}(t)\mathbf{r}_a = H(t)\mathbf{x}_a$$

We obtain straightforwardly the Hubble-Lemaître law.

- We can, however, understand the acceleration of other frames as the direct effect of gravity, the only force in the interactions; thus the equivalence principle tells us that all particles are dinamically equivalent.



Discrete Newtonian Cosmology

The Cosmology

From the first and second equations of the classical mechanics of gravitation we get

$$m_a \mathbf{r}_a \ddot{a}(t) = -\sum_{b \neq a} G m_a m_b \frac{(\mathbf{r}_a - \mathbf{r}_b)}{a^2(t)\,|\,\mathbf{r}_a - \mathbf{r}_b\,|^3}.$$

Defining $C(t) := a^2(t)\ddot{a}(t)$, we can rewrite

$$C(t) m_a \mathbf{r}_a = -\sum_{b \neq a} G m_a m_b \frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\,\mathbf{r}_a - \mathbf{r}_b\,|^3},$$

Discrete Newtonian Cosmology

The Cosmology

From the conservation of mass and the comoving coordinates,

$$\frac{\partial}{\partial t}(C(t)m_a\mathbf{r}_a) = 0 \implies C(t) = \text{cte} =: -G\tilde{M},$$

where $[\tilde{M}] = \text{mass} \times \text{distance}^{-3}$.

Discrete Newtonian Cosmology

The Cosmology

This leads to the **Central Configuration Equation**:

$$\tilde{M} m_a \mathbf{r}_a = -\sum_{b \neq a} G m_a m_b \frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3}.$$

For systems of a few particles, this shows the particles must obey the homothetic symmetry (the configuration of particles in space will remain in the shape of a regular polyhedron); when $N \to \infty$, the structure will approach spatial homogeneity.

Discrete Newtonian Cosmology

The Cosmology

Central Configuration Equation:

$$\tilde{M} m_a \mathbf{r}_a = -\sum_{b \neq a} G m_a m_b \frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3}.$$

This equation is a consistency condition for the existence of a solution to the equations of motion.

- In the case of approximately spherically symmetric distributions of particles, it can be show that the constant factor $\tilde{M}$ is given by

$$\tilde{M} = \frac{m(r)}{r^3},$$

where $m(r)$ is the mass inside a radius r.

Discrete Newtonian Cosmology

The Cosmology

Central Configuration Equation:

$$\tilde{M} m_a \mathbf{r}_a = - \sum_{b \neq a} G m_a m_b \frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3}.$$

Thus, for a spherical distribution, the homogeneous density of the central configuration is given by

$$\tilde{\rho} = \frac{3\tilde{M}}{4\pi},$$

whereas the physical density is given in relation to the physical coordinates:

$$\rho = \frac{3\tilde{M}}{4\pi a^2(t)}$$

Discrete Newtonian Cosmology

The Cosmology

A Virial like relation

We can define an effective potential $\tilde{V}$ and moment of inertia I_0 by

$$\tilde{V} \equiv -\sum_a \sum_{b \neq a} \frac{G m_a m_b}{|\mathbf{r}_a - \mathbf{r}_b|}$$

$$\tilde{I}_0 \equiv \frac{1}{2} \sum_a m_a |r_a|^2,$$

both are constant, and an argument similar to the proof of the Virial theorem shows that

$$2G\tilde{M}\tilde{I}_0 = -\tilde{V}$$

Discrete Newtonian Cosmology

The Cosmology

Energy Conservation and Friedmann Equation

From the definition of kinetic energy we obtain

$$T = \frac{1}{2}\sum_a m_a |\dot{\mathbf{x}}_a|^2 = \frac{1}{2}\dot{a}^2(t)\sum_a m_a |\dot{\mathbf{r}}_a|^2 = \dot{a}^2\tilde{I}_0,$$

and from the potential

$$V = -\sum_a \sum_{b\neq a} \frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|} = -a(t)^{-1}\sum_a \sum_{b\neq a} \frac{Gm_a m_b}{|\mathbf{r}_a - \mathbf{r}_b|} = \frac{\tilde{V}}{a}$$

