Mathematical and Physical Foundation of Extended Gravity (III)

-Constraining models by Gravity Probe B and LARES-

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Summary

- *Extended Gravity*

  The general case: Scalar-tensor-higher-order gravity
  The case of non-commutative spectral geometry

- *The weak field limit*

  The Newtonian limit
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- *The body motions in the weak gravitational field*

  Circular rotation curves in a spherically symmetric field
  Rotating sources and orbital parameters

- *Experimental constrains*

  Gravity Probe B and LARES

- *Conclusions*

Why extending General Relativity?

- ✓ Several issues in modern Astrophysics ask for new paradigms.
- ✓ No final evidence for Dark Energy and Dark Matter at fundamental level (LHC, astroparticle physics, ground based experiments, LUX…).
- ✓ Such problems could be framed extending GR at infrared scales.
- ✓ GR does not work at ultraviolet scales (no Quantum Gravity).
- ✓ ETGs as minimal extension of GR considering Quantum Fields in Curved Spaces
- ✓ Big issue: Is it possible to find out probes and test-beds for ETGs?
- ✓ Further modes of gravitational waves!
- ✓ Constraints at Newtonian and post-Newtonian level could come from:
  - Geodesic motions around compact objects e.g- SgrA*
  - Lense-Thirring effect
  - Exact torsion-balance experiments
  - Microgravity experiments from atomic physics
  - Violation of Equivalence Principle (effective masses related to further gravitational degrees of freedom)

Main role of GPB and LARES satellites

The general case: Scalar-tensor-higher-order gravity

Action

$$\mathcal{S} = \int d^4x \sqrt{-g}[f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} + \mathcal{X}\mathcal{L}_m],$$

Field Equations

$$\begin{aligned} & f_R R_{\mu\nu} - \frac{f + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}}{2} g_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu}\Box f_R \\ & + 2f_Y R_\mu{}^\alpha R_{\alpha\nu} - 2[f_Y R^\alpha{}_{(\mu}]_{;\nu)\alpha} + \Box[f_Y R_{\mu\nu}] \\ & + [f_Y R_{\alpha\beta}]^{;\alpha\beta} g_{\mu\nu} + \omega(\phi)\phi_{;\mu}\phi_{;\nu} = \mathcal{X}T_{\mu\nu}, \end{aligned}$$

The trace of the field equation

$$\begin{aligned} & f_R R + 2f_Y R_{\alpha\beta}R^{\alpha\beta} - 2f + \Box[3f_R + f_Y R] + 2[f_Y R^{\alpha\beta}]_{;\alpha\beta} \\ & - \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} = \mathcal{X}T, \end{aligned}$$

the Klein-Gordon equation

$$2\omega(\phi)\Box\phi + \omega_\phi(\phi)\phi_{;\alpha}\phi^{;\alpha} - f_\phi = 0,$$

An example: Non-Commutative Spectral Geometry

For almost-commutative manifolds, the geometry is described by the tensor product $M\times F$ of a 4D compact Riemannian manifold M and a discrete non-commutative space F, with M describing the geometry of spacetime and F the internal space of the particle physics model.

The non-commutative nature of F is encoded in the spectral triple (A_F, H_F, D_F)

The algebra $A_F = C^\infty(M)$ of smooth functions on M, playing the role of the algebra of coordinates, is an involution of operators on the finite-dimensional Hilbert space H_F of Euclidean fermions.

The operator D_F is the Dirac operator

$$\partial\!\!\!/_{\mathcal{M}} = \sqrt{-1}\gamma^\mu \nabla_\mu^s$$

on the spin manifold M; it corresponds to the inverse of the Euclidean propagator of fermions and is given by the Yukawa coupling matrix and the Kobayashi-Maskawa mixing parameters.

The algebra A_F has to be chosen so that it can lead to the Standard Model of particle physics, while it must also fulfill non-commutative geometry requirements.

The case of Non-Commutative Spectral Geometry

It is chosen to be $\mathcal{A}_{\mathcal{F}} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C})$.

with k=2a; H is the algebra of quaternions, which encodes the non-commutativity of the manifold.

The first possible value for k is 2, corresponding to the Hilbert space of four fermions; it is ruled out from the existence of quarks.

The minimum possible value for k is 4 leading to the correct number of k^2 =16 fermions in each of the three generations.

