Quantum Gravity and Cosmology

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Outline of the course

- Introduction: Quantum Gravity & Cosmology
- Constrained Hamiltonian Systems
- Hamiltonian Formalism of GR
- Connecting with observations

Quantum Gravity

- Why should quantize gravity?
- What means quantum gravity?
- Quantum cosmology versus quantum gravity
- Some conceptual issues in quantum cosmology

Why should quantize gravity?

- It is not obvious at all... gravity is spacetime
- Is quantum nature more fundamental? In which sense?
- Quantum mechanics (QFT) defined on Hilbert space but based on Minkowski S-T
- Notwithstanding, it might be a reasonable idea...
    - Theoretical unification
    - Source term in GR ($T^{\mu\nu}$) is quantum
    - Black Hole Thermodynamics (entropy) suggests $\exists$ microstates
    - Singularity theorems of GR
    - Initial conditions in cosmology (what's the I.C. of the universe?)
    - Problem of time (external in QFT, arbitrary in GR)

What means quantum gravity?

- Several approaches
- semiclassical schemes (QFT in curved S-T)
- Supergravity
- String Theory
- Asymptotic safety
- Causal dynamical triangulations
- Noncommutative geometry
- Canonical quantization
  - Quantum geometrodynamics: Wheeler-DeWitt (WDW)
  - Loop quantum gravity (LQG)

Conceptual issues in quantum cosmology

Quantum cosmology is an attempt to take into account quantum effects in models of the universe.

- Patching models $\times$ the wave function of the universe
- Is quantum mechanics consistent for closed systems? (how happens measurements?)
- Emergence of classical spacetime
  - WKB approximation

    peaked about classical spacetimes: $\Psi \approx e^{iS}$

    nonclassical spacetimes: $\Psi \approx e^{-I}$
  - decoherence to avoid quantum interference
- How to define probability measure (WDW is not positive, similarly to KG)
- Singularity avoidance depends on the interpretation of quantum mechanics

Minisuperspace is quantum cosmology for the FLRW universe.

Standard Model

- Spacetime geometry:

$$ds^2 = a^2(t)\left[d\eta^2 - \frac{dr^2}{1-kr^2} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$

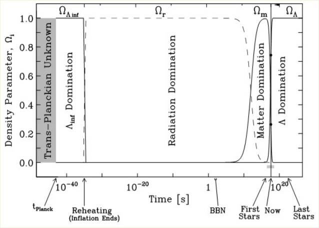
- Dynamics equations

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{k}{a^2} \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i (\rho_i + 3p_i)$$

if $\rho + 3p > 0$ the expansion is decelerated $\quad (\ddot{a} < 0)$

General Properties

- homogeneous and isotropic > 200 Mpc
- exist for at least $13,7$ billions of years
- Expanding: denser and hotter in the past
- 3 eras: radiation / dust / dark energy
- $\sim$ flat ($\Omega_k < 0.005$) / Is it (in)finite?



$$\rho_{rad} \backsim a^{-4} \quad / \quad \rho_{mat} \backsim a^{-3}$$

Thermal History (Microphysics)

- Below a few $MeV's$: *photons, baryons, electrons, neutrinos, dark sector*

$T \backsim$ [$\backsim$ 3 billions of years] galaxies formation

$T \backsim 0.25\,eV$ $\left[10^{12}\,sec\right]$ hydrogen rec. CMB decoupling

$T \backsim 1\,eV$ $\left[10^{11}\,sec\right]$ radiation-dust era transition

$T \backsim 50\,KeV$ $[300\,sec]$ light elements formation

$T \backsim 0.5\,MeV$ $[1\,sec]$ Freeze-out $e^{-} + e^{+} \leftrightarrows \gamma$

$T \backsim 1\,MeV$ $[0.2\,sec]$ neutrino decoupling

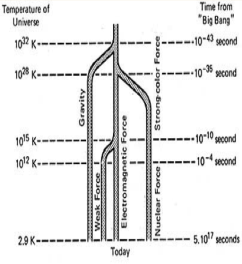
$T_\nu \approx 1.401 T_\gamma$ $(p/n$ free-out)

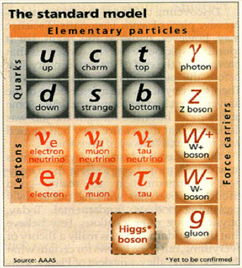
$T \backsim 200\,MeV$ $\left[10^{-5} sec\right]$ formation of baryons
quarks and gluons confinement

$T \backsim 100\,GeV - 10\,TeV$ $\left[10^{-10} - 10^{-14} sec\right]$
particle Standard Model still valid

$T \backsim 10\,TeV - 10^{19} GeV$ $\left[10^{-14} - 10^{-44} sec\right]$
beyond SM but still classical gravity

$T > 10^{19} GeV$ what comes next?





Gauge theories as constrained systems

- more dynamical variables than physical degrees of freedom
- Covariant form of Electromagnetism: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

- In Flat S-T, the symmetry group is given by the Poincaré group

$$\begin{aligned} \vec{E}' &= \gamma\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) - \frac{\gamma^2}{\gamma+1}\left(\frac{\vec{v}.\vec{E}}{c}\right)\frac{\vec{v}}{c} \\ \vec{B}' &= \gamma\left(\vec{B} - \frac{\vec{v}}{c} \times \vec{E}\right) - \frac{\gamma^2}{\gamma+1}\left(\frac{\vec{v}.\vec{B}}{c}\right)\frac{\vec{v}}{c} \end{aligned}$$

- gauge transformation: $A_\mu \; \rightarrow \; A_\mu + \partial_\mu \Lambda(x)$ but $F_{\mu\nu} \; \rightarrow \; F_{\mu\nu}$
- what's an observable? (deterministic dynamics)

Constrained Dynamical Systems

- Lagrangian system of point-like particles with n degrees of freedom.

$$S = \int \mathrm{d}t \, L\left(\vec{q}, \dot{\vec{q}}\right) \qquad \mathrm{i} = 1, \ldots, n$$

Variational principle $\delta S = 0$, Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \quad \Longrightarrow \quad \mathbf{W} \cdot \ddot{\vec{q}} = \vec{V} \tag{1}$$

with

$$W_{ik}\left(\vec{q}, \dot{\vec{q}}\right) \equiv \frac{\partial^2 L}{\partial \dot{q}^k \, \partial \dot{q}^i} \qquad V_i\left(\vec{q}, \dot{\vec{q}}\right) \equiv \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial q^k \, \partial \dot{q}^i} \dot{q}^k$$

If exists $\mathbf{W}^{-1}$,

$$\ddot{\vec{q}} = \mathbf{W}^{-1} \cdot \vec{V} \quad \Longrightarrow \quad \ddot{q}_k = \ddot{q}_k\left(\vec{q}, \dot{\vec{q}}\right) \qquad \text{regular systems}$$

If $\det \mathbf{W} = 0$,

$$\ddot{\vec{q}} \quad \left\{ \begin{array}{l} \text{cannot be described by } \left(\vec{q}, \dot{\vec{q}}\right) \\ \text{evolution is not univocally determined} \end{array} \right\} \quad \text{singular systems}$$

Hamiltonian formalism

- Legendre transformation defines the momenta

$$p_k \equiv \frac{\partial L}{\partial \dot{q}^k}\left(\vec{q}, \dot{\vec{q}}\right)$$

- Hamiltonian is defined as

$$H\left(\vec{q}, \vec{p}\right) = \vec{p} \cdot \dot{\vec{q}} - L\left(\vec{q}, \dot{\vec{q}}\right) \qquad \text{it is assumed} \quad \dot{\vec{q}} = \dot{\vec{q}}\left(\vec{q}, \vec{p}\right)$$

- Evolution is given by

$$\dot{q}^i = \left\{q^i, H\right\} = \frac{\partial H}{\partial p_i} \qquad , \qquad \dot{p}_j = \left\{p_j, H\right\} = -\frac{\partial H}{\partial q^j}$$

- The transformation $x = (q, \dot{q}) \longrightarrow y = (q, p)$ requires well defined jacobian $J = \frac{\partial y}{\partial x}$

$$J_{ik} = \begin{pmatrix} \delta_{ik} & 0 \\ \frac{\partial p_i}{\partial q_k} & \frac{\partial p_i}{\partial \dot{q}_k} \end{pmatrix} \qquad \text{note that} \qquad \frac{\partial p_i}{\partial \dot{q}_k} = \frac{\partial^2 L}{\partial \dot{q}_i \dot{q}_k} = W_{ik}$$

Constrained Dynamical Systems

- Let the Hessian be such that $\det \mathbf{W} = 0$
- rank r smaller than its dimensions n, there are $m = n - r$

$$\sum_i Y_j{}^i(q,\dot q)\, W_{ik}(q,\dot q) = 0 \qquad \text{with} \quad j = 1, \ldots, m$$

Euler-Lagrange equations imply

$$\sum_i Y_j{}^i \left(\sum_k W_{ik}\, \ddot q^k - V_i \right) = 0$$

- Thus, there are $m' \leq m$ equations

$$\phi_j(\vec q,\, \dot{\vec q}) \equiv \sum_i Y_j{}^i\, V_i = 0 \quad \text{set of independent eq.'s out of}$$