Discrete Newtonian Cosmology

The Cosmology

Energy Conservation and Friedmann Equation

Being a Newtonian mechanical system, the energy E must be conserved, thus

$$E = T + V = \dot{a}^2(t) I_0 + \frac{\tilde{V}}{a(t)} = E_0,$$

which, if we use the virial-like relation gives us

$$\frac{1}{2}\frac{\dot{a}^2(t)}{a^2(t)} = \frac{G\tilde{M}}{a^3(t)} + \frac{E_0}{2I_0 a^2(t)}$$

Discrete Newtonian Cosmology

The Cosmology

Raychaudhuri equation and second Friedmann equation

From the definition of $\tilde{M}$ we get

$$-G\tilde{M} = a^2(t)\ddot{a}(t) \implies \ddot{a}(t) = -\frac{G\tilde{M}}{a^2(t)}$$

which is a form of the Raychaudhuri equation for FLRW spacetimes.
If we use the first Friedmann equation, which we just obtained, we can write

$$\ddot{a}(t) = -\frac{G\tilde{M}}{a^3(t)} \implies \frac{\ddot{a}(t)}{a(t)} = -\frac{G\tilde{M}}{a^3(t)}$$

Discrete Newtonian Cosmology

The Cosmology

Theorem:

There is an exact homothetic solution for a finite system of particles under the Newtonian gravitational law of attraction, given that the set satisfies the central configuration equation. The function $a(t)$ evolves im accordance to the FLRW solution cosmological equations for pressureless matter fields.

Discrete Newtonian Cosmology

Cosmological Constant and solutions

- The equations for models with cosmological constant can be obtained in the exact same by the way of an additional term in the law of inertia:

$$m_a \ddot{\mathbf{x}}_a = -\frac{G m_b}{|\mathbf{x}_a - \mathbf{x}_b|^3}(\mathbf{x}_a - \mathbf{x}_b) + \frac{\Lambda m_a \mathbf{x}_a}{3},$$

the equations follow exactly:

$$\frac{1}{2}\frac{\dot{a}^2(t)}{a^2(t)} = \frac{G\tilde{M}}{a^3(t)} + \frac{E_0}{2I_0 a^2(t)} + \frac{\Lambda}{6},$$

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{G\tilde{M}}{a^3(t)} + \frac{\Lambda}{3}$$

Discrete Newtonian Cosmology

Cosmological Constant and solutions

- **Static solutions** (Einstein Universe):

$$\dot{a}(t) = 0 \implies \frac{\Lambda}{3} = \frac{G\tilde{M}}{a_0^3} > 0$$

- **Expanding solutions**:

From the result for spherical symmetry we obtained for $\tilde{M}$, we can, for a (approximately) spherically symmetric distribution of particles (as in RW metric), write

$$H^2 = \frac{8\pi G\rho}{3} + \frac{k}{a^2(t)} + \frac{\Lambda}{3},$$

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G\rho}{3} + \frac{\Lambda}{3}$$

Discrete Newtonian Cosmology

Cosmological Constant and solutions

- **Einstein-de Sitter** ($\Lambda = k = 0$):

$$a(t) = a_0 t^{2/3}, a_0 = (\frac{4\pi G\rho}{3})^{1/3}$$

- **Milne** ($\Lambda = 0, k > 0$):

$$a(t) = Ht, H^2 = 2E$$

- **de Sitter** ($\Lambda > 0 = E = k$):

$$a(t) = a_0 \exp(\frac{\Lambda t}{3})$$

Discrete Newtonian Cosmology

Cosmological Constant and solutions

- **Einstein-de Sitter** $(\Lambda = k = 0)$:

$$a(t) = a_0 t^{2/3}, a_0 = (\frac{4\pi G\rho}{3})^{1/3}$$

- **Milne** $(\Lambda = 0, k > 0)$:

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$$a(t) = a_0 \exp(\frac{\Lambda t}{3})$$

Discrete Newtonian Cosmology

Issues and further analyses

- **Fine Tuning:**

  The central configuration equation must be strictly satisfied for any solution to replicate FLRW cosmology.

- Non pontual particles (tidal forces)
- Inhomogeneous solutions
- External gravitational field

Discrete Newtonian Cosmology

Issues and further analyses

- **Perturbations:**

  No cosmological model is complete, however heuristic, without perturbation theory

- Non pontual particles (tidal forces)
- Inhomogeneous solutions
- External gravitational field

Discrete Newtonian Cosmology

Thank you!