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Hawking radiation for non-asymptotically flat dilatonic black holes using gravitational anomaly

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Abstract:

The d-dimensional scalar field action may be reduced, in the background geometry of a black hole, to a twodimensional effective action. In the near-horizon region, it appears a gravitational anomaly: the energy-momentum tensor of the scalar field is not conserved anymore. This anomaly is removed by introducing a term related to the Hawking temperature of the black hole. Even if the temperature term introduced is not covariant, a gauge transformation may restore the covariance.We apply this method to compute the temperature of the dilatonic non-asymptotically flat black holes. We compare the results with those obtained through other methods.

Review of the methodology:

The static spherical symmetric D-dimensional metric is given by

$$ds^2 = f(r)\,dt^2 - f(r)^{-1}\,dr^2 - r^2 d\Omega^2_{(D-2)} \quad , \tag{1}$$

with $f(r)$ admitting a horizon at $f(r = rH) = 0$.
The action of the scalar field, for the metric (1), is given by

$$\begin{aligned} S[\varphi] &= \frac{1}{2}\int d^D x\sqrt{-g}\varphi\nabla^2\varphi \\ &= \frac{1}{2}\int d^D x r^{D-2}\sqrt{\gamma}\varphi\left[\frac{1}{f}\,\partial_t^2\varphi\right. \\ &- \left.\frac{1}{r^{D-2}}\partial_r\left(r^{D-2} f\partial_r\varphi\right) - \frac{1}{r^2}\nabla_\Omega\varphi\right] \end{aligned} \tag{2}$$

γ being the determinant of the angular section $d\Omega^2$ and ∇_Ω a collection of angular derivatives.
Employing the tortoise radial coordinate r^* defined as

$$dr^* = \frac{1}{f(r)}\, dr \ ,$$

the action in the limit $r \to r_H$ becomes

$$\begin{aligned} S[\varphi] &= \frac{1}{2}\int dtdr^* d^{D-2}x r^{D-2}\sqrt{\gamma}\varphi\left[\partial_t^2\varphi\right. \\ &- \left.\frac{1}{r^{D-2}}\partial_r^*\left(r^{D-2}\partial_r^*\varphi\right) - \frac{f(r)}{r^2}\nabla_\Omega\varphi\right] \\ &\approx \sum_n \frac{r_H^{D-2}}{2}\int dtdr\,\varphi_n\left[\frac{1}{f}\,\partial_t^2\varphi_n - \partial_r\left(f\partial_r\varphi_n\right)\right] \ . \qquad (3) \end{aligned}$$

The scalar field φ has been expanded in terms of spherical harmonic functions in $D-2$ dimensions labelled by the indice n, and a constant term that appears in the near-horizon approximation has been discarded since it does not contribute to the equations of motion. The orthogonality of the spherical harmonic functions has been used also. Hence, near the horizon, the field can be described by an infinite collection of fields living in the space-time $(1+1)$ given by the metric,

$$ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 \quad . \tag{4}$$

We end up with a two-dimensional action, corresponding to an effective theory near the horizon. In this action we can impose the condition the ingoing modes are absent, since we are interested in the radiation arriving at infinity.

The gravitational anomaly to be treated here is represented by the non-conservation of the energy-momentum tensor in two dimensions,

$$\nabla_\mu T^\mu_\nu(r_H) \equiv \Xi_\nu(r) \equiv \frac{1}{\sqrt{-g}}\,\partial_\mu N^\mu_\nu \quad , \tag{5}$$

where,

$$N^\mu_\nu = \frac{1}{96\pi}\,\epsilon^{\beta\mu}\partial_\alpha\Gamma^\alpha_{\nu\beta} \rightarrow N^r_t = \frac{1}{192\pi}\left(ff'' + f'^{\,2}\right) \quad , \tag{6}$$

with $\epsilon^{01} = \epsilon^{tr} = 1$ and $(')$ is a derivative with respect to r.
For the metric (4), it is easy to see that it is a timelike anomaly, $\Xi_t \neq 0$ and $\Xi_r = 0$.

The equation (5) must be solved in two regions, $r_H < r \leq r_H + \delta$, where the anomalies appear, and $r > r_H + \delta$, where there is no anomaly.