- Main issue.... cannot invert the acceleration in terms of coordinates and momenta

Hamiltonian for Constrained Systems

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- Legendre transformation defines the momenta

$$p_k \equiv \frac{\partial L}{\partial \dot{q}^k}\left(\vec{q}\,, \dot{\vec{q}}\right)$$

- let assume all m constraints are independent

$$\phi_j\left(\vec{q}\,, \dot{\vec{q}}\right) \approx 0 \qquad \text{primary constraints}$$

- Phase space $\mathscr{P}_h$ $(dim = 2n)$ reduces to Γ $(dim = 2n - m)$
- Poisson brackets give variation out of Γ.
- $F(\vec{q}, \vec{p})$ can be zero in Γ but nonzero poisson bracket
- Definition

$$F\Big|_\Gamma = 0 \qquad \text{weakly zero denoted by} \qquad F \approx 0$$

$$\left(F, \frac{\partial F}{\partial q}, \frac{\partial F}{\partial p}\right)\Big|_\Gamma = 0 \qquad \text{strongly zero denoted by} \qquad F = 0$$

Hamiltonian for Constrained Systems

- The variation of the Hamiltonian $H(\vec{q},\vec{p}) = \vec{p}\cdot\dot{\vec{q}} - L\left(\vec{q},\dot{\vec{q}}\right)$

$$\delta H(\vec{q},\vec{p}) = \sum_k \left[\dot{q}^k\delta p_k + \overbrace{\cancel{\left(p_k - \frac{\partial L}{\partial \dot{q}^k}\right)}}^{0}\delta\dot{q}^k - \frac{\partial L}{\partial q^k}\delta q^k\right]$$

$$= \sum_k \left(-\frac{\partial L}{\partial q^k}\delta q^k + \dot{q}^k\delta p_k\right)$$

- it is assumed the Euler-Lagrange equations, i.e. H is defined only on Γ.
- We need to extend the Hamiltonian for the phase space $\mathscr{P}_h$ in order to calculate evolution through poisson brackets
- Correct way is to define

$$\boxed{H_T \equiv H + \sum_j \mathscr{U}^j\phi_j}$$

$\mathscr{U}^k$ are arbitrary functions of (q^k, p_j)

Dirac-Bergmann Algorithm

$$H_T \equiv H + \sum_j \mathscr{U}^j \phi_j$$

- Consistency rule: constraint must always be satisfied

$$\dot{\phi}_j \approx \{\phi_j, H_T\} \approx \{\phi_j, H\} + \sum_j \{\phi_j, \mathscr{U}^k\} \cancelto{\approx 0}{\phi_k} + \sum_j \mathscr{U}^k \{\phi_j, \phi_k\} \approx 0$$

- Solve for $\mathscr{U}^k$

$$\mathbf{A} \cdot \vec{\mathscr{U}} \approx \vec{b} \qquad \text{with} \qquad A_{jk} = \{\phi_j, \phi_k\}, \quad b_j = \{\phi_j, H\}$$

- if $\det \mathbf{A} \neq 0$ ok! $\longrightarrow$ $\vec{\mathscr{U}} \approx \mathbf{A}^{-1} \cdot \vec{b}$
- if $\det \mathbf{A} \approx 0$ ok! $\longrightarrow$ $\tilde{\phi}_k$ secondary constraints
- Check consistency for $\tilde{\phi}_k$ and everything goes as before ... ...

An Example: Newtonian Mechanics

- Newtonian Mechanics with $L\left(q^k, \dot{q}^k, t\right)$ can be parameterized as

$$S = \int \mathrm{d}\lambda \frac{\mathrm{d}t}{\mathrm{d}\lambda} L\left(\vec{q}, \dot{\vec{q}}, t\right) = \int \mathrm{d}\lambda\, t'\, L\left(\vec{q}, \frac{\vec{q}'}{t'}, t\right) = \int d\lambda\, \mathscr{L}\left(x^k, x'^k\right)$$

where $\quad x^k = (t, q^k) \quad x'^k = \dfrac{\mathrm{d}x^k}{\mathrm{d}\lambda}, \quad \dot{q}^k = \dfrac{q'^k}{t'}$

- The momenta are

$$\pi_i = \frac{\partial \mathscr{L}}{\partial x'^i} = t' \frac{\partial L}{\partial \dot{q}^k} \cdot \frac{\partial \dot{q}^k}{\partial q'^i} = t' \frac{\partial L}{\partial \dot{q}^k} \cdot \frac{\delta^k{}_i}{t'} = p_i$$

$$\pi_0 = \frac{\partial \mathscr{L}}{\partial t'} = L + t' \cdot \sum_i \frac{\partial L}{\partial \dot{q}^i} \cdot \frac{\partial \dot{q}^i}{\partial t'} = L - \sum_i p_i \dot{q}^i = -H$$

- Thus, there is a constraint $\quad \phi_0 = \pi_0 + H\left(q^i, p_i, t\right) = 0$

An Example: Newtonian Mechanics

- Note that the Hamiltonian vanishes! $\mathscr{H} = \vec{\pi} \cdot \vec{q} - \mathscr{L}$

$$\mathscr{H} = \pi_0 \cdot t' + \pi_i \cdot q'^i - \mathscr{L} = t'\left(\pi_0 \cdot +p_i \cdot \dot{q}^i - L\right) = t'\left(\pi_0 + H\right) = 0$$

- The evolution is given by $\mathscr{H}_T = \mathscr{H} + u^0\phi_0 = u^0\left[\pi_0 + H\left(q_i, \pi^i, t\right)\right]$
- It is straightforward to show that $\phi'_0 = [\phi_0\ ,\ u_0\phi_0] \approx 0$
- Equations of motions are

$$\begin{aligned}
t' &= \{t, \mathscr{H}_T\} = u_0 & \longrightarrow && u_0 &= \frac{dt}{d\lambda}\\
\pi'_0 &= \{\pi_0, \mathscr{H}_T\} = -u_0 \cdot \frac{\partial H}{\partial t} & \longrightarrow && \frac{dH}{dt} &= \frac{\partial H}{\partial t}\\
x'^i &= \left\{q^i, \mathscr{H}_T\right\} = u_0 \cdot \frac{\partial H}{\partial p^i} & \longrightarrow && \dot{q}_i &= \frac{\partial H}{\partial p^i}\\
\pi'_i &= \{p_i, \mathscr{H}_T\} = -u_0 \cdot \frac{\partial H}{\partial q_i} & \longrightarrow && \dot{p}^i &= -\frac{\partial H}{\partial q_i}
\end{aligned}$$

FLRW with Scalar Field

- The line element of a FLRW universe is

$$ds^2 = N^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + sin^2(\theta)\, d\phi^2\right)\right]$$

- N is the lapse function, a is the scale factor and $k = 0, \pm 1$
- The action of the gravitational sector is ($l_{Pl}^2 = 1$)

$$\begin{aligned} S &= \int \mathrm{d}^4x\,\sqrt{-g}\left(\frac{R}{2} - \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi + V(\varphi)\right) \\ &= \int \mathrm{d}^4x\,\sqrt{\gamma}Na^3\left(\frac{3a'^2}{a^2N^2} - \frac{3k}{a^2} - \frac{1}{2}\frac{\varphi'^2}{N^2} + V(\varphi)\right) \\ &= V_D\int \mathrm{d}tN\left(\frac{3aa'^2}{N^2} - 3ak - \frac{a^3}{2}\frac{\varphi'^2}{N^2} + a^3V(\varphi)\right) \end{aligned}$$

- The momenta

$$P_N = \frac{\partial L}{\partial N'} = 0 \qquad P_a = \frac{\partial L}{\partial a'} = 6\frac{aa'}{N} \qquad P_\varphi = \frac{\partial L}{\partial \varphi'} = -\frac{a^3\varphi'}{N}$$

FLRW with Scalar Field

- The Hamiltonian for FLRW with scalar field reads

$$H = P_a \dot{a} + P_\varphi \dot{\varphi} - L = N \underbrace{\left(\frac{P_a^2}{12a} - \frac{P_\varphi^2}{2a^3} + 3ka - a^3 V(\varphi) \right)}_{\mathscr{H}}$$

- The total Hamiltonian is $\mathscr{H}_T = N\mathscr{H} + \mathscr{U}\, P_N$
- Consistency check

$$P_N' \approx \{P_N, \mathscr{H}_T\} \approx \cancelto{-1}{\{P_N, N\}}\, \mathscr{H} + \{P_N, \mathscr{U}\}\, \cancelto{\approx 0}{P_N} \qquad \Rightarrow \qquad \boxed{\mathscr{H} \approx 0}$$

- The equation of motion are

$$N' = \{N, \mathscr{H}_T\} \approx \mathscr{U}\,, \quad a' = \frac{N}{6a} P_a\,, \quad \varphi' = -\frac{N}{a^3} P_\varphi\,, \quad P_N' = -\mathscr{H} \approx 0$$

$$P_\varphi' = N a^3 V_\varphi\,, \quad P_a' = \frac{N}{a} \left(\frac{P_a^2}{12a} - 3\frac{P_\varphi^2}{2a^3} - 3ka + 3a^3 V(\varphi) \right)$$

