Entropy bounds, black holes and nonlinear electrodynamics

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PPGCosmo Workshop - 2020

UFES, Vitória - Espírito Santo

Outline

- Introduction entropy bounds & black holes
- Inequalities for nonlinear electrodynamics (flat spacetime)
- Entropy bounds & nonlinear electrodynamics

Black Hole & Thermodynamics

- Area Theorem: the area of a Black Hole never decreases
- Surface gravity κ is constant over the horizon
- For a stationary, axisymmetric spacetime with a BH

$$\delta M = \frac{1}{8\kappa}\delta A + \Omega_H \, \delta J_H$$

Ω_H : angular velocity of the Horizon
J_H : angular momentum of the BH

- Correspondance between Black Hole Mechanics and Thermodynamics

	Black Hole	Thermodynamics
zeroth law	κ is constant over horizon	T is constant over body in equilibrium
First law	$\delta M = \frac{1}{8\kappa}\delta A + \Omega_H \, \delta J_H$	$\mathrm{d}\mathcal{U} = T\mathrm{d}S - P\mathrm{d}V$
Second law	$\delta A \geq 0$	$\delta S \geq 0$

Bekenstein Bounds & Inequalities

Puzzle-1

- If $S_{BH} = \frac{k_B c^3}{4G\hbar} A$, then it seems that the second law is violated
    - Falling matter into the BH diminishes S_{mat}
    - Hawking radiation reduce $M_{BH} \longrightarrow \delta A < 0 \longrightarrow$ decrease of S_{BH}

Tentative answer: Generalized Second Law (GSL): $\delta(S_{BH} + S_{mat}) \geq 0$

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Puzzle-2

- BH entropy seems much larger than ordinary systems entropy
  - Solar-mass BH has an entropy 10^{20} larger than the sun's thermal entropy
  - thermodynamics $S \propto \mathcal{E}$ while in BH $S \propto M^2$

Tentative answer: upper bound to the entropy-energy ratio $S/\mathcal{E} \leq 2\pi\mathcal{R}$

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Puzzle-3

- How to express BH entropy in statistical terms, meaning describing it from microscopic configurations

Tentative answer: Quantum Gravity?

- Classicaly:
  - entropy is a measure of how much of phase space is available
  - for fixed energy (max. momentum), if the system is spatially bounded
  - if $\mathcal{E} \nearrow$ then $S \nearrow$, but why $S \propto \mathcal{E}$

- GSL points to a specific relation
  - stationary BH absorbs a body (neglible self-gravity) of energy $\mathcal{E}$ and radius $\mathcal{R}$, the area of BH increase by at least $2\pi\mathcal{E}\mathcal{R}$
  - GSL would be violated if the body's entropy could exceed $2\pi\mathcal{E}\mathcal{R}$

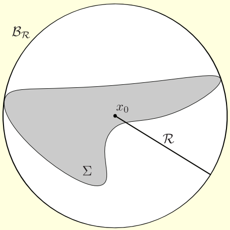
$$\boxed{\frac{S}{\mathcal{E}} \leq \frac{2\pi\, k_B}{\hbar c}\mathcal{R}}$$

Bekenstein Bounds & Inequalities

- Tighting the bound (Zaslavskii, Hod, Bekenstein & Mayo)

$$\frac{\hbar c}{2\pi\kappa_B} S \leq \sqrt{(\mathcal{E}\mathcal{R})^2 - c^2 J^2 - \frac{Q^2}{8\pi}} \, .$$

- Total charge $Q = \int_\Sigma \rho.$
- total energy $\mathcal{E} = \int_\Sigma T_{\alpha\beta} v^\alpha v^\beta.$
- total angular momentum $J = \int_\Sigma T_{\alpha\beta} v^\alpha \omega^\beta.$



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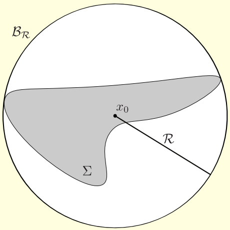

- Assuming that entropy is nonnegative $(c = 1)$

$$\boxed{\mathcal{E}^2 \geq \frac{Q^4}{64\pi^2\mathcal{R}^2} + \frac{J^2}{\mathcal{R}^2}} \quad , \quad \boxed{\mathcal{E} \geq \frac{Q^2}{8\pi\mathcal{R}}} \quad , \quad \boxed{\mathcal{E}(\Sigma) \geq \frac{|J(\Sigma)|}{\mathcal{R}}}$$

causality?