- For $r > r_H + \delta$,

$$\partial_r T^r_t(\infty) = 0 \rightarrow T^r_t(\infty) = a_0 \quad . \tag{7}$$

- For $r_H < r \leq r_H + \delta$,

$$\partial_r T^r_t(r_H) = \Xi_t \rightarrow T^r_t(r_H) = a_H + N^r_t(r) - N^r_t(r_H) \quad . \tag{8}$$

In the expressions above a_0 and a_H are integration constants.

We combine the energy-momentum tensor in the two regions, leading to,

$$T^{\mu}_{\nu} = T^{\mu}_{\nu}(\infty)\Theta_H(r) + T^{\mu}_{\nu}(r_H)H(r) \quad , \tag{9}$$

where $\Theta_H(r) = \Theta(r - r_H - \delta)$ is the step function and $H(r) = 1 - \Theta_H(r)$ a hat function which is 1 in the region $r_H < r \leq r_H + \delta$ and zero outside this region.

Through the transformation, $x' = x - \xi \rightarrow \delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$, the anomaly of the effective action (suppressing the ingoing modes **as described above**) is given by

$$\begin{aligned} -\delta W &= \int d^2x \sqrt{-g^{(2)}} \xi^t \nabla_\mu T^\mu_t \\ &= \int d^2x\, \xi^t \Big\{ \partial_r \Big[N^r_t(r) H(r) \Big] \\ &+ \Big(a_0 - a_H + N^r_t(r_H) \Big) \delta(r - r_H - \delta) \Big\} \quad . \end{aligned} \tag{10}$$

The first term is zero since it can be reduced to a surface term (ξ^t is constant). But, in order to cancel the second term, it is necessary to consider the quantum effects associated to the ingoing modes, given by,

$$a_0 = a_H - N^r_t(r_H) \quad . \tag{11}$$

One consequence of the equation (5) is the absence of covariance. In order to restore the covariance, is necessary to add a new energy-momentum tensor such that,

$$\tilde{T}^{\mu}_{\nu} = T^{\mu}_{\nu} + \tilde{N}^{\mu}_{\nu} \quad , \tag{12}$$

where

$$\tilde{N}^{r}_{t} = \frac{1}{192\pi}\left(ff'' - 2(f')^2\right) \quad . \tag{13}$$

The value of the constant a_H is fixed by imposing that the covariant energy-momentum tensor is zero over the horizon ($\tilde{T}^{r}_{t}(r_H) = 0$). Following this procedure, we find a_H:

$$\begin{aligned} \tilde{T}^{r}_{t}(r_H) &= T^{r}_{t}(r_H) + \tilde{N}^{r}_{t}(r_H) = 0 \\ a_H &= \frac{1}{96\pi}\left(f'|_{r=r_H}\right)^2 \quad . \end{aligned} \tag{14}$$

Remark that $\tilde{N}^r_t(r_H) = -2N^r_t(r_H)$, since $f(r_H) = 0$. The equation (11) may be written as,

$$\begin{aligned} a_0 &= a_H - N^r_t(r_H) \\ &= \frac{1}{48\pi}\,\kappa^2 \quad , \end{aligned}$$

where $\kappa = \partial_r f|_{r=r_H}$ is the surface gravity. Using now the expression for the temperature of the black hole $T_H = \kappa/2\pi$, we can express the flux of energy-momentum as,

$$a_0 = \frac{\pi}{12}\,T_H^2 \quad . \tag{15}$$

The main point is the that the temperature term is essential to cancel the gravitational anomaly in the near horizon region.

The NAF Dilatonic black holes:

The action describing the dilatonic field coupled to gravity and to a gauge field $U(1)$ in $3+1$ dimensions is given by,

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - 2\partial_\mu\phi\partial^\mu\phi - e^{-2\alpha\phi}F_{\mu\nu}F^{\mu\nu}\right) ,$$

where ϕ is the dilatonic field, $F_{\mu\nu}$ the $U(1)$ gauge field, and α is a coupling constant. Among these solutions there is the linear dilatonic case ($\alpha = 1$), a static, spherical solution non asymptotically flat solution, described by the following expressions,

$$ds^2 = \frac{(r-b)}{r_0}dt^2 - \frac{r_0}{(r-b)}dr^2 - rr_0 d\Omega^2 \quad , \tag{16}$$

$$F^{tr} = \frac{Q}{r_0^2} , \quad e^{2\phi(r)} = \frac{r}{r_0} \quad . \tag{17}$$