FLRW with Scalar Field

- Combining

$$\varphi' \quad \text{with} \quad P'_\varphi : \quad \frac{1}{N}\left(\frac{\varphi'}{N}\right)' + 3\frac{a'}{Na}\frac{\varphi'}{N} + V_\varphi = 0 \qquad \text{Klein-Gordon}$$

$$P'_N : \quad \left(\frac{a'}{aN}\right)^2 + \frac{k}{a^2} = \frac{1}{3}\left(\frac{1}{2}\frac{\varphi'^2}{N^2} + V\right) \qquad \text{Friedmann-1}$$

$$a' \quad \text{with} \quad P'_a : \quad \frac{1}{Na}\left(\frac{a'}{N}\right)' = -\frac{1}{6}\left(\frac{2\varphi'^2}{N^2} - 2V\right) \qquad \text{Friedmann-2}$$

- For a scalar field $\rho = \frac{\varphi'^2}{2N} + V$ and $p = \frac{\varphi'^2}{2N} - V$

- For a cosmic time gauge ($N = 1$)

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} + V_\varphi = 0 \qquad \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{1}{3}\rho \qquad \frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p)$$

GR Hamiltonian Formalism

- It is a constrained hamiltonian system
  Symmetry group identified with the covariance group: $\mathscr{M}\mathscr{M}\mathscr{G}$

- Einstein equation has no evolutionary parameter

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} \quad \Rightarrow \quad g_{\mu\nu}\left(x^0, x^1, x^2, x^3\right)$$

Basic variables:

- Arnowitt-Deser-Misner (ADM) formalism: Geometrodynamics
  The tri-metric h_{ij} is the dynamic variable

- Loop quantum gravity:
  The densitized triads E^a_i and connection $A^a_i = \Gamma^a_i + \gamma K^a_i$

ADM Formalism

- Suppose a topology $R \otimes \Sigma^3$, hence $\exists$ time-like congruence $\vec{e}_0$
  The subspace Σ can be expanded by $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$

$$\vec{e}_\mu \cdot \vec{e}_\nu \equiv g\left(\vec{e}_\mu, \vec{e}_\nu\right) = \eta_{\mu\nu}$$

- Given an arbitrary (t, x^i), the vector $\vec{T} \equiv \frac{\partial}{\partial t} = N\,\vec{e}_0 + N^k\,\vec{e}_k$

Projecting the metric

$$\begin{aligned} h_{ij} &\equiv g_{\alpha\beta}\; e^\alpha{}_i\; e^\beta{}_j \\ g_{0i} &\equiv g_{\alpha\beta}\; T^\alpha\; e^\beta{}_i = N^a\; h_{ai} = N_i \\ g_{00} &\equiv g_{\alpha\beta}\; T^\alpha\; T^\beta = N^2 - N^a N_a \end{aligned} \qquad g_{\mu\nu} = \begin{pmatrix} N^2 - N^a N_a & N_i \\ N_j & h_{ij} \end{pmatrix}$$

- how h_{ij} is immersed: extrinsic curvature

$$K_{\mu\nu} \doteq -\frac{1}{2} \perp_{\alpha\mu} \perp_\nu{}^\beta\; \nabla_{(\beta} e^\alpha{}_{0)} = -\frac{1}{2} \perp_\mu{}^\alpha \perp_\nu{}^\beta \mathscr{L}_{\vec{e}_0}\left(g_{\alpha\beta}\right)$$

$$K_{ab} = -N\,\Gamma^0_{ab} \qquad K_{0b} = N^a\; K_{ab} \qquad K_{00} = N^a N^b\; K_{ab}$$

ADM Formalism

- The action reads

$$S = \int d^4x \;\sqrt{-g}\,R = \int dt d^3x\, N\sqrt{h}\left(K^{ij}K_{ij} - K^2 + {}^{(3)}R\right)$$

- Equations of motion

$$\frac{\delta S}{\delta N} = 0 \Rightarrow \quad \sqrt{h}\left(K^{ij}K_{ij} - K^2 - {}^{(3)}R\right) = 0 \qquad G_{\mu\nu}\, T^{\mu}T^{\nu} = 0$$

$$\frac{\delta S}{\delta N_i} = 0 \Rightarrow \quad 2\sqrt{h}\,\nabla_j\left(K_i{}^j - \delta_i{}^j K\right) = 0 \qquad G_{\mu\nu}\, T^{\mu}\perp^{\nu}{}_{\alpha} = 0$$

$$\frac{\delta S}{\delta h_{ij}} = 0 \Rightarrow \qquad G_{\mu\nu}\,\perp^{\mu}{}_{\beta}\perp^{\nu}{}_{\alpha} = 0$$

$$\Rightarrow \quad \dot{K}_{ij} = -N\left[{}^{(3)}R_{ij} + KK_{ij} - 2K_i^m K_{mj}\right] + \nabla_j\nabla_i N - \nabla_{(i}N^m K_{j)m} - N^m\nabla_m\nabla_j K_i$$

ADM Formalism

- The momenta

$$P = \frac{\partial \mathscr{L}}{\partial \dot{N}} = 0 \quad , \quad P^i = \frac{\partial \mathscr{L}}{\partial \dot{N}_i} = 0 \quad , \quad \Pi^{ij} = \frac{\partial \mathscr{L}}{\partial \dot{h}_{ij}} = -\sqrt{h}\left(K^{ij} - h^{ij} K\right)$$

- The total hamiltonian

$$\mathscr{H}_T = \int dt\, d^3x \left(N\, \mathscr{H}_0 + N_i\, \mathscr{H}^i + \lambda\, P + \lambda_i\, P^i\right)$$

$$\mathscr{H}_0 = G_{ijkl}\, \Pi^{ij}\, \Pi^{kl} - {}^{(3)}R\, \sqrt{h} \qquad \text{(super-hamiltoniana)}$$

$$\mathscr{H}^i = -2 \nabla_j\, \Pi^{ij} \qquad \text{(super-momentum)}$$

$$G_{ijkl} = \frac{1}{2\sqrt{h}}\left(h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl}\right)$$

- Preservation of constraints

$$\dot{P} \approx 0 \quad \Rightarrow \quad \mathscr{H}_0 \approx 0 \qquad \dot{P}^i \approx 0 \quad \Rightarrow \quad \mathscr{H}^i \approx 0$$

- Dynamics

$$\dot{N} = \lambda \qquad \dot{N}^i = \lambda^i \qquad \dot{h}_{ij} = \ldots \qquad \dot{\Pi}_{ij} = \ldots$$

LQG Formalism

- The basic variables (build explicitly SU(2))

$$E^a{}_i = \sqrt{h} e^a{}_i \quad \Rightarrow \quad h h^{ab} = E^{ai} E_i{}^b \qquad \text{densitised triads}$$

$$K^i{}_a = K_a{}^b e^i{}_b \qquad \text{projected extrinsic curvature}$$

- The poisson brackets reads $\left\{K^i{}_a(x)\,,\, E^b{}_j(y)\right\} = \delta(x-y)\delta^b{}_a\delta^i{}_j$
- but none of them are connection, hence

$$A^i{}_a = \Gamma^i{}_a + \gamma K^i{}_a \qquad \begin{cases} \gamma \text{ is Barbero-Immirzi parameter} \\ \Gamma^i{}_a \text{ is spin connection} \end{cases}$$

- defining a new $\tilde{E}^a{}_i \equiv \gamma^{-1} E^a{}_i$

$$\left\{A^i{}_a(x)\,,\, \tilde{E}^b{}_j(y)\right\} = \delta(x-y)\delta^b{}_a\delta^i{}_j$$

- Area and volume

$$\mathbf{A}(S) = \int_S d^2\sigma\ |E| \qquad , \qquad \mathbf{V}(R) = \int_R d^3\tau\ \sqrt{|\det E|}$$

Quantizing Constrained Systems

- Non-relativistic dynamics reads

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = H(\hat{q},\hat{p})\,\psi(x,t)$$

- Canonical quantization: dynamical variable $\rightarrow$ operator
- Poisson brackets goes to commutation relations

$$[\hat{q},\hat{p}] = i\hbar$$

- We cannot translate the constraints as relations for operators

  Example: $\phi(q,p) = p \quad \longrightarrow \quad \phi(\hat{q},\hat{p}) = 0 \quad \Rightarrow \quad \hat{p} = 0$
  But

$$[\hat{q},\hat{p}] = i\hbar = \hat{q}\hat{p} - \hat{p}\hat{q} = 0 \qquad \Rightarrow \qquad i\hbar = 0! \quad \text{inconsistency}$$

- Constraints restricts the possible physical states, hence

$$\boxed{\phi(\hat{q},\hat{p})\psi = 0} \qquad \text{must annihilate the wave function}$$

- Non-commutability introduces extra conditions
- Let $\phi_j(\hat{q},\hat{p})$ and $\phi_k(\hat{q},\hat{p})$ be two different constraints

$$\phi_j(\hat{q},\hat{p})\psi = 0 \qquad \text{and} \qquad \phi_k(\hat{q},\hat{p})\psi = 0$$