Sergio Dain proved that Maxwell electrodynamics satisfies all these inequalities in flat spacetime - Phys. Rev. 92, 044033 (2015).

Nonlinear Electrodynamics

Bekenstein inequalities and nonlinear electrodynamics
M. L. Peafiel and F. T. Falciano - Phys. Rev. D 96, 125011 (2017).

- The most general Lorentz invariant lagrangian $\mathcal{L} = \mathcal{L}(F, G)$.

Breaking Bekenstein's Inequalities

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Breaking Bekenstein's Inequalities

Example: ***Born-Infeld Electrodynamics***

$$S = \int \mathrm{d}\tau\, \beta^2 \left(1 - \sqrt{1 + \frac{F}{\beta^2} - \frac{G^2}{\beta^4}} \right) \quad \text{where} \quad \begin{cases} F \equiv \frac{1}{2} F^{\mu\nu} F_{\mu\nu} \\ G \equiv \frac{1}{2} \widetilde{F}^{\mu\nu} F_{\mu\nu} \end{cases}$$

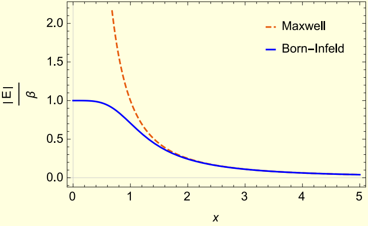
Fields equations:

$$\nabla \cdot \mathbf{D} = 4\pi\rho \quad , \quad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{j}$$

with

$$\mathbf{D} \equiv \frac{1}{\sqrt{U}} \left(\mathbf{E} + \frac{(\mathbf{E} \cdot \mathbf{B})}{\beta^2} \mathbf{B} \right)$$

$$\mathbf{H} \equiv \frac{1}{\sqrt{U}} \left(\mathbf{B} - \frac{(\mathbf{E} \cdot \mathbf{B})}{\beta^2} \mathbf{B} \right) ,$$



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Example: ***Born-Infeld Electrodynamics***

- Violates: $\mathcal{E} \geq Q^2/8\pi\mathcal{R}$ X
- satisfies: $\mathcal{E}(\Sigma) \geq |J(\Sigma)|/\mathcal{R}$ ✓ ... indeed DEC is sufficient!

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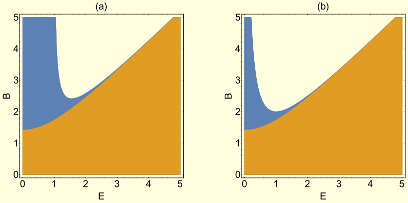
- Example: $\mathcal{L} = -\frac{F}{2} e^{-\frac{F}{2\beta^2}}$
- causality implies $\boxed{B^2 \leq 2\beta^2 + E^2}$
- It can be shown

$$\mathcal{E}(\Sigma) - \frac{|J(\Sigma)|}{\mathcal{R}} \geq \int_\Sigma e^{-\frac{F}{2\beta^2}} \mu(x, E, B)$$

$$\mu = \frac{B^2}{2} + \left(1 - \frac{F}{2\beta^2}\right)\left(E^2 + \frac{x}{\mathcal{R}} EB|\sin\theta|\right)$$

$(a)\ \theta = 0$ (b) $\theta = \pi/2$



Blue : $\mu(x, E, B) > 0 \rightarrow$ Inequality holds
Orange : causality holds

Test particle in the vicinity of a black hole

- Maxwell in Schwarzschild -

Black holes & electrodynamics

- The Schwarzschild spacetime (*isotropic coordinate*)

$$\mathrm{d}s^2 = \left(\frac{1 - r_s/4r}{1 + r_s/4r}\right)^2 c^2 \mathrm{d}t^2 - (1 + r_s/4r)^4 \left[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2\right]$$