In these expressions, b are r_0 are related to the mass and charge of the black hole:

$$M = \frac{1}{4}\, b \;, \quad Q = \sqrt{\frac{1}{2}}\, r_0 \quad . \tag{18}$$

Near the horizon the effective two-dimensional metric is given by,

$$ds^2 = \frac{(r-b)}{r_0} dt^2 - \frac{r_0}{(r-b)} dr^2 \quad . \tag{19}$$

This solution contains a gauge term. As for the pure gravitational system, now there is an anomaly in presence of gauge fields which is given by,

$$\nabla_\mu J^\mu = -\frac{e^2}{4\pi\sqrt{-g}}\, \epsilon^{\mu\nu} \partial_\mu A_\nu \quad , \tag{20}$$

where J^μ is the non-conserved four-current and A_ν the electromagnetic potential, we are interested in the outgoing modes only.

We consider a charged scalar field living in the spacetime determined by the metric (19). An energy-momentum tensor and a current are associated to this charged scalar field. The equation (20) must be solved in the two regions we are interested in:

- $r > r_H + \delta$, where the conservation laws are satisfied,

$$\begin{aligned} \nabla_\mu J^\mu(\infty) &= \partial_\mu J^\mu(\infty) = \partial_r J^r(\infty) = 0\,, \\ J^r(\infty) &= c_0 \quad . \end{aligned} \tag{21}$$

- $r_H < r \leq r_H + \delta$, where the anomaly appears,

$$\begin{aligned} \partial_r J^r(r_H) &= -\frac{e^2}{4\pi}\,\epsilon^{rt}\partial_r A_t\,, \\ J^r(r_H) &= c_H + \frac{e^2}{4\pi}\left(A_t(r) - A_t(r_H)\right) \quad . \end{aligned} \tag{22}$$

In the expressions above c_0 and c_H are integration constants.

Under an infinitesimal gauge transformation, after omitting the ingoing modes near the horizon, the effective action is,

$$-\delta W = \int d^2x \sqrt{-g_{(2)}} \lambda \nabla_\mu J^\mu_{(2)} \quad ,$$

where λ is the gauge parameter. As in the gravitational anomaly case, the current is the sum of the currents in the two regions,

$$J^\mu = J^\mu(\infty)\Theta_H(r) + J^\mu(r_H)H(r) \quad , \tag{23}$$

where $\Theta_H(r) = \Theta_H(r - r_H - \delta)$ and $H(r) = 1 - \Theta_H(r)$.

Using the equations (21) and (22), we find,

$$\begin{aligned}\int d^2x\sqrt{-g_{(2)}}\lambda\nabla_\mu J^\mu_{(2)} &= \int d^2x\lambda\Big\{\partial_r\left(\frac{e^2}{4\pi}A_t(r)H(r)\right) \\ &+ \left(c_0 - c_H + \frac{e^2}{4\pi}A_t(r_H)\right)\delta(r-r_H-\delta)\Big\}.\end{aligned}$$

The first term is zero classicaly. But, in order to cancell the second term, it is necessary to take into account the quantum terms:

$$c_0 = c_H - \frac{e^2}{4\pi}A_t(r_H) \quad . \tag{24}$$

Again, as in the case of the energy-momentum tensor anomaly, the equation (20) does not transform covariantly. In order to restore the covariance, we must add a new current:

$$\tilde{J}^{\mu} = J^{\mu} + \frac{e^2}{4\pi\sqrt{-g}}\,\epsilon^{\lambda\mu}A_{\lambda} \quad . \tag{25}$$

In order to determine the value of the constant c_H, it is required the the current (25) is zero near the horizon. Hence,

$$c_H = -\frac{e^2}{4\pi}A_t(r_H) \quad . \tag{26}$$

Substituting c_H in the equation (24), we find the charge flux,

$$c_0 = -\frac{e^2}{2\pi}A_t(r_H = b) \quad . \tag{27}$$

The potential given by the equations (17) and (18), is given by,

$$A_t(r_H) = \frac{b}{\sqrt{2}\, r_0} \quad . \tag{28}$$

The charge flux is,

$$c_0 = -\frac{e^2}{2\pi}\frac{b}{\sqrt{2}\, r_0} \quad . \tag{29}$$

-*Gravitational Anomaly*: In presence of a current, even at classical level the energy-momentum tensor is not conserved:

$$\nabla_\mu T^\mu{}_\nu = F_{\mu\nu} J^\mu \quad . \tag{30}$$

Hence, taking into account all fields present in the action, computing the variation with respect $S[g_{\mu\nu}, A_\mu, \varphi, \sigma]$, taking into account the Ward's identities, we obtain the following expression for the variation of the energy-momentum tensor after adding the gravitational anomaly term:

$$\nabla_\mu T^\mu_\nu = F_{\mu\nu} J^\mu + A_\nu \nabla_\mu J^\mu + \Xi_\nu \quad , \tag{31}$$