- Then

$$\phi_k\phi_j\psi = 0 \qquad \text{and} \qquad \left(\phi_k\phi_j - \phi_k\phi_j\right)\psi = 0$$

- In order not to create extra conditions, we need

$$\left[\phi_j(\hat{q},\hat{p}), \phi_k(\hat{q},\hat{p})\right] = C^m_{jk}(\hat{q},\hat{p})\,\phi_m(\hat{q},\hat{p}) \tag{2}$$

- We cannot guarantee eq.(2), but let suppose there is an operator ordering that implement this relation

Quantizing Constrained Systems

- The constraints must be valid for all times...
- At a time t we have
$$\phi_k(\hat{q},\hat{p})\,\psi(t) = 0$$
- It follows that
$$\hat{H}\,\hat{\phi}_k\,\psi(t) = 0$$
- Let's taylor expand the evolution, and use Schrödinger equation
$$\hat{\phi}_k\,\psi(t+dt) = \hat{\phi}_k\,\psi(t) - \frac{i}{\hbar}\hat{\phi}_k\,\hat{H}\,\psi(t)$$
- For the constraint to be valid at $t+dt$
$$\hat{\phi}_k\,\hat{H}\,\psi(t) = 0$$
- Therefore,
$$\left[\phi_j(\hat{q},\hat{p}),H(\hat{q},\hat{p})\right] = D_j^m(\hat{q},\hat{p})\;\phi_m(\hat{q},\hat{p}) \tag{3}$$
- Again, we cannot guarantee eq.(3), but let suppose there is an operator ordering that implement this relation

Quantum Cosmology

There are a few issues....

- What is the meaning to quantize spacetime?

- The problem of time in quantum gravity

$$i\hbar\frac{\delta}{\delta\tau}\Psi\left(\varphi, A_\mu, h_{\mu\nu}\right) = \hat{H}\,\Psi\left(\varphi, A_\mu, h_{\mu\nu}\right) = 0 \qquad \text{no evolution?}$$

Quantum cosmology issues....

- Scale factor is defined in the half-line $\mathbb{R}^+$

- How to make predictions? (Interpretations of quantum mechanics)

Quantization on the half-line

- The scale factor is defined on the half-line $a \in (0, \infty)$
- Thus, operators generically are no longer self-adjoint
- The most important example: unitary evolution requires self-adjoint hamiltonians
- We need to impose boundary conditions.... example $\hat{H} = \hat{P}_a^2 + k\hat{a}^2$

$$\left\langle \psi_1 \middle| \hat{H}\psi_2 \right\rangle = \left\langle \hat{H}\psi_1 \middle| \psi_2 \right\rangle$$

symmetric condition

$$\int_0^\infty \mathrm{d}a\, \psi_1^* \frac{\mathrm{d}^2\psi_2}{\mathrm{d}a^2} = \int_0^\infty \mathrm{d}a\, \psi_2 \frac{\mathrm{d}^2\psi_1^*}{\mathrm{d}a^2}$$

square integrable

$$\left(\cancel{\psi_1^* \frac{\mathrm{d}\psi_2}{\mathrm{d}a}} - \cancel{\psi_2 \frac{\mathrm{d}\psi_1^*}{\mathrm{d}a}}\right)\Bigg|_{a=\infty} = \left(\psi_2 \frac{\mathrm{d}\psi_1^*}{\mathrm{d}a} - \psi_1^* \frac{\mathrm{d}\psi_2}{\mathrm{d}a}\right)\Bigg|_{a=0} \Rightarrow \boxed{\left.\frac{\mathrm{d}\psi}{\mathrm{d}a}\right|_{a=0} = r\,\psi\Big|_{a=0} \quad r \in \mathbb{R}}$$

Quantization on the half-line

Extreme cases

$$r = 0 \quad \rightarrow \quad \left.\frac{d\psi}{da}\right|_{a=0} = 0 \qquad , \qquad r = \infty \quad \rightarrow \quad \psi\big|_{a=0} = 0$$

- These conditions reflects on the green function (evolution propagator of the solution)
- If $G(x, x', t)$ is the conventional time evolution for the given system, then we need

$$G^{(r=0)}(x, x', t) = G(x, x', t) + G(x, -x', t)$$
$$G^{(r=\infty)}(x, x', t) = G(x, x', t) - G(x, -x', t)$$

- given an initial wave function $\psi_0(a)$, the solution reads

$$\psi^{(r)}(a, t) = \int_0^\infty da' G^{(r)}(a, a', t)\psi_0^{(r)}(a')$$

- An alternative approach is to change variables... Define

$$a \equiv e^\alpha \quad \Leftrightarrow \quad \alpha = \ln(a) \quad \Rightarrow \quad \alpha \in \mathbb{R} \qquad \text{whole-line}$$

Interpretations of Quantum Mechanics

- Copenhagen interpretation doesn't work for cosmology (cannot be applied!)
- Many-worlds interpretation
- Consistent histories interpretation
- Bohm-De Broglie Interpretation

Copenhagen Interpretations of QM

- Copenhagen interpretation doesn't work for cosmology (cannot be applied!)
- Let $\mathscr{O}_n$ be an observable with discrete spectrum given by n.

$$\hat{\mathscr{O}}_n\left|\lambda_n\right\rangle = n\left|\lambda_n\right\rangle$$

- We can defined an average value of $\mathscr{O}_n$ in the state

$$\left\langle\psi\middle|\hat{\mathscr{O}}_n\middle|\psi\right\rangle = \sum n.\left|\left\langle\psi\middle|\lambda_n\right\rangle\right|^2 \qquad \text{if } \left|\lambda_n\right\rangle \text{ forms a complete set of eigen vectors}$$

- After a measurement, if the result is k

$$\left|\psi\right\rangle \qquad \Longrightarrow \qquad P^{\mathscr{O}_n}_{n=k}\left|\psi\right\rangle = \left|\lambda_k\right\rangle \qquad \text{(Colapse of Wave)}$$

- To makes sense needs a classical domain to perform the measurement
- Indeterminance is explained through the complementary principle
- Decoherence is not enough (keeps all diagonal terms of density matrix)

Many-Worlds Interpretations of QM

- Everett's original idea: reformulate quantum mechanics that can describe closed systems (like the universe)
- "*This paper proposes to regard pure wave mechanics... as a complete theory*"
  H. Everett, *'Relative State' Formulation of QM*, *Rev. Mod. Phys. 29 (1957), 454*.
- There is no independent existence: "... a constituent subsystem cannot be said to be in any single well-defined state, independently of the remainder of the composite system."
- Relative States: "To any arbitrarily chosen relative state for one subsystem there will correspond a unique relative state for the remainder of the composite system."

Many-Worlds Interpretations of QM

- Max Tegmark's reading[1]

"Everett postulate: All isolated systems evolve according to the Schrödinger equation"

Corollaries...

1. The entire Universe evolves according to the Schrodinger equation, since it is by definition an isolated system.
2. There can be no definite outcome of quantum measurements (wavefunction collapse), since this would violate the Everett postulate
3. "It doesnt explain why we dont perceive weird superpositions"

[1] Max Tegmark, "The Interpretation of Quantum Mechanics: many worlds or many words?, ArXiv:quant-ph/9709032

Many-Worlds Interpretations of QM

- Most commonly... Frank Tipler's reading[2]
- Two state measurement: $|\uparrow\rangle$ and $|\downarrow\rangle$
- Total state: $|total\rangle = |system\rangle \otimes |detector\rangle$

$$\hat{M}\,|\uparrow\rangle \otimes |none\rangle = |\uparrow\rangle \otimes |up\rangle \qquad , \qquad \hat{M}\,|\downarrow\rangle \otimes |none\rangle = |\downarrow\rangle \otimes |down\rangle$$

- For a generic state: $|\psi\rangle = \alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle$

$$\hat{M}\,|\psi\rangle \otimes |none\rangle = \alpha\,(|\uparrow\rangle \otimes |up\rangle) + \beta\,(|\downarrow\rangle \otimes |down\rangle)$$

"If we regard human beings and more generally macroscopic objects as quantum mechanical objects - if for example $|up\rangle$ and $|down\rangle$ are regarded as mental states of human beings - then we are led inexorably by the last expression to the conclusion that an observation by a human being on some system which is not in an eigenstate of the observable will result in the human observer being 'split' by the observation: in one 'world' the system will be in state $|\uparrow\rangle$ and the observer will measure it as being in state $|\uparrow\rangle$ (that is the observer will be in state $|up\rangle$), and in the other 'world' the system will be in state $|\downarrow\rangle$ and the observer will measure it as being in state $|\downarrow\rangle$. Acceptance of this counter-intuitive (!!) conclusion is the many-worlds interpretation."

[2] F. J. Tipler, "The many-worlds interpretation of quantum mechanics in quantum cosmology", ch.13 - Quantum Concpets in Space and Time, ed. Penrose & Isham

Consistent Histories Interpretations of QM

- Attempts to reformulate Everett's idea to describe the emergence of a "quasi-classical domain" and the meaning of branching of the system without recurring to external influences.
- Aims at a coherent formulation of QM for closed system
- A set of alternative histories is defined by giving exhaustive sets of exclusive alternatives results at a sequence of time

Example: Two slits experiment....
Possible alternatives....