  Note that in isotropic coordinate system the horizon is located at $r_h = r_s/4 = GM/2c^2$. Maxwell equation Maxwell's equations read

$$\partial_\mu \left(\sqrt{-g} F^{\mu\nu}\right) = 4\pi \sqrt{-g}\, j^\nu$$

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- Electrostatic case $\mathbf{E} = -\boldsymbol{\nabla}\phi$ reads

$$\Delta\phi + \frac{(1 - r_s/4r)}{(1 + r_s/4r)^3} \frac{\partial}{\partial r}\left(\frac{(1 + r_s/4r)^3}{(1 - r_s/4r)}\right)\frac{\partial\phi}{\partial r} = -4\pi\left(1 - \frac{r_s^2}{16r^2}\right)^2 \rho$$

  where Δ is the flat spacetime Laplacian operator.

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- The solution given by Copson(1928) and Linet(1976) for a particle at $(r = a, \theta = 0)$ is

$$\psi(r,\theta) = V_c(r,\theta) + V_p(r,\theta) = \frac{q\,(1 + r_s/4a)^{-2}}{r\,\mu\,(r,\theta)} \left(\frac{\mu\,(r,\theta) + r_s/4a}{1 + r_s/4r}\right)^2$$

$$\mu\,(r,\theta) = \sqrt{\frac{(r - b)^2 + 2br\,(1 - \cos\theta)}{(r - a)^2 + 2ar\,(1 - \cos\theta)}} \quad , \quad b \equiv \frac{r_s^2}{16a}$$

Black holes & electrodynamics

- Lowering the test particle with (m, q)

$$S = \int d\tau \left(mc \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} + \frac{q}{c} \dot{x}_\mu \hat{A}^\mu \right)$$

$\hat{A}^\mu$ is the background electromagnetic potential

$$\hat{A}_0 = \hat{A}_0^{BH} + \frac{1}{2} A_0 \qquad \text{Schw:} \quad \hat{A}_0^{BH} = 0$$

- Stationarity gives the conserved quantity

$$\begin{aligned} \mathcal{E} = c\, p_\mu \xi^\mu &= mc\, g_{0\beta}\, \dot{x}^\beta + \frac{q}{2} A_0 \,, \\ &= mc^2 \frac{(1 - r_s/4a)}{(1 + r_s/4a)} + \frac{q}{2} \psi(r \to a, \theta \to 0) \end{aligned}$$

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- Change of coordinate adapted to the particle

$$r \cos\theta \ \to a + \rho \cos\vartheta \ , \quad r \ \to \sqrt{a^2 + \rho^2 + 2a\rho \cos\vartheta}$$

The charged particle is located at $\rho = 0$

$$\psi(\rho, \vartheta, \varphi) = \frac{(1 - r_s/4a)}{(1 + r_s/4a)^3} \frac{q}{\rho} + \frac{q\, r_s}{2a^2\, (1 + r_s/4a)^4} + \mathcal{O}(\rho)$$

Coulmob $+$ horizon polarization

Black holes & electrodynamics

- proper distance close to the horizon

$$L \equiv \int_{r_s/4}^{a} \mathrm{d}r \sqrt{g_{rr}} \approx 4\,(a - r_s/4)$$

hence

$$\mathcal{E} = \frac{2mc^2 L + q^2}{4r_s} + \mathcal{O}\left(\frac{L}{M}\right)^2 \geq \frac{2mc^2 \mathcal{R} + q^2}{4r_s} + \mathcal{O}\left(\frac{\mathcal{R}}{M}\right)^2$$

The black hole area increment is

$$\delta A = \frac{8\pi G}{c^4}\left(2r_s\mathcal{E} - q^2\right) + \mathcal{O}(3) \geq \frac{4\pi G}{c^4}\left[2mc^2\mathcal{R} - q^2\right]$$

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- Generalized Second Law of thermodynamics (GSL)

$$\delta\left(S_{BH} + S_p\right) = \frac{\delta A}{4\hbar} - S_p \geq 0 \qquad \Rightarrow \qquad S_p \leq \frac{2\pi G}{\hbar c^2} m\mathcal{R}\left[1 - \frac{q^2}{2mc^2\mathcal{R}}\right]$$