The flux of the energy-momentum tensor is independent of the dilation field σ. It must be stressed that the space-time is static and that only the outgoing modes are taked into account.
Solving the equation (31) for the two relevant regions, for $\nu = t$

- em $r > r_H + \delta$,

$$\begin{aligned} \partial_r T^r_t(\infty) &= F_{rt} J^r(\infty) \\ T^r_t(\infty) &= a_0 + c_0 A_t(r) \quad . \end{aligned} \tag{32}$$

- In $r_H < r \leq r_H + \delta$,

$$\begin{aligned} \partial_r T^r_t(r_H) &= F_{rt} J^r(r_H) + A_t(r)\partial_r J^r(r_H) + \Xi_t \\ T^r_t(r_H) &= a_H + \frac{e^2}{4\pi}\Big[A^2_t(r) - 2A_t(r)A_t(r_H)\Big] \\ &+ N^r_t(r) \\ &= a_H + \frac{e^2}{4\pi} A^2_t(r) + c_0 A_t(r) + N^r_t(r) \quad , \end{aligned} \tag{33}$$

where the equations (22), (26) and (27) have been used.

Above, a_0 and a_H are integration constants.

Collecting all terms, the total energy-momentum tensor can be written as

$$T^r_t(r) = \left(\tilde{T}^r_t(r_H) + \bar{T}^r_t(r_H)\right)H(r - r_h - \delta) + \left(\tilde{T}^r_t(\infty) + \bar{T}^r_t(\infty)\right)\Theta(r - r_h - \delta), \qquad (34)$$

where The terms $\tilde{T}^r_t$ collect the constants contributions and the $\bar{T}^r_t$ contains the terms that are r dependent.

Following the same steps as before, the anomaly part of the effective action is given by,

$$
\begin{aligned}
\int d^2x\sqrt{-g_{(2)}}\xi^t\nabla_\mu T^\mu{}_t &= \int d^2x\,\xi^t\Big\{\nabla_r\Big[\bar{T}^r_t(r_H)H(r-r_H-\delta) \\
&+ \bar{T}^r_t(\infty)\Theta(r-r_H-\delta)\Big] \\
&+ (a_0-a_H+N^r_t(r_H))\times\delta(r-r_H-\delta)\Big\}\ .
\end{aligned}
$$

To cancel the anomaly, it is required that,

$$a_0 = a_H - N^r_t(r_H) \quad . \tag{35}$$

Equation (5) does not transform covariantly. In order to restore the covariance, a new energy-momentum tensor is added:

$$\tilde{T}^{\mu}_{\nu} = T^{\mu}_{\nu} + \tilde{N}^{\mu}_{\nu} \quad , \tag{36}$$

where

$$\tilde{N}^{r}_{t} = \frac{1}{192\pi}\left(ff'' - 2(f')^2\right) \quad . \tag{37}$$

Imposing that the covariant energy-momentum tensor (36) be zero on the horizon, the flux of the energy-momentum tensor (35) becomes,

$$a_H = -\tilde{N}^{r}_{t}(r_H). \tag{38}$$

Hence,

$$a_0 = -\tilde{N}_t^r(r_H) - N_t^r(r_H) = \frac{1}{192\pi}\frac{1}{r_0^2}, \quad . \tag{39}$$

implying the temperature,

$$T_H = \frac{1}{4\pi r_0}, \tag{40}$$

confirming the temperature found by **G. Clément, J. C. Fabris and G. T. Marques, Phys. Lett. B 651 (2007) 54.**

The covariant anomaly:

The anamalies treated in this paper obey strong consistency conditions. For this reason, they are called *consitents anomalies*. But, in this formulation, expressed by equations (5) and (20), do not transform covariantly under coordinate transformation. For this reason, the new energy-momentum tensor (36) and currents (25), are defined, leading to new anomaly equations, which are now invariant under coordinate and gauge transformations. Hence, we will perform an analysis of covariant equations for the gauge and gravitational anomaly for the metric (19).