Higher values of k can lead to particle physics models beyond the Standard Model

The spectral geometry in the product M× F is given by the product rules:

where $L^2(M, S)$ is the Hilbert space of L^2 spinors and D_M is the Dirac operator of the Levi-Cività spin connection on M →

$$\mathcal{A} = C^\infty(\mathcal{M}) \oplus \mathcal{A}_F,$$
$$\mathcal{H} = L^2(\mathcal{M}, S) \oplus \mathcal{H}_F,$$
$$\mathcal{D} = \mathcal{D}_M \oplus 1 + \gamma_5 \oplus \mathcal{D}_F.$$

Applying the spectral action principle to the product geometry M×F leads to the NCSG action

$$\mathrm{Tr}(f(D_A/\Lambda)) + (1/2)\langle J\psi, D\psi\rangle$$

split into the bare bosonic action and the fermionic one. Note that $D_A = D + A + \epsilon' JAJ^{-1}$ are unimodular inner fluctuations, f is a cutoff function, Λ fixes the energy scale, J is the real structure on the spectral triple and ψ is a spinor in the Hilbert space H of the quarks and leptons.

The case of Non-Commutative Spectral Geometry

Considering the bosonic part of the action, seen as the bare action at the mass scale Λ which includes the eigenvalues of the Dirac operator that are smaller than the cutoff scale Λ, considered as the grand unification scale.

Using heat kernel methods, the trace $Tr(fD_A/\Lambda)$ can be written in terms of the geometrical Seeley–de Witt coefficients known for any second-order elliptic differential operator, as $\Sigma\infty_{n=0}F_{4-n}\Lambda^{4-n}a_n$ where the function F is defined such that $F(D^2{}_A)=f(D_A)$.

Considering the Riemannian geometry to be four dimensional, the asymptotic expansion of the trace reads

$$\mathrm{Tr}(f(\mathcal{D}_A/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 + \cdots + \Lambda^{-2k} f_{-2k} a_{4+2k} + \cdots ,$$

where f_k are the momenta of the smooth even test (cutoff) function which decays fast at infinity, and only enters in the multiplicative factors:

$$f_0 = f(0),$$

$$f_2 = \int_0^\infty f(u)u\,\mathrm{d}u,$$

$$f_4 = \int_0^\infty f(u)u^3\,\mathrm{d}u,$$

$$f_{-2k} = (-1)^k \frac{k!}{(2k)!} f^{(2k)}(0)$$

The case of Non-Commutative Spectral Geometry

Since the Taylor expansion of the f function vanishes at zero, the asymptotic expansion of the spectral action reduces to

$$\mathrm{Tr}(f(\mathcal{D}_{\mathcal{A}}/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4$$

Hence, the cutoff function f plays a role only through its momenta. f_0 , f_2 , f_4 are three real parameters, related to the coupling constants at unification, the gravitational constant, and the cosmological constant, respectively

The NCSG model lives by construction at the grand unification scale, hence providing a framework to study early Universe cosmology

The gravitational part of the asymptotic expression for the bosonic sector of the NCSG action, including the coupling between the Higgs field φ and the Ricci curvature scalar R, in Lorentzian signature, obtained through a Wick rotation in imaginary time, reads

$$\mathcal{S}^{\mathrm{L}}_{\mathrm{grav}} = \int \mathrm{d}^4 x \sqrt{-g}\left[\frac{R}{2\kappa_0^2} + \alpha_0 C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + \tau_0 R^\star R^\star - \xi_0 R|\mathbf{H}|^2\right];$$

$$\mathbf{H} = (\sqrt{a f_0}/\pi)\phi$$

With a parameter related to fermion and lepton masses and lepton mixing

At unification scale (set up by Λ), $\alpha_0 = -3f_0/(10\pi^2)$, $\xi_0 = 1/12$.

The case of non-commutative spectral geometry

The square of the Weyl tensor can be expressed in terms of R^2 and $R_{\alpha\beta}R^{\alpha\beta}$ as

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = 2R_{\alpha\beta}R^{\alpha\beta} - \frac{2}{3}R^2$$

***The above action is clearly a particular case of the above action describing a general model of ETG** Phys.Rev. D91 (2015) 044012*

As we will show, it may lead to effects observable at local scales (in particular at Solar System scales); hence it may be tested against current gravitational data by GPB and LARES.

IN OTHER WORDS, WE CAN USE GPB AND LARES TO TEST FUNDAMENTAL PHYSICS!!!

The weak field limit

The weak field limit

- *The typical values of the Newtonian gravitational potential Φ are larger (in modulus) than 10^{-5} in the Solar System (in geometrized units, Φ is dimensionless).*
- *Planetary velocities satisfy the condition $v^2 \lesssim -\Phi$, while the matter pressure P experienced inside the Sun and the planets is generally smaller than the matter gravitational energy density $-\rho\,\Phi$; in other words $P/\rho \lesssim -\Phi$*
- *As matter of fact, one can consider that these quantities, as a function of the velocity, give second-order contributions as $-\Phi \sim v^2 \sim O(2)$*
- *Then we can set, as a perturbation scheme of the metric tensor, the following expression*