1. Whether or not the observer decided to measure from which slit the electron goes.
2. Whether the electron goes through the upper or lower slit.
3. The positions $\Delta y_1, \ldots, \Delta y_n$ that the electron can arrived at the screen.

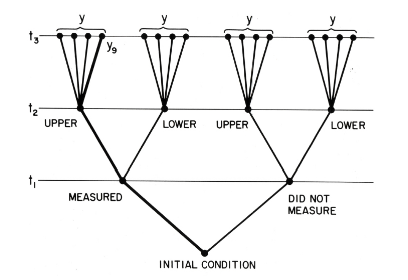


Figure: J. B. Hartle [gr-qc]/9304006

Consistent Histories Interpretations of QM

- Quantum mechanics does not satisfy the probability sum rule (due to interference)

$$\left|\psi_{upper} + \psi_{lower}\right|^2 \neq \left|\psi_{upper}\right|^2 + |\psi_{lower}|^2 \quad \Rightarrow \quad p(\Delta y_k) \neq p(\Delta y_k/upper) + p(\Delta y_k/lower)$$

- Challenge: assign probabilities that satisfy the probability sum rules - *branching of states*
- Probabilities for sets of histories if interference is negligible
- Alternatives are represented by exhaustive sets of orthogonal projectors

$$\sum_{\alpha_k} P^k_{\alpha_k}(t_k) = 1 \qquad \text{with} \quad P^k_{\alpha_k}(t_k)\, P^k_{\bar{\alpha}_k}(t_k) = \delta_{\alpha_k \bar{\alpha}_k} P^k_{\alpha_k}(t_k)$$

- An individual history is given by $C_\alpha = P^n_{\alpha_n}(t_n)\ldots P^1_{\alpha_1}(t_1)$
- Given an initial state $|\psi\rangle$, a branch is defined as $C_\alpha |\psi\rangle$
- The probability for each history is $p(\alpha) = \| C_{\bar{\alpha}} |\psi\rangle \|^2$
- Predictions only for set of decoherent histories

$$\boxed{d(\alpha, \bar{\alpha}) \equiv \langle\psi| C^\dagger_\alpha C_{\bar{\alpha}} |\psi\rangle \approx 0}$$

Bohm-de Broglie Interpretations of QM

- Dynamical evolution given by Schrödinger equation

$$i\hbar\frac{\partial\Psi(\vec{x},t)}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + V(\vec{x},t)\right)\Psi(\vec{x},t).$$

- Polar form $\Psi(\vec{x},t) = R(\vec{x},t)\, e^{i S(\vec{x},t)/\hbar} \quad \Rightarrow \quad \boxed{R^2 = \|\Psi\|^2}$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V + Q = 0$$ Hamilton-Jacobi-like

$$\frac{\partial R^2}{\partial t} + \nabla\left(R^2\frac{\nabla S}{m}\right) = 0$$ Continuity-like

$$\boxed{Q(\vec{x},t) \equiv -\frac{\hbar^2}{2m}\frac{\nabla^2 R(\vec{x},t)}{R(\vec{x},t)}}$$ Quantum Potential

- Introducing extra ingredients...

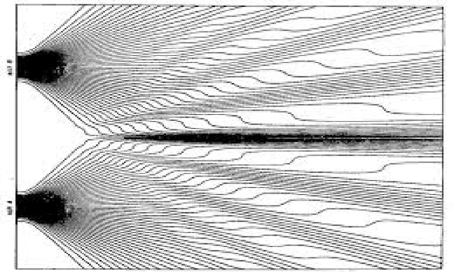
$$\vec{p}(\vec{x},t) = \vec{\nabla} S(\vec{x},t)$$ canonical momentum

$$\dot{\vec{x}} = \frac{1}{m}\nabla S(\vec{x},t)$$ particle's velocity

Bohm-de Broglie Interpretations of QM

- No particle-wave duality.... there is particle AND wave (both are real!)
- It is a causal interpretation of QM (never collapse)
- The probability to **find** the particle is $R^2(\vec{x}, t)$
- Quantum potential carries no energy but function as propagator of information (intrinsically **non-local**)

$$Q(\vec{x}_1, \ldots, , \vec{x}_N, t) \equiv -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\nabla_i^2 R(\vec{x}_1, \ldots, , \vec{x}_N, t)}{R(\vec{x}_1, \ldots, , \vec{x}_N, t)} \quad \text{many-particles system}$$



Quantum Cosmology with Scalar Field

Wheeler-DeWitt Quantization can Solve the Singularity Problem

N. Pinto-Neto, etal. - PRD 86 (2012) 063504.

- Ashtekar, Corichi, Craig and Singh claim that WDW is generically singular (probability=1) independently of the wave-function
- Craig and Singh - ***Free Scalar Field using Consistent History***
- Change of variables... $\boxed{\alpha \equiv \log a}$

$$\mathscr{H} = \frac{e^{-3\alpha}}{2}\left(-\frac{4\pi G}{3}p_\alpha^2 + p_\phi^2\right) \approx 0.$$

- Wheeler-deWitt equation

$$\left(\partial_\phi^2 - \frac{4\pi G}{3}\partial_\alpha^2\right)\Psi(\alpha,\phi) = 0, \ \Rightarrow$$

$$\pm\, i\partial_\phi \Psi(\alpha.\phi) = \sqrt{\hat{\Theta}}\,\Psi(\alpha,\phi) \quad \text{with} \quad \Theta := -\frac{4\pi G}{3}\partial_\alpha^2$$

- ϕ plays the role of time, and $\pm$ to positive and negative frequency sector

Quantum Cosmology with Scalar Field

- The action of $\sqrt{\hat{\Theta}}$ can be defined using Fourier space

$$\langle\alpha|\, k\rangle = \frac{1}{\sqrt{2\pi}} e^{ik\alpha}, \qquad \hat{\Theta}\,|k\rangle = \omega^2|k\rangle \qquad \omega := \sqrt{\frac{4\pi G}{3}}|k|.$$

The physical scalar product is given by

$$\langle\Phi|\,\Psi\rangle := \int d\alpha\, \Phi^*(\alpha, \phi_0)\Psi(\alpha, \phi_0), \tag{4}$$

- this scalar product is independent of choice of ϕ_0
- Positive and negative sectors are orthogonal w.r.t. (4)
- Restricting to the positive frequency sector, evolution is given by the propagator

$$U(\phi - \phi_0) = e^{i\sqrt{\Theta}(\phi-\phi_0)}$$

Quantum Cosmology with Scalar Field

- The set of histories and the corresponding decoherence functional

$$d(h,h') := \langle \Psi_{h'} | \Psi_h \rangle, \qquad \Psi_h := C_h^\dagger |\Psi\rangle$$

Ψ is the initial state and C_h is the class operator defining the history h

$$C_h := P_{\Delta\alpha_1}(t_1) \ldots P_{\Delta\alpha_n}(t_n)$$

- Using Heisenberg operators

$$P_{\Delta\alpha}(t) := U^\dagger(t)\, P_{\Delta\alpha}\, U(t)$$

the time independent projector is

$$P_{\Delta\alpha} = \int_{\Delta\alpha} d\alpha\, |\alpha\rangle \langle\alpha|$$

- The set of histories are composed by two times

$$C_{S-S}(-\infty,\infty) = \lim_{\phi\to\infty} P_{\Delta\alpha_1}(-\phi) P_{\Delta\alpha_2}(\phi)$$

$$C_{S-B}(-\infty,\infty) = \lim_{\phi\to\infty} P_{\Delta\alpha_1}(-\phi) P_{\overline{\Delta}\alpha_2}(\phi)$$

$$C_{B-S}(-\infty,\infty) = \lim_{\phi\to\infty} P_{\overline{\Delta}\alpha_1}(-\phi) P_{\Delta\alpha_2}(\phi)$$

$$C_{B-B}(-\infty,\infty) = \lim_{\phi\to\infty} P_{\overline{\Delta}\alpha_1}(-\phi) P_{\overline{\Delta}\alpha_2}(\phi)$$

S = arbitrary small $\alpha \to 0$

B = arbitrary large $\alpha \to \infty$

$$\overline{\Delta}\alpha \equiv \mathbb{R} - \Delta\alpha$$

Quantum Cosmology with Scalar Field

- Decoherence Test: $d(h', h) = 0?$ Note that $\left(P_{\Delta\alpha}P_{\bar{\Delta}\alpha} = 0\right)$

$$d(h_{S-S}, h_{S-S}) = d(h_{S-S}, h_{B-B}) = d(h_{B-B}, h_{B-B}) = 0$$

- The only non-trivial terms are $d(h_{S-B}, h_{B-B})$ and $d(h_{S-S}, h_{B-S})$.

$$\begin{aligned} d(h_{S-B}, h_{B-B}) &= \lim_{\phi\to\infty} \langle\Psi| P_{\bar{\Delta}\alpha_1}(-\phi)P_{\bar{\Delta}\alpha_2}(\phi)P_{\bar{\Delta}\alpha_2}(\phi)P_{\Delta\alpha_1}(-\phi)\,|\Psi\rangle \\ &= \lim_{\phi\to\infty} \langle\Psi| P_{\bar{\Delta}\alpha_1}(-\phi)P_{\bar{\Delta}\alpha_2}(\phi)P_{\Delta\alpha_1}(-\phi)\,|\Psi\rangle \end{aligned}$$