-*Gauge covariant anomaly*: Substituting equation (20) in equation (25), we obtain,

$$\nabla_\mu \tilde{J}^\mu = -\frac{e^2}{4\pi\sqrt{-g}}\epsilon^{\mu\nu}F_{\mu\nu} \quad , \tag{41}$$

which is now gauge invariant.
For the metric (19) the equations (41) can be integrated , resulting in

$$\begin{aligned} \partial_r \tilde{J}^r(r) &= \frac{e^2}{2\pi}\partial_r A_t(r) \\ \tilde{J}^r(\infty) - \tilde{J}^r(r_H) &= \frac{e^2}{2\pi}(A_t(\infty) - A_t(r_H)) \quad . \end{aligned} \tag{42}$$

Imposing that $\tilde{J}^r(r_H)$ is zero over the horizon and that $A_t(\infty)$ is not zero at infinity, since the metric is not asymptotically flat at the infinity, the charge flux becomes,

$$F_c = \tilde{J}^r(\infty) - \frac{e^2}{2\pi}A_t(\infty) = -\frac{e^2}{2\pi}A_t(r_H) = -\frac{e^2}{2\pi}\frac{b}{\sqrt{2}\,r_0} \quad . \qquad (43)$$

This confirms the universality of the expression for the charge flux, in the asymptotic limit ($r \to \infty$), via anomalies.

-*Covariant gravitational anomaly*: The covariant gravitational anomaly is given by the equation,

$$\begin{aligned} \nabla_\mu \tilde{T}^\mu_\nu &= \frac{1}{96\pi}\epsilon_{\mu\nu}\partial^\mu R \quad , \\ \partial_r \tilde{T}^r_t &= \frac{1}{96\pi} f(r)\partial^3_r f(r) = \partial_r \bar{N}^r_t \quad , \end{aligned} \tag{44}$$

where R is the Ricci scalar and

$$\bar{N}^r_t = \frac{1}{96\pi}\left(ff'' - \frac{f'^2}{2}\right) \quad ,$$

where $' = d/dr$.
The Ward's identities for the covariant anomaly case is given by

$$\nabla_\mu \tilde{T}^\mu_\nu = F_{\mu\nu}\tilde{J}^\mu + \partial_\mu \bar{N}^\mu_\nu \quad . \tag{45}$$

Using equation (42), we have

$$\begin{aligned}\partial_r \tilde{T}^r_t(r) &= \partial_r A_t(r)\left(\frac{e^2}{2\pi}A_t(r)+d_0\right)+\partial_r \bar{N}^r_t(r)\\ \tilde{T}^r_t(\infty)-\tilde{T}^r_t(r_H) &= \int_{r_H}^{\infty} d\left(\frac{e^2}{4\pi}A_t(r)^2+d_0 A_t(r)+\bar{N}^r_t(r)\right)\end{aligned} \tag{46}$$

Through equation (42) it is possible to obtain the value of the constant d_0 which is $-e^2\,A_t(r_H)/2\pi$. Hence, after integration of the right hand side of the equation (46), we find

$$\begin{aligned}\tilde{T}^r_t(\infty) &= \frac{e^2}{4\pi}A_t(\infty)^2-\frac{e^2}{2\pi}A_t(\infty)A_t(r_H)+\bar{N}^r_t(\infty)\\ &+ \frac{e^2}{4\pi}A_t(r_H)^2-\bar{N}^r_t(r_H) \quad ,\end{aligned}$$

where it was used the conditioin that $\tilde{T}^r_t$ is zero over the horizon.

Since the metric (19) is not asymptotically flat the term $\bar{N}_t^r(\infty)$ is not zero.
In this way, a flux of energy and momentum at infinity is defined:

$$\begin{aligned} F_{me} &= T_t^r(\infty) - \frac{e^2}{4\pi}A_t(\infty)^2 + \frac{e^2}{2\pi}A_t(\infty)A_t(r_H) - \bar{N}_t^r(\infty) \\ &= \frac{e^2}{4\pi}A_t(r_H)^2 - \bar{N}_t^r(r_H) = \frac{e^2 b^2}{8\pi r_0^2} + \frac{\pi}{12}\left(\frac{1}{4\pi r_0}\right)^2 \quad . \end{aligned} \tag{47}$$

This confirms the equation (38), showing the universality of the method of calculation of the Hawking temperature via anomalies. Hence, the convariant anomaly method allows also to cancel the anomalies near the horizon.

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