$$g_{\mu\nu} \sim \begin{pmatrix} 1 + g_{tt}^{(2)}(t,\mathbf{x}) + g_{tt}^{(4)}(t,\mathbf{x}) + \ldots & g_{ti}^{(3)}(t,\mathbf{x}) + \ldots \\ g_{ti}^{(3)}(t,\mathbf{x}) + \ldots & -\delta_{ij} + g_{ij}^{(2)}(t,\mathbf{x}) + \ldots \end{pmatrix} = \begin{pmatrix} 1 + 2\Phi + 2\Xi & 2A_i \\ 2A_i & -\delta_{ij} + 2\Psi\delta_{ij} \end{pmatrix}$$

$$\phi \sim \phi^{(0)} + \phi^{(2)} + \ldots = \phi^{(0)} + \varphi,$$

- *Φ, Ψ, ϕ are proportional to the power c^{-2} (Newtonian limit) while A_i is proportional to c^{-3} and Ξ to c^{-4} (post-Newtonian limit)*

The weak field limit

The function f, up to the c^{-4} order, can be developed as

$$f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi) = f_R(0,0,\phi^{(0)})R + \frac{f_{RR}(0,0,\phi^{(0)})}{2}R^2 + \frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2}(\phi - \phi^{(0)})^2 + f_{R\phi}(0,0,\phi^{(0)})R\phi + f_Y(0,0,\phi^{(0)})R_{\alpha\beta}R^{\alpha\beta},$$

while all other possible contributions in f are negligible

The field equations hence read

$$f_R(0,0,\phi^{(0)})\left[R_{tt} - \frac{R}{2}\right] - f_Y(0,0,\phi^{(0)})\triangle R_{tt} - \left[f_{RR}(0,0,\phi^{(0)}) + \frac{f_Y(0,0,\phi^{(0)})}{2}\right]\triangle R - f_{R\phi}(0,0,\phi^{(0)})\triangle\varphi = \mathcal{X}T_{tt},$$

$$f_R(0,0,\phi^{(0)})\left[R_{ij} + \frac{R}{2}\delta_{ij}\right] - f_Y(0,0,\phi^{(0)})\triangle R_{ij} + \left[f_{RR}(0,0,\phi^{(0)}) + \frac{f_Y(0,0,\phi^{(0)})}{2}\right]\delta_{ij}\triangle R - f_{RR}(0,0,\phi^{(0)})R_{,ij} - 2f_Y(0,0,\phi^{(0)})R^{\alpha}_{(i,j)\alpha} - f_{R\phi}(0,0,\phi^{(0)})(\partial^2_{ij} - \delta_{ij}\triangle)\varphi = \mathcal{X}T_{ij},$$

$$f_R(0,0,\phi^{(0)})R_{ti} - f_Y(0,0,\phi^{(0)})\triangle R_{ti} - f_{RR}(0,0,\phi^{(0)})R_{,ti} - 2f_Y(0,0,\phi^{(0)})R^{\alpha}{}_{(t,i)\alpha} - f_{R\phi}(0,0,\phi^{(0)})\varphi_{,ti} = \mathcal{X}T_{ti}, f_R(0,0,\phi^{(0)})R + [3f_{RR}(0,0,\phi^{(0)}) + 2f_Y(0,0,\phi^{(0)})]\triangle R + 3f_{R\phi}(0,0,\phi^{(0)})\triangle\varphi = -\mathcal{X}T,$$

$$2\omega(\phi^{(0)})\triangle\varphi + f_{\phi\phi}(0,0,\phi^{(0)})\varphi + f_{R\phi}(0,0,\phi^{(0)})R = 0,$$

where $\triangle$ is the Laplace operator in the flat space

The weak field limit

*The geometric quantities $R_{\mu\nu}$ and R are evaluated at the first order with respect to the metric potentials Φ, Ψ and A_i. By introducing the **effective masses***

$$m_R{}^2 \doteq -\frac{f_R(0,0,\phi^{(0)})}{3f_{RR}(0,0,\phi^{(0)})+2f_Y(0,0,\phi^{(0)})}, \qquad m_Y{}^2 \doteq \frac{f_R(0,0,\phi^{(0)})}{f_Y(0,0,\phi^{(0)})}, \qquad m_\phi{}^2 \doteq -\frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2\omega(\phi^{(0)})},$$

and setting $f_R(0, 0, \varphi^{(0)})=1$, $\omega(\varphi^{(0)})= 1/2$ for simplicity, we get the complete set of differential equations