- Let us as a first step study the behavior of $P_{\Delta\alpha}(\phi)\,|\Psi\rangle$ and $P_{\bar{\Delta}\alpha}(\phi)$ for $\phi \to \pm\infty$.

$$\lim_{\phi\to+\infty} P_{\Delta\alpha}(\phi)\,|\Psi\rangle = |\Psi_L\rangle \qquad \lim_{\phi\to-\infty} P_{\Delta\alpha}(\phi)\,|\Psi\rangle = |\Psi_R\rangle$$

$$\lim_{\phi\to+\infty} P_{\bar{\Delta}\alpha}(\phi)\,|\Psi\rangle = |\Psi_R\rangle \qquad \lim_{\phi\to-\infty} P_{\bar{\Delta}\alpha}(\phi)\,|\Psi\rangle = |\Psi_L\rangle\,.$$

We have used the left/right-moving decomposition of the wave function

$$\Psi(\alpha,\phi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\,\Psi(k)e^{ik\alpha}e^{i\omega\phi} \propto \int_{-\infty}^{0} dk\,\Psi(k)e^{ik(\alpha-\phi)} + \int_{0}^{\infty} dk\,\Psi(k)e^{ik(\alpha+\phi)} =$$

$$= \Psi_R(v_r) + \Psi_L(v_l), \qquad \text{where} \qquad v_r := \alpha - \phi, v_l := \alpha + \phi$$

Quantum Cosmology with Scalar Field

- Since right and left moving sectors are orthogonal,

$$
\begin{aligned}
d(h_{S-B}, h_{B-B}) &= \lim_{\phi\to\infty} \langle\Psi|\, P_{\overline{\Delta\alpha_1}}(-\phi) P_{\overline{\Delta\alpha_2}}(\phi) P_{\overline{\Delta\alpha_2}}(\phi) P_{\Delta\alpha_1}(-\phi)\,|\Psi\rangle \\
&= \lim_{\phi\to\infty} \langle\Psi_L|\; P_{\overline{\Delta\alpha_2}}(\phi)\; |\Psi_R\rangle = \langle\Psi_L|\;|\Psi_R\rangle = 0
\end{aligned}
$$

- Mutatis mutandis $d(h_{S-S}, h_{B-S}) = 0$
- The probability for a bounce is

$$
\begin{aligned}
p(h_{B-B}) = \|C^{\dagger}_{B-B}|\Psi\rangle\|^2 &= \lim_{\phi\to\infty} \langle\Psi| P_{\overline{\Delta\alpha_1}}(-\phi) P_{\overline{\Delta\alpha_2}}(\phi)\, P_{\overline{\Delta\alpha_2}}(\phi) P_{\overline{\Delta\alpha_1}}(-\phi)\,|\Psi\rangle \\
&= \lim_{\phi\to\infty} \langle\Psi_L| P_{\overline{\Delta\alpha_2}}(\phi)\,|\Psi_L\rangle = 0
\end{aligned}
$$

- For two times, every solution is singular
- Three times the decoherence functional is generically nonzero (nothing can be said!)

Quantum Cosmology with Scalar Field

The de Broglie-Bohm theory applied to quantum cosmology

- Recall the WDW equation reads $\left(-\partial_\alpha^2 + \partial_\phi^2\right)\Psi = 0$
- Using the polar form $\Psi = R\, e^{iS}$

$$-\left(\frac{\partial S}{\partial \alpha}\right)^2 + \left(\frac{\partial S}{\partial \phi}\right)^2 + Q(q_\mu) = 0 \quad , \qquad Q(\alpha,\phi) := \frac{1}{R}\left[\frac{\partial^2 R}{\partial \alpha^2} - \frac{\partial^2 R}{\partial \phi^2}\right]$$

$$\frac{\partial}{\partial \phi}\left(R^2 \frac{\partial S}{\partial \phi}\right) - \frac{\partial}{\partial \alpha}\left(R^2 \frac{\partial S}{\partial \alpha}\right) = 0 \quad ,$$

- The guidance relations are

$$\dot{\alpha} = -e^{-3\alpha}\frac{\partial S}{\partial \alpha} \quad , \qquad \dot{\phi} = e^{-3\alpha}\frac{\partial S}{\partial \phi}$$

- The general solution is

$$\Psi(v_l, v_r) = \int_{-\infty}^{\infty} dk U(k)\, e^{ikv_l} + \int_{-\infty}^{\infty} dk V(k)\, e^{ikv_r} \quad , \quad \text{with } U \text{ and } V \text{ arbitrary}$$

Quantum Cosmology with Scalar Field

The de Broglie-Bohm theory applied to quantum cosmology

- Choosing only left or right moving components $Q = 0$
- In this case, only classical trajectories (singular) are allowed.
- Avoidance of singularities: iff Ψ depends on both v_l and v_r.
- Restricting to positive frequency only

$$\Psi(v_l, v_r) = \int_0^{\infty} dk \Psi(k)\, e^{ikv_l} + \int_{-\infty}^{0} dk \Psi(k)\, e^{ikv_r} \quad .$$

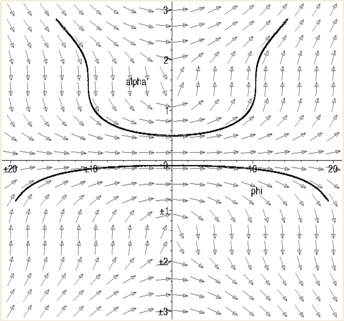
$$\frac{d\alpha}{d\phi} = -\frac{\partial S/\partial\alpha}{\partial S/\partial\phi} = -\left(\Psi\frac{\partial\Psi^*}{\partial\alpha} - \Psi^*\frac{\partial\Psi}{\partial\alpha}\right)\left(\Psi\frac{\partial\Psi^*}{\partial\phi} - \Psi^*\frac{\partial\Psi}{\partial\phi}\right)^{-1}$$

$$\left.\begin{array}{llll} v_r \to \pm\infty & \frac{d\alpha}{d\phi} = -1 & \Rightarrow & \alpha + \phi = v_l = \text{const} \\ v_l \to \pm\infty & \frac{d\alpha}{d\phi} = 1 & \Rightarrow & \alpha - \phi = v_r = \text{const} \end{array}\right\} \text{Classical solutions}$$

Quantum Cosmology with Scalar Field

The de Broglie-Bohm theory applied to quantum cosmology

- A particular solution $\Psi(k) = e^{-(|k|-d)^2/\sigma^2}$ with $\sigma << 1$ and $d \geq 1$.
- Two sharply peaked gaussians at $\pm d$



- but there is no continuity equation

$$\frac{\partial}{\partial\phi}\left(R^2\frac{\partial S}{\partial\phi}\right) - \frac{\partial}{\partial\alpha}\left(R^2\frac{\partial S}{\partial\alpha}\right)$$
$$\neq \frac{\partial R^2}{\partial\phi} - \frac{\partial}{\partial\alpha}\left(R^2\frac{\mathrm{d}\alpha}{\mathrm{d}\phi}\right)$$

- no probability measure for bohmian trajectories

Concluding remarks

- Consistent histories: have probabilities but unable to investigate singularity for more than two times.
- de Broglie-Bohm: know the entire evolution, but looses probability

Outline of the course

- Introduction: Quantum Gravity & Cosmology ✓
- Constrained Hamiltonian Systems ✓
- Hamiltonian Formalism of GR ✓
- Connecting with observations

Distinguishing Primordial Universe Scenarios with Observations

F. T. Falciano/ L. F. Guimarães/ G. Brando

Brazilian Center for Research in Physics - CBPF

International PhD Program in Astrophysics, Cosmology and Gravitation - PPGCosmo

ArXiv:1902.05031 [gr-qc]

Primordial Universe

- Planck $\oplus$ WMAP agree on the 6−parameter ΛCDM model
- CMB is the farthest observable
- Inflation is the most popular primordial universe scenario
- Competitive alternatives: Bounce models, pre-big-bang, ekpyrotic, string-gas cosmology, etc...
- Three main observables:

  scalar spectral index (n_s) tensor-to-scalar ratio (r) non-gaussianity (f_{NL})

- No clear theoretical prerogative
- Current status: there is no reason to privilege one scenario over the others
- Prove by contradiction to distinguish primordial scenarios: a constructive method

Standard Model

CBPF 70 ANOS CENTRO BRASILEIRO DE PESQUISAS FÍSICAS MINISTÉRIO DA CIÊNCIA, TECNOLOGIA, INOVAÇÕES E COMUNICAÇÕES

PPGCosmo

- Spacetime geometry:

$$ds^2 = a^2(t)\left[d\eta^2 - \frac{dr^2}{1-kr^2} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$