Test particle in the vicinity of a black hole

- Born-Infeld in Schwarzschild -

Entropy bounds & nonlinear electrodynamics

F. T. Falciano, M. L. Peafiel and E. P. Bergliaffa

Phys. Rev. D 100, 125008 (2019)

Black holes & nonlinear electrodynamics

- Intermezzo: Born-Infled in flat spacetime

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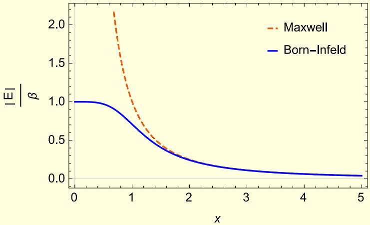
$\nabla \cdot \mathbf{D} = 4\pi\rho$, spherically symmetric electrostatic



$$\mathbf{D} = \frac{q}{(r-a)^3}(\mathbf{r}-\mathbf{a}) = \frac{\beta}{x^2}\hat{\mathbf{x}}$$

$$\mathbf{E} = \frac{\beta}{\sqrt{1+x^4}}\hat{\mathbf{x}}$$

where $x \equiv \sqrt{\frac{\beta}{q}}\,(r-a)$

$$\phi_{\rm BI}(r) = -\int_\infty^r \mathrm{d}\mathbf{r}\cdot\mathbf{E} = \frac{\sqrt{q\beta}}{x}\,{}_2F_1\left[\frac{1}{4},\frac{1}{2},\frac{5}{4},-\frac{1}{x^4}\right]$$

in particular, over the particle

$$\phi_{\rm BI}(a) = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\sqrt{\pi}}\sqrt{q\beta} \equiv \frac{q}{\lambda_{\rm BI}} \quad \text{where} \quad \lambda_{\rm BI} \approx 0.54\sqrt{\frac{q}{\beta}}$$

Black holes & nonlinear electrodynamics

- The Schwarzschild spacetime (*isotropic coordinate*)

$$\mathrm{d}s^2 = \left(\frac{1 - r_s/4r}{1 + r_s/4r}\right)^2 c^2 \mathrm{d}t^2 - (1 + r_s/4r)^4 \left[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2\right]$$

Nonlinear electrodynamics $\mathcal{L}(F, G)$ dynamics

$$\partial_\mu (\sqrt{-g}\, E^{\mu\nu}) = -4\pi \sqrt{-g}\, j^\nu$$

where

$$E^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} = 2\left(\mathcal{L}_F F^{\mu\nu} + \mathcal{L}_G \widetilde{F}^{\mu\nu}\right) \qquad \text{excitation tensor}$$

$$D^\mu = -E^\mu{}_\nu v^\nu = -2\left(\mathcal{L}_F E^\mu + \mathcal{L}_G B^\mu\right) \qquad \text{displacement vector}$$

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Theorem

The electrostatic potential $\phi(x)$ produced by a charged particle in a generic NLED theory $\mathcal{L}(F, G)$ in Schwarzschild spacetime is entirely specified by the electrostatic potential $\psi(x)$ satisfying Maxwell's electromagnetism in the same background. The displacement vector is curl-free and given by $\mathbf{D} = -\boldsymbol{\nabla}\psi(x)$.

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Proof. —
The electrostatic implies $\mathbf{D} = -2\mathcal{L}_F(E)\,\mathbf{E}$, with $\mathbf{E} = -\boldsymbol{\nabla}\phi$

$$\psi(x) = -2\int \mathcal{L}_F(\boldsymbol{\nabla}\phi)\boldsymbol{\nabla}\phi \cdot \mathrm{d}\mathbf{l} \qquad \Leftrightarrow \qquad \boldsymbol{\nabla}\psi(x) = -2\mathcal{L}_F(\boldsymbol{\nabla}\phi)\boldsymbol{\nabla}\phi$$

$$\Delta\psi + \frac{(1 - r_s/4r)}{(1 + r_s/4r)^3}\frac{\partial}{\partial r}\left(\frac{(1 + r_s/4r)^3}{(1 - r_s/4r)}\right)\frac{\partial\psi}{\partial r} = -4\pi\left(1 - \frac{r_s^2}{16r^2}\right)^2 \rho$$