$$(\triangle - m_Y{}^2)R_{tt} + \left[\frac{m_Y{}^2}{2} - \frac{m_R{}^2+2m_Y{}^2}{6m_R{}^2}\triangle\right]R + m_Y{}^2 f_{R\phi}(0,0,\phi^{(0)})\triangle\varphi$$
$$= -m_Y{}^2\mathcal{X}T_{tt}, (\triangle - m_Y{}^2)R_{ij} + \left[\frac{m_R{}^2-m_Y{}^2}{3m_R{}^2}\partial^2_{ij} - \delta_{ij}\left(\frac{m_Y{}^2}{2} - \frac{m_R{}^2+2m_Y{}^2}{6m_R{}^2}\triangle\right)\right]R + m_Y{}^2 f_{R\phi}(0,0,\phi^{(0)})(\partial^2_{ij} - \delta_{ij}\triangle)\varphi$$
$$= -m_Y{}^2\mathcal{X}T_{ij}, (\triangle - m_Y{}^2)R_{ti} + \frac{m_R{}^2-m_Y{}^2}{3m_R{}^2}R_{,ti} + m_Y{}^2 f_{R\phi}(0,0,\phi^{(0)})\varphi_{,ti}$$
$$= -m_Y{}^2\mathcal{X}T_{ti}, (\triangle - m_R{}^2)R - 3m_R{}^2 f_{R\phi}(0,0,\phi^{(0)})\triangle\varphi = m_R{}^2\mathcal{X}T, (\triangle - m_\phi{}^2)\varphi + f_{R\phi}(0,0,\phi^{(0)})R = 0.$$

The components of the Ricci tensor in the weak-field limit

$$R_{tt} = \frac{1}{2}\triangle g^{(2)}_{tt} = \triangle\Phi,$$
$$R_{ij} = \frac{1}{2}g^{(2)}_{ij,mm} - \frac{1}{2}g^{(2)}_{im,mj} - \frac{1}{2}g^{(2)}_{jm,mi} - \frac{1}{2}g^{(2)}_{tt,ij} + \frac{1}{2}g^{(2)}_{mm,ij} = \triangle\Psi\delta_{ij} + (\Psi - \Phi)_{,ij},$$
$$R_{ti} = \frac{1}{2}g^{(3)}_{ti,mm} - \frac{1}{2}g^{(2)}_{im,mt} - \frac{1}{2}g^{(3)}_{mt,mi} + \frac{1}{2}g^{(2)}_{mm,ti} = \triangle A_i + \Psi_{,ti}.$$

The weak field limit

Expansion of the energy momentum tensor $T_{\mu\nu}$

The pressure is negligible in the weak field limit, it reads $T_{\mu\nu} = \rho u_\mu u_\nu$ with $u_\sigma u^\sigma = 1$

Starting at the zeroth order, it is $T_{tt} = T^{(0)}{}_{tt} = \rho$, $T_{ij} = T^{(0)}{}_{ij} = 0$ and $T_{ti} = T^{(1)}{}_{ti} = \rho v_i$, where ρ is the density mass and v_i is the velocity of the source

$T_{\mu\nu}$ is independent of metric potentials and satisfies the Bianchi identities

$$T^{\mu\nu}{}_{,\mu} = 0$$

Equations read

$$(\triangle - m_Y{}^2)\triangle\Phi + \left[\frac{m_Y{}^2}{2} - \frac{m_R{}^2 + 2m_Y{}^2}{6m_R{}^2}\triangle\right]R + m_Y{}^2 f_{R\phi}(0,0,\phi^{(0)})\triangle\varphi = -m_Y{}^2\mathcal{X}\rho,$$

$$\left\{(\triangle - m_Y{}^2)\triangle\Psi - \left[\frac{m_Y{}^2}{2} - \frac{m_R{}^2 + 2m_Y{}^2}{6m_R{}^2}\triangle\right]R - m_Y{}^2 f_{R\phi}(0,0,\phi^{(0)})\triangle\varphi\right\}\delta_{ij}$$
$$+ \left\{(\triangle - m_Y{}^2)(\Psi - \Phi) + \frac{m_R{}^2 - m_Y{}^2}{3m_R{}^2}R + m_Y{}^2 f_{R\phi}(0,0,\phi^{(0)})\varphi\right\}_{,ij} = 0,$$

$$\left\{(\triangle - m_Y{}^2)\triangle A_i + m_Y{}^2\mathcal{X}\rho v_i\right\} + \left\{(\triangle - m_Y{}^2)\Psi + \frac{m_R{}^2 - m_Y{}^2}{3m_R{}^2}R + m_Y{}^2 f_{R\phi}(0,0,\phi^{(0)})\varphi\right\}_{,ti} = 0,$$

$$(\triangle - m_R{}^2)R - 3m_R{}^2 f_{R\phi}(0,0,\phi^{(0)})\triangle\varphi = m_R{}^2\mathcal{X}\rho,$$

$$(\triangle - m_\phi{}^2)\varphi + f_{R\phi}(0,0,\phi^{(0)})R = 0.$$

Solutions for fields Φ, ϕ and R

The above equations are a coupled system and, for a pointlike source $\rho(x) = M\,\delta(x)$, admit the solutions