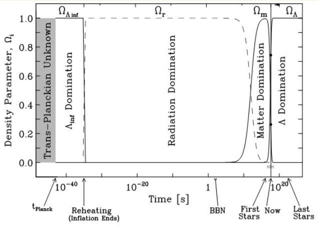
- Dynamics equations

$$H^2 \equiv \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{k}{a^2} \qquad \frac{\ddot a}{a} = -\frac{4\pi G}{3}\sum_i(\rho_i + 3p_i)$$

if $\rho + 3p > 0$ the expansion is decelerated $(\ddot a < 0)$

General Properties

- homogeneous and isotropic > 200 Mpc
- exist for at least 13, 7 billions of years
- Expanding: denser and hotter in the past
- 3 eras: radiation / dust / dark energy
- $\sim$ flat ($\Omega_k < 0.005$) / Is it (in)finite?



$$\rho_{rad} \backsim a^{-4} \quad / \quad \rho_{mat} \backsim a^{-3}$$

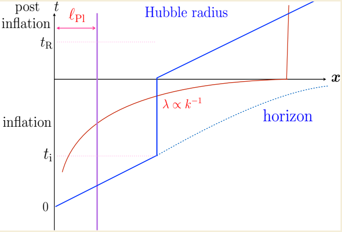
Testing Primordial Universe (Theory)

Shortcomings of the Standard Model

- physical mechanism to produce the expansion
- matter / anti-matter asymmetry
- cosmic singularity in a finite past time
- particle horizon
- flatness of spatial sections
- initial condition for structure formation

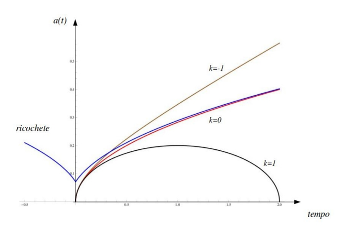
Inflation

accelerated expansion phase



Bounce

contraction → expansion



Testing Primordial Universe (Theory)

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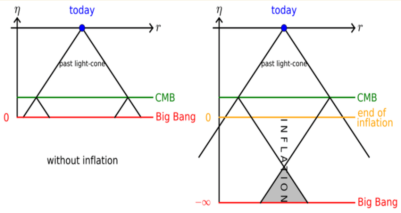
$$\Delta r = \int \frac{dt}{a}$$

$$\Rightarrow \frac{R_{LSS}}{R_H} \sim 10^2$$

Inflation

accelerated expansion phase

$$t_0 = 0 \quad \text{but} \quad a \simeq e^{Ht} \quad \rightarrow \eta_0 = -\infty$$



Bounce

contraction → expansion

$$t_0 = -\infty$$

"All the time of eternity""

Testing Primordial Universe (Theory)

Shortcomings of the Standard Model

- physical mechanism to produce the expansion
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- initial condition for structure formation

$$\frac{\mathrm{d}}{\mathrm{d}t}||\Omega - 1|| = -\frac{2\ddot{a}}{\dot{a}^3}$$

Inflation

accelerated expansion phase

$$\ddot{a} > 0 \quad \text{and} \quad \dot{a} > 0$$

Bounce

contraction $\rightarrow$ expansion

$$\ddot{a} < 0 \quad \text{but} \quad \dot{a} < 0$$

Testing Primordial Universe (Theory)

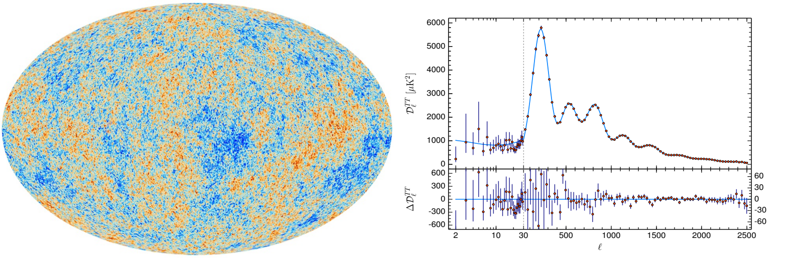
Shortcomings of the Standard Model

- physical mechanism to produce the expansion
- matter / anti-matter asymmetry
- cosmic singularity in a finite past time
- particle horizon
- flatness of spatial sections
- initial condition for structure formation

Inflation	Bounce
accelerated expansion phase	contraction $\rightarrow$ expansion
Quantum Vacuum Fluctuations	Quantum Vacuum Fluctuations
physical length much smaller than the curvature scale	Universe is immense with negligible curvature
as in Flat Spacetime	very close to Flat Spacetime

Testing Primordial Universe (Observations)

- While waiting for primordial gravitational waves....
- The farthest observable is the CMB



- Three main observables:

scalar spectral index	tensor-to-scalar ratio	non-gaussianity
$n_s = \dfrac{\mathrm{d}\log \mathrm{P}_{\mathscr{R}}}{\mathrm{d}\log k}$	$r = \dfrac{\mathrm{P}_{\mathscr{R}}}{\mathrm{P}_{\mathscr{T}}}$	f_{NL}

- but there is also

B-mode polarization, tensor spectral index, running, etc...

Testing Primordial Universe (Observations)

Connection with observations[3] : Correlation functions

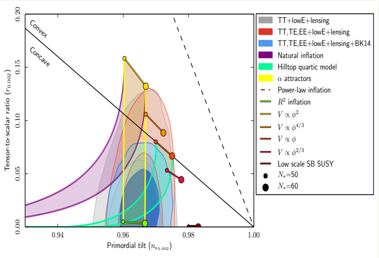
two point $$\langle \zeta^*(\vec{k}_1)\,\zeta(\vec{k}_2)\rangle = \frac{(2\pi)^2}{k^3}\delta(\vec{k}_1 - \vec{k}_2)P_\zeta(k_1)$$

bispectrum $$\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle = (2\pi)^7\delta\left(\vec{k}_1 + \vec{k}_2 + \vec{k}_3\right)\frac{P_\zeta^2}{\Pi\, k_i^3}\mathscr{A}\left(\vec{k}_1, \vec{k}_2, \vec{k}_3\right)$$

- Power spectrum parametrization

$$\mathrm{P}_{\mathscr{R}} = A_s\left(\frac{k}{k_*}\right)^{n_s - 1} \qquad n_s - 1 = \frac{\mathrm{d}\ln \mathrm{P}_{\mathscr{R}}}{\mathrm{d}\ln k}$$

$$\mathrm{P}_{\mathscr{T}} = A_t\left(\frac{k}{k_*}\right)^{n_t} \qquad k_* = 0.05\mathrm{Mpc}^{-1}$$



- Tensor-to-scalar ratio

$$r \equiv \frac{\mathrm{P}_{\mathscr{T}}}{\mathrm{P}_{\mathscr{R}}} \qquad k_* = 0.002\mathrm{Mpc}^{-1}$$

$$n_s = 0.9649 \pm 0.0042\,, \quad r_{0.002} < 0.064$$

[3] Planck 2018 results. X. Constraints on inflation arXiv:1807.06211v1 [astro-ph.CO]
Planck2018 results. VI. Cosmological parameters arXiv:1807.06209v1 [astro-ph.CO]

Testing Primordial Universe (Observations)

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General Relativity

- scalar perturbations (fourier mode)

$$\boxed{v_k'' + \left(k^2 - \mu_s^2\right) v_k = 0 \qquad \mu_s^2 = \frac{z_s''}{z_s}} \qquad v \equiv z_s \mathscr{R} \qquad z_s = a\frac{\dot{\varphi}}{H}$$

- gravitational waves

$$\boxed{v_k^{\lambda\,''} + \left(k^2 - \mu_t^2\right) v_k^\lambda = 0 \qquad \mu_t^2 = \frac{a''}{a}} \qquad v_k^\lambda = \frac{a}{2} h_k^\lambda \qquad z_t = a$$

$f(R)$ theories

- scalar perturbations $\qquad Q_s = 3M_{\rm Pl}^2 F'^2/2F\left[\mathscr{H} + \left(\frac{F'}{2F}\right)\right]^{-2}$

$$\boxed{v_k'' + \left(k^2 - \mu_{fs}^2\right) v_k = 0 \qquad \mu_{fs}^2 = \frac{z_{fs}''}{z_{fs}}} \qquad z_{fs} = a\sqrt{Q_s} \quad v_k = z_{fs}\mathscr{R}_k$$

- gravitational waves

$$\boxed{v_k^{\lambda\,''} + \left(k^2 - \mu_{ft}^2\right) v_k^\lambda = 0 \qquad \mu_{ft}^2 = \frac{z_{ft}''}{z_{ft}}} \qquad z_{ft} = a\sqrt{F} \quad v_k^\lambda = \frac{M_{\rm Pl}}{2} z_{ft} h_k^\lambda$$

Wands' Duality

- There is a hidden symmetry in the dynamic equation

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0$$

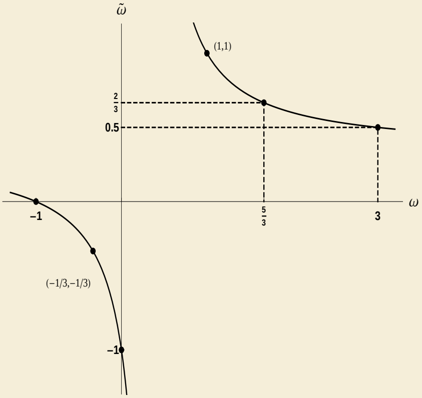
- invariant under transformation

$$\tilde{z}(\eta) = c_0\, z(\eta) \int \frac{d\eta'}{z^2(\eta')}$$

Example: perfect fluid ($p = \omega\,\rho$)

- slow-roll $\quad H \approx \dot{\varphi} \quad (z \approx a)$

$\omega \to \tilde{\omega} = \frac{1+\omega}{-1+3\omega}$



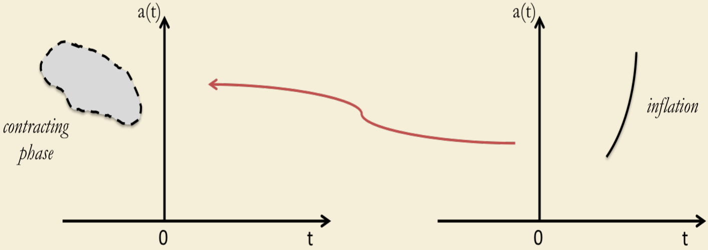
dust	de Sitter
$a \propto \eta^2 \Leftrightarrow$	$\tilde{a} \propto \dfrac{1}{\eta}$

Mimicking Starobinsky Inflation

What's the embedding dynamics?