Black holes & nonlinear electrodynamics

- Born-Infeld in Schwarzschild

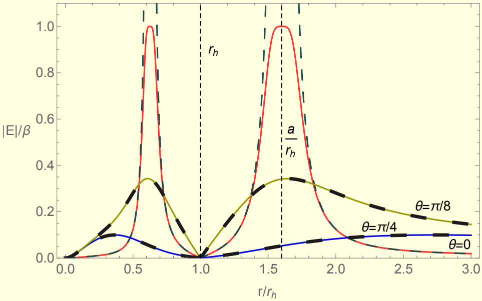
$$\mathbf{E} = -\nabla\phi(r,\theta)$$
$$= \frac{\mathbf{D}}{\sqrt{1+|\mathbf{D}|^2\beta^{-2}}}$$

theorem guarantees

$$\mathbf{D}(r,\theta) = -\nabla\psi(r,\theta)$$
$$\nabla\phi(r,\theta) = \frac{\nabla\psi(r,\theta)}{\sqrt{1+|\nabla\psi(r,\theta)|^2\beta^{-2}}}$$

where $\psi(r,\theta)$ is Linet's solution

Unfortunately, there is no analytical solution....



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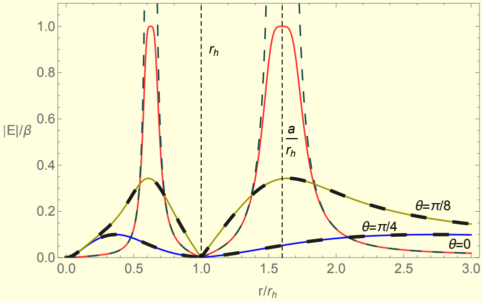


Unfortunately, there is no analytical solution.... formal solution

$$\phi(r,\theta) = \int_\infty^r \frac{\mathrm{d}\mathbf{x}.\nabla\psi(x,\theta)}{\sqrt{1+|\nabla\psi(x,\theta)|^2\beta^{-2}}}$$

- Born-Infeld in Schwarzschild

first approximation

$$\mathbf{E}(r,\theta) = -\nabla\phi \approx -\frac{\nabla V_c}{\sqrt{1+|\nabla V_c|^2\beta^{-2}}} - \nabla V_p \qquad \text{recall} \qquad \psi = V_c + V_p$$

which is equivalent to

$$\phi(r,\theta) \approx \phi_c(r,\theta) + V_p(r,\theta) \qquad \text{with} \qquad \nabla\phi_c = \frac{\nabla V_c}{\sqrt{1+|\nabla V_c|^2\beta^{-2}}}$$

second approximation...

$$\lim_{r\to a} |\nabla V_c| \approx \beta \frac{V_c^2}{\Sigma^2} \qquad \text{with} \qquad \Sigma^2 \equiv \frac{\beta q(1-b/a)}{(1+r_s/4a)^2}$$

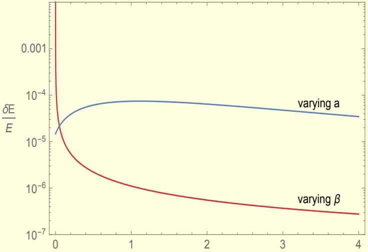
$$\phi_c(r,\theta) = \Sigma\left(\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\sqrt{\pi}} - \frac{\Sigma}{V_c}\,{}_2F_1\left[\frac{1}{4},\frac{1}{2},\frac{5}{4},-\left(\frac{\Sigma}{V_c}\right)^4\right]\right)$$