$$\varphi(\mathbf{x}) = \sqrt{\frac{\xi}{3}}\frac{r_g}{|\mathbf{x}|}\frac{e^{-m_R\tilde{k}_R|\mathbf{x}|} - e^{-m_R\tilde{k}_\phi|\mathbf{x}|}}{\tilde{k}_R^2 - \tilde{k}_\phi^2},$$

$$R(\mathbf{x}) = -m_R{}^2\frac{r_g}{|\mathbf{x}|}\frac{(\tilde{k}_R^2 - \eta^2)e^{-m_R\tilde{k}_R|\mathbf{x}|} - (\tilde{k}_\phi^2 - \eta^2)e^{-m_R\tilde{k}_\phi|\mathbf{x}|}}{\tilde{k}_R^2 - \tilde{k}_\phi^2}$$

where r_g is the Schwarzschild radius

$$\tilde{k}_{R,\phi}^2 = \frac{1-\xi+\eta^2\pm\sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}$$

and $\xi = 3 f_{R\phi}(0,0,\phi^{(0)})^2$ *and* $\eta = \dfrac{m_\phi}{m_R}$

ξ and η satisfy the condition $(\eta-1)^2 - \xi > 0$

The solution of the gravitational potential Φ reads

$$\Phi(\mathbf{x}) = \frac{-1}{16\pi^2}\int\frac{d^3\mathbf{x}'d^3\mathbf{x}''}{|\mathbf{x}-\mathbf{x}'|}\frac{e^{-m_Y|\mathbf{x}'-\mathbf{x}''|}}{|\mathbf{x}'-\mathbf{x}''|}\left[\frac{4m_Y{}^2 - m_R{}^2}{6}\mathcal{X}\rho(\mathbf{x}'') + \frac{m_Y{}^2 - m_R{}^2(1-\xi)}{6}R(\mathbf{x}'') - \frac{m_R{}^4\eta^2}{2\sqrt{3}}\xi^{1/2}\varphi(\mathbf{x}'')\right],$$

Solutions for fields Φ, ϕ and R

for a pointlike source, it is

$$\Phi(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}\left[1 + g(\xi,\eta)e^{-m_R\tilde{k}_R|\mathbf{x}|} + \left[\frac{1}{3} - g(\xi,\eta)\right]e^{-m_R\tilde{k}_\phi|\mathbf{x}|} - \frac{4}{3}e^{-m_Y|\mathbf{x}|}\right]$$

where

$$g(\xi,\eta) = \frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}$$

For $f_Y \to 0$ i.e. $m_Y \to \infty$, we obtain the same outcome for the gravitational potential of $f(R, \varphi)$-theory

Solutions for fields Ψ and A_i

Solution for A_i,

$$A_i(\mathbf{x}) = -\frac{{m_Y}^2 \mathcal{X}}{16\pi^2} \int d^3\mathbf{x}' d^3\mathbf{x}'' \frac{e^{-m_Y|\mathbf{x}'-\mathbf{x}''|}}{|\mathbf{x}-\mathbf{x}'||\mathbf{x}'-\mathbf{x}''|} \rho(\mathbf{x}'') v_i''$$

In Fourier space, solution presents the massless pole of GR and a massive one induced by the $R^{\alpha\beta}R_{\alpha\beta}$ term

The solution is the sum of GR contributions and massive modes

$$A_i(\mathbf{x}) = -\frac{\mathcal{X}}{4\pi} \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}') v_i'}{|\mathbf{x}-\mathbf{x}'|} + \frac{\mathcal{X}}{4\pi} \int d^3\mathbf{x}' \frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \rho(\mathbf{x}') v_i'$$

For a spherically symmetric system ($|x| = r$) at rest and rotating with angular frequency $\Omega(r)$, the energy momentum tensor T_{ti} is

$$\begin{aligned} T_{ti} &= \rho(\mathbf{x}) v_i = T_{tt}(r)[\Omega(r) \times \mathbf{x}]_i \\ &= \frac{3M}{4\pi\mathcal{R}^3} \Theta(\mathcal{R} - r)[\Omega(r) \times \mathbf{x}]_i, \end{aligned}$$

where R is the radius of the body and Θ is the Heaviside function

Solutions for fields Ψ and A_i

In fact for any term $\propto \frac{e^{-mr}}{r}$ *there is a geometric factor multiplying the Yukawa term, namely*

$$F(m\mathcal{R}) = 3\,\frac{m\mathcal{R}\cosh m\mathcal{R} - \sinh m\mathcal{R}}{m^3\mathcal{R}^3}$$