- GR with perfect fluid / scalar field / ...
- F(R) without matter content / with conventional matter / ...
- Collapsing with scalar field: $z_s^B = a_B\,\dot{\varphi}/H$
- Quasi-de Sitter should be mapped into a quasi-matter dominated universe

$$\dot{\varphi}^2 \simeq 2V \qquad \Rightarrow \qquad \dot{\varphi} \simeq \sqrt{3}H \qquad \Rightarrow \qquad z_s^B(\eta) \approx a_s^B(\eta)$$



Mimicking Starobinsky Inflation

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$$\dot{\varphi}^2 \simeq 2V \qquad \Rightarrow \qquad \dot{\varphi} \simeq \sqrt{3}H \qquad \Rightarrow \qquad z_s^B(\eta) \approx a_s^B(\eta)$$

- consider the background dynamics

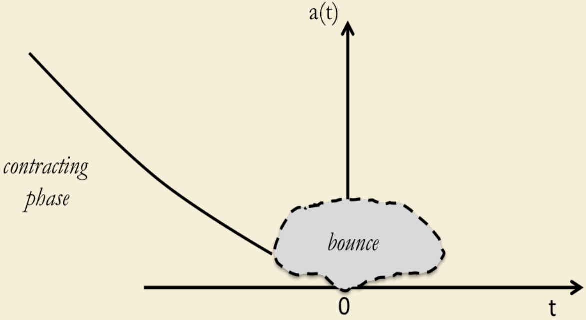
$$a_B(\eta) = a_{B0}\,\eta^2 \ln(\bar{\eta}) \left[1 - \frac{2}{3\ln(\bar{\eta})} + \mathcal{O}\left(\frac{1}{\ln^2(\bar{\eta})}\right)\right]$$

$$\left.\begin{aligned} V(\eta) &= \frac{6}{a_{B0}^2}\frac{1}{\eta^6 \log^2(\bar{\eta})}\left[1 + \frac{15}{6\ln(\bar{\eta})} + \mathcal{O}\left(\epsilon^2\right)\right] \\ \varphi &= -\sqrt{12}\ln\left[\bar{\eta}\ln^{5/12}(\bar{\eta})\right] + \mathcal{O}\left(\epsilon^2\right) \end{aligned}\right\}$$

$$\boxed{V(\varphi) = V_0\sqrt{1 - \varphi/\varphi_*}\; e^{\sqrt{3}\varphi}}$$

Quantum Bounce

- Quantum mechanisms: Wheeler-deWitt, Superstring motivated (Ekpyrotic), LQC



Quantum Bounce

- Loop Quantum Cosmology: semi-classical states have a master equation
- LQC has an analytical solution for scalar field

$$H^2 = \frac{M_{\rm Pl}^{-2}}{3}\rho\left(1-\frac{\rho}{\rho_c}\right), \quad \dot{H} = -\frac{M_{\rm Pl}^{-2}}{6}(\rho+p)\left(1-\frac{2\rho}{\rho_c}\right), \quad \dot{\rho}+3H(\rho+p)=0$$

- scalar field such that $p = \omega\rho \quad \Rightarrow \quad \rho = \rho_c\left(\frac{a_B}{a}\right)^{3(1+\omega)}$

$$a(t) = a_B\left(1+\alpha^2(t-t_B)^2\right)^{1/3(1+\omega)} \qquad \alpha = \frac{\sqrt{3\rho_c}}{2}(1+\omega)M_{\rm Pl}$$

$$\varphi(t)-\varphi_B = \frac{\sqrt{\rho_c(1+\omega)}}{\alpha}\operatorname{arcsinh}\left(\alpha(t-t_B)\right)$$

$$V = \frac{\rho_c(1-\omega)}{2}\operatorname{sech}^2\left[\frac{\alpha(\varphi-\varphi_B)}{\sqrt{\rho_c(1+\omega)}}\right]$$

Quantum Bounce

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- Our solution:

$$V(\varphi) = V_0 \sqrt{1-\varphi/\varphi_*}\, e^{\sqrt{3}\varphi} \quad \Rightarrow \quad \omega = \frac{\varphi'^2 - 2a^2 V}{\varphi'^2 + 2a^2 V} = -\frac{1}{6\ln(\bar{\eta})} \equiv -\frac{1}{6}\epsilon_c$$

- Scalar perturbations

$$v'' + \left[\left(1 - \frac{2\rho}{\rho_c}\right)k^2 - \frac{z''}{z}\right]v = 0 \qquad z = \frac{\sqrt{\rho + p}}{H}a$$

- Matching Conditions

before crossing $\quad v^{in}(\eta) = \sqrt{\frac{-\pi\eta}{4}}\,\mathrm{H}_\gamma^{(1)}(-k\eta) \qquad \gamma = \frac{3}{2} + \epsilon_c$

through the bounce $\quad v(\eta) \approx B_1 z + B_2 z \int^{\eta} \frac{d\bar{\eta}}{z^2}$

- power spectrum

$$\mathrm{P}_{\mathscr{R}} = \frac{k^3}{2\pi^2}\left|\frac{v}{z}\right|^2 \approx 1.736 \times 10^{-3} \left(\frac{\rho_c}{M_{\mathrm{Pl}}^4}\right) k^{-2\epsilon_c} \qquad \boxed{n_s - 1 = -2\epsilon_c}$$

Quantum Bounce

- Tensor perturbations

$$v'' + \left[\left(1 - \frac{2\rho}{\rho_c}\right)k^2 - \frac{z_T''}{z_T}\right]v = 0 \qquad z_T = \frac{a}{\sqrt{1 - 2\rho/\rho_c}}$$

- Matching Conditions

before crossing $\quad v^{in}(\eta) = \sqrt{\frac{-\pi\eta}{4}}\,\mathrm{H}_\gamma^{(1)}(-k\eta) \qquad \gamma = \frac{3}{2} + \epsilon_c$

through the bounce $\quad v(\eta) \approx B_1 z_T + B_2 z_T \int^\eta \frac{\mathrm{d}\bar{\eta}}{z_T^2}$

- power spectrum

$$\mathrm{P}_{\mathscr{T}} = 2 \times \frac{k^3}{2\pi^2}\left(2M_{\mathrm{Pl}}\left|\frac{v}{z_T}\right|\right)^2 \approx 4.645\,\epsilon_c^2 \times 10^{-3}\left(\frac{\rho_c}{M_{\mathrm{Pl}}^4}\right)k^{-\epsilon_c} \qquad \boxed{n_t = -2\epsilon_c}$$

- tensor-to-scalar ratio

$$\boxed{r = \frac{\mathrm{P}_{\mathscr{T}}}{\mathrm{P}_{\mathscr{R}}} = \frac{8}{3}\epsilon_c^2}$$

Overview

- Starobinsky Inflation

$$n_s - 1 = -\frac{2}{N} \quad , \quad r = \frac{12}{N^2} \quad \Rightarrow \quad r = 3\,(n_s-1)^2$$

Consistency (single field slow-roll): $n_t = -r/8 \;\Rightarrow\; n_t = -\frac{3}{8}\,(n_s-1)^2$

- Our model

$$n_s - 1 = -2\epsilon_c\,, \quad n_t = -2\epsilon_c\,, \quad r = \frac{8}{3}\epsilon_c^2 \quad \Rightarrow \quad r = \frac{2}{3}\,(n_s-1)^2$$

- we are a factor $2/9$ smaller!

Convolution of two contributions

The mapping: inflation $\longrightarrow$ bounce

Starobinsky: $(z_s/z_T)^2 = \frac{Q_s}{F} \approx \frac{3}{2}M_{\rm Pl}^2\epsilon_c^2$

matter bounce: $(z_s/z_T)^2 = 3M_{\rm Pl}^2$

This gives a factor $2/\epsilon_c^2$

The bounce

extra factor of $\epsilon_c^2/9$

Overview

- We constructed the "best-fit to data" matter-bounce model
- Control of each contribution to the power spectrum
- In the near future, we expect to have decisive new observational data of the very early universe.
- We need ways to discriminate primordial universe scenarios
- Constructive method might give a systematic route...

Thanks for your attention!