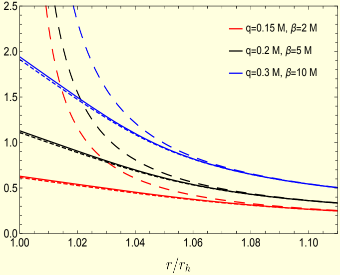
Thus,

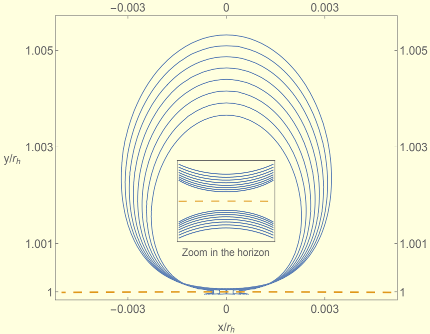
$$\phi(a,0) = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\sqrt{\pi}}\sqrt{q\beta}\sqrt{\frac{(1-r_s/4a)}{(1+r_s/4a)}} + \frac{qr_s}{2a^2\,(1+r_s/4a)^2}$$

Black holes & nonlinear electrodynamics

- Goodness of the approximation







Born-Infeld region
(nonlinearities are important)

$$|\nabla\psi| \geq \beta$$

Black holes & nonlinear electrodynamics

PPGCosmo CBPF 70 ANOS CENTRO BRASILEIRO DE PESQUISAS FÍSICAS MINISTÉRIO DA CIÊNCIA, TECNOLOGIA, INOVAÇÕES E COMUNICAÇÕES

- Lowering the test particle with (m, q)
- Stationarity: $\mathcal{E} = c\, p_\mu\, \xi^\mu = mc\, g_{0\beta}\, \dot{x}^\beta + \frac{q}{2} A_0$

$$\mathcal{E} = mc^2 \frac{(1 - r_s/4a)}{(1 + r_s/4a)} + \frac{q}{2}\left[\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\sqrt{\pi}}\sqrt{q\beta}\sqrt{\frac{(1 - r_s/4a)}{(1 + r_s/4a)}} + \frac{q r_s}{2a^2\,(1 + r_s/4a)^2}\right]$$

Close to the horizon $a \to r_s/4$

$$\mathcal{E} = \frac{1}{4r_s}\left[2mc^2 L + q^2\left(1 + \sqrt{\frac{2Lr_s}{\lambda_{\mathrm{BI}}^2}}\right)\right]$$

$$\geq \frac{1}{4r_s}\left[2mc^2\mathcal{R} + q^2\left(1 + \sqrt{\frac{2\mathcal{R}r_s}{\lambda_{\mathrm{BI}}^2}}\right)\right] \qquad \text{with} \qquad \lambda_{\mathrm{BI}} = \sqrt{\frac{q}{\beta}}$$

the change in the black hole area is

$$\delta A = \frac{8\pi G}{c^4}\left(2r_s\mathcal{E} - q^2\right) + \mathcal{O}(3) \geq 4\pi\left[\frac{2Gm}{c^2}\mathcal{R} - \frac{Gq^2}{c^4}\left(1 - \sqrt{\frac{2\mathcal{R}\, r_s}{\lambda_{\mathrm{BI}}^2}}\right)\right]$$

thus, assuming GSL

$$S_{\mathrm{BI}} \leq \frac{2\pi k_B}{\hbar c}\left[mc^2\mathcal{R} + \frac{q^2}{2}\left(-1 + \sqrt{2\frac{\mathcal{R}\, r_s}{\lambda_{\mathrm{BI}}^2}}\right)\right] \qquad r_s \geq \mathcal{R}$$

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Conclusion

- Flat spacetime

$$\begin{cases} \text{Maxwell's electrodynamics satisfies} \\ \text{Nonlinear electrodynamics does not} \\ \mathcal{E}(\Sigma) \geq \frac{|J(\Sigma)|}{\mathcal{R}} \qquad \text{open physical meaning} \end{cases}$$

- Entropy bound for nonlinear electrodynamics in Schwarzschild black hole doesn't work

$$\begin{cases} \delta A \quad \text{depends on Black hole mass} \\ \text{A loose inequality violates Bekenstein previous bound} \end{cases}$$

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Open questions:

- Can we generalize entropy bound to include nonlinear electrodynamics?
- Should we abandon Generalized Second Law of thermodynamics?

Thanks for your attention!