We get

$$\Phi_{\text{ball}}(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}\left[1 + g(\xi,\eta)F(m_R\tilde{k}_R\mathcal{R})e^{-m_R\tilde{k}_R|\mathbf{x}|} + \left[\frac{1}{3} - g(\xi,\eta)\right]F(m_R\tilde{k}_\phi\mathcal{R})e^{-m_R\tilde{k}_\phi|\mathbf{x}|} - \frac{4F(m_Y\mathcal{R})}{3}e^{-m_Y|\mathbf{x}|}\right]$$

$$\Psi_{\text{ball}}(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}\left[1 - g(\xi,\eta)F(m_R\tilde{k}_R\mathcal{R})e^{-m_R\tilde{k}_R|\mathbf{x}|} - \left[\frac{1}{3} - g(\xi,\eta)\right]F(m_R\tilde{k}_\phi\mathcal{R})e^{-m_R\tilde{k}_\phi|\mathbf{x}|} - \frac{2F(m_Y\mathcal{R})}{3}e^{-m_Y|\mathbf{x}|}\right].$$

For $\Omega(r)=\Omega_0$, the metric potential is

$$\mathbf{A}(\mathbf{x}) = -\frac{3MG}{2\pi\mathcal{R}^3}\Omega_0 \times \int d^3\mathbf{x}'\,\frac{1-e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\Theta(\mathcal{R}-r')\mathbf{x}'.$$

in the approximation

$$\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \sim \frac{e^{-m_Y r}}{r} + \frac{e^{-m_Y r}(1+m_Y r)\cos\alpha}{r}\frac{r'}{r} + \mathcal{O}\left(\frac{r'^2}{r^2}\right)$$

α is the angle between the vectors x, x', with x = r x where ˆx =(sin θ cos φ;, sin θ sin φ, cos θ) and, at the first order of r'/r, we can evaluate the integration in the vacuum (r > R) as

$$\int d^3\mathbf{x}'\,\frac{e^{-m_Y|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\Theta(\mathcal{R}-r')\mathbf{x}' = \frac{4\pi}{15}\frac{(1+m_Y r)e^{-m_Y r}\mathcal{R}^5}{r^3}\mathbf{x}.$$

Solutions for fields Ψ and A_i

The field A outside the sphere is

$$\mathbf{A}(\mathbf{x}) = \frac{G}{|\mathbf{x}|^2}[1 - (1 + m_Y|\mathbf{x}|)e^{-m_Y|\mathbf{x}|}]\hat{\mathbf{x}} \times \mathbf{J}.$$

where $J = 2MR^2\Omega_0/5$ is the angular momentum of the ball

The modification with respect to GR has the same feature as the one generated by the pointlike source

From the definition of m_R and m_Y, the presence of a Ricci scalar function [$f_{RR'}(0) \neq 0$] appears only in m_R

Considering f(R)-gravity ($m_Y \to \infty$), the above solution is unaffected by the modification in the Hilbert-Einstein action.

The body motion in the weak gravitational field

The body motion in the weak gravitational field

Let us consider the geodesic equations

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0$$

Where $ds=\sqrt{g_{\alpha\beta}dx^\alpha dx^\beta}$

In terms of the potentials generated by the ball source with radius R, the components of the metric $g_{\mu\nu}$ ***read***

$$g_{tt}=1+2\Phi_{\rm ball}(\mathbf{x})=1-\frac{2GM}{|\mathbf{x}|}\left[1+g(\xi,\eta)F(m_R\tilde{k}_R\mathcal{R})e^{-m_R\tilde{k}_R|\mathbf{x}|}+[1/3-g(\xi,\eta)]F(m_R\tilde{k}_\phi\mathcal{R})e^{-m_R\tilde{k}_\phi|\mathbf{x}|}-\frac{4F(m_Y\mathcal{R})}{3}e^{-m_Y|\mathbf{x}|}\right]$$

$$g_{ti}=2A_i(\mathbf{x})=\frac{2G}{|\mathbf{x}|^2}[1-(1+m_Y|\mathbf{x}|)e^{-m_Y|\mathbf{x}|}]\hat{\mathbf{x}}\times\mathbf{J},$$

$$g_{ij}=-\delta_{ij}+2\Psi_{\rm ball}(\mathbf{x})\delta_{ij}=-\delta_{ij}-\frac{2GM}{|\mathbf{x}|}\left[1-g(\xi,\eta)F(m_R\tilde{k}_R\mathcal{R})e^{-m_R\tilde{k}_R|\mathbf{x}|}-[1/3-g(\xi,\eta)]F(m_R\tilde{k}_\phi\mathcal{R})e^{-m_R\tilde{k}_\phi|\mathbf{x}|}\right.$$
$$\left.-\frac{2F(m_Y\mathcal{R})}{3}e^{-m_Y|\mathbf{x}|}\right]\delta_{ij},$$

and the non-vanishing Christoffel symbols read

$$\Gamma^t_{ti}=\Gamma^i_{tt}=\partial_i\Phi_{\rm ball},\qquad \Gamma^i_{tj}=\frac{\partial_iA_j-\partial_jA_i}{2},\qquad \Gamma^i_{jk}=\delta_{jk}\partial_i\Psi_{\rm ball}-\delta_{ij}\partial_k\Psi_{\rm ball}-\delta_{ik}\partial_j\Psi_{\rm ball}.$$

Circular rotation curves in a spherically symmetric field

In the Newtonian limit, neglecting the rotating component of the source, leads to the equation of motion

$$\frac{d^2\mathbf{x}}{dt^2} = -\nabla\Phi_{\rm ball}(\mathbf{x})$$

Our aim is to evaluate the corrections to the classical motion in the easiest situation, namely the circular motion, in which case we do not consider radial and vertical motions.

The condition of stationary motion on the circular orbit reads

$$v_{\rm c}(r) = \sqrt{r\frac{\partial\Phi(r)}{\partial r}},$$

Circular rotation curves in a spherically symmetric field

Let us consider the phenomenological potential

$$\Phi_{SP}(r) = -\frac{GM}{r}\left[1 + \alpha e^{-m_S r}\right]$$

With α *and* m_S *free parameters. Sanders tried to fit galactic rotation curves of spiral galaxies in the absence of dark matter, within the modified Newtonian dynamics (MOND) proposal by Milgrom.*

The parameters selected by Sanders were $\alpha \simeq -0.92$ *and* $1/m_S \simeq 40$ *Kpc*

This potential can be used also for fitting elliptical galaxies (SC et al. ApJ (2012))

In both cases, assuming a negative value for α*, an almost constant profile for rotation curve is recovered* (SC and De Laurentis, Annalen Phys. 2012).

Circular rotation curves in a spherically symmetric field

Setting the gravitational constant equal to

$$G_0 = \frac{2\omega(\phi^{(0)})\phi^{(0)}-4}{2\omega(\phi^{(0)})\phi^{(0)}-3}\frac{G_\infty}{\phi^{(0)}}$$

where G_∞ is the gravitational constant as measured at infinity, and imposing

$$\alpha^{-1} = 3 - 2\omega(\phi^{(0)})\phi^{(0)}$$

the potential becomes

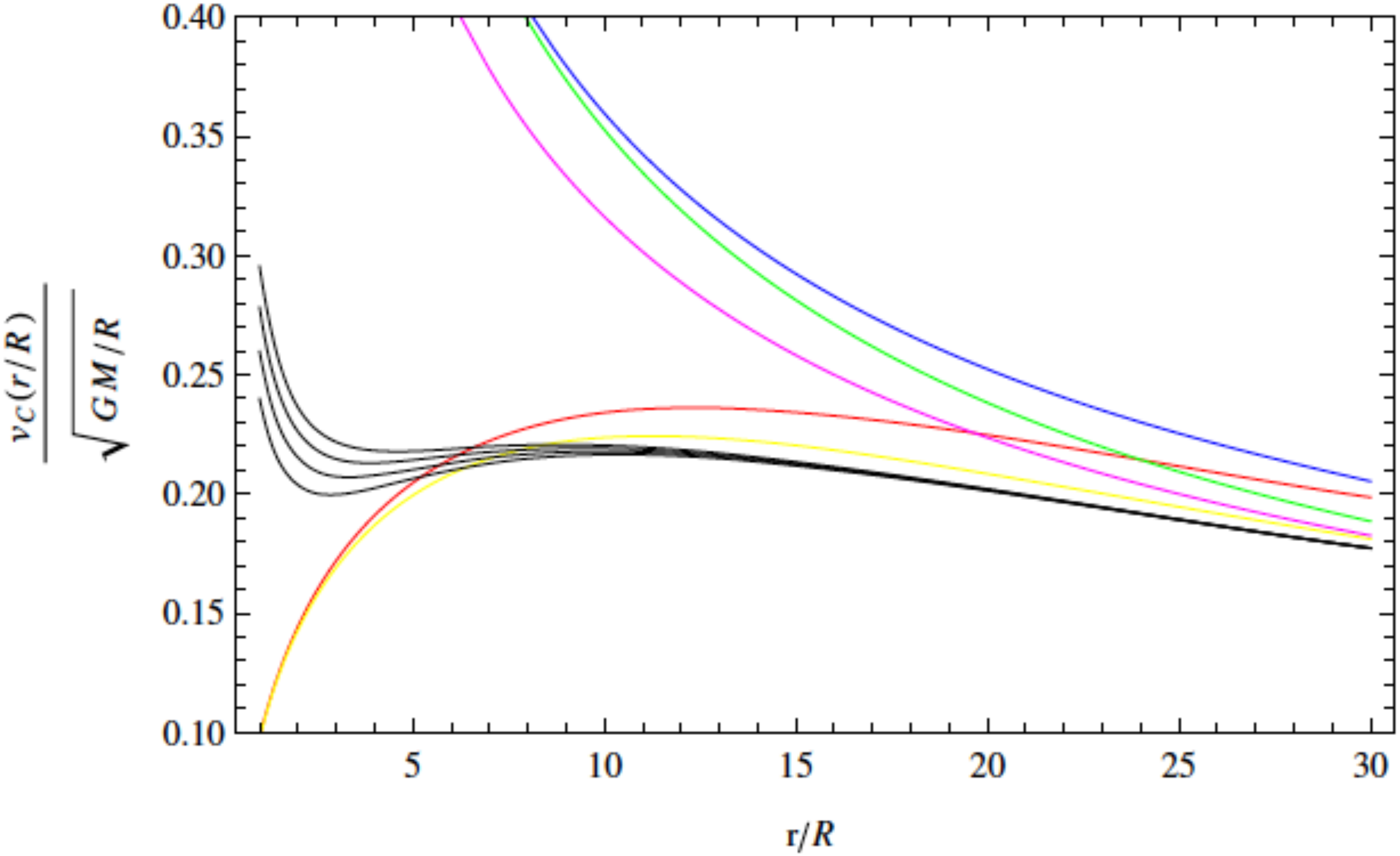
$$\Phi(r) = -\frac{G_\infty M}{r}\left\{1 + \alpha e^{-\sqrt{1-3\alpha}m_\phi r}\right\}$$

and then the Sanders potential can be recovered.

In Fig. below we show the radial behavior of the circular velocity induced by the presence of a ball source in the case of the Sanders potential and of potentials shown in next Table.

TABLE I. Table of fourth-order gravity models analyzed in the Newtonian limit for gravitational potentials generated by a pointlike source Eq. (17). The range of validity of cases C, D is $(\eta-1)^2-\xi>0$. We set $f_R(0,0,\phi^{(0)})=1$.

Case	Theory	Gravitational potential	Free parameters
A	$f(R)$	$-\frac{GM}{\lvert\mathbf{x}\rvert}[1+\frac{1}{3}e^{-m_R\lvert\mathbf{x}\rvert}]$	$m_R{}^2=-\frac{1}{3f_{RR}(0)}$
B	$f(R,R_{\alpha\beta}R^{\alpha\beta})$	$-\frac{GM}{\lvert\mathbf{x}\rvert}[1+\frac{1}{3}e^{-m_R\lvert\mathbf{x}\rvert}-\frac{4}{3}e^{-m_Y\lvert\mathbf{x}\rvert}]$	$m_R{}^2=-\frac{1}{3f_{RR}(0,0)+2f_Y(0,0)}$ $m_Y{}^2=\frac{1}{f_Y(0,0)}$
C	$f(R,\phi)+\omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$-\frac{GM}{\lvert\mathbf{x}\rvert}[1+g(\xi,\eta)e^{-m_R\tilde{k}_R\lvert\mathbf{x}\rvert}$ $+[1/3-g(\xi,\eta)]e^{-m_R\tilde{k}_\phi\lvert\mathbf{x}\rvert}]$	$m_R{}^2=-\frac{1}{3f_{RR}(0,\phi^{(0)})}$ $m_\phi{}^2=-\frac{f_{\phi\phi}(0,\phi^{(0)})}{2\omega(\phi^{(0)})}$ $\xi=\frac{3f_{R\phi}(0,\phi^{(0)})^2}{2\omega(\phi^{(0)})}$ $\eta=\frac{m_\phi}{m_R}$ $g(\xi,\eta)=\frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}$ $\tilde{k}^2_{R,\phi}=\frac{1-\xi+\eta^2\pm\sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}$
D	$f(R,R_{\alpha\beta}R^{\alpha\beta},\phi)+\omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$-\frac{GM}{\lvert\mathbf{x}\rvert}[1+g(\xi,\eta)e^{-m_R\tilde{k}_R\lvert\mathbf{x}\rvert}$ $+[1/3-g(\xi,\eta)]e^{-m_R\tilde{k}_\phi\lvert\mathbf{x}\rvert}-\frac{4}{3}e^{-m_Y\lvert\mathbf{x}\rvert}]$	$m_R{}^2=-\frac{1}{3f_{RR}(0,0,\phi^{(0)})+2f_Y(0,0,\phi^{(0)})}$ $m_Y{}^2=\frac{1}{f_Y(0,0,\phi^{(0)})}$ $m_\phi{}^2=-\frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2\omega(\phi^{(0)})}$ $\xi=\frac{3f_{R\phi}(0,0,\phi^{(0)})^2}{2\omega(\phi^{(0)})}$ $\eta=\frac{m_\phi}{m_R}$ $g(\xi,\eta)=\frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}$ $\tilde{k}^2_{R,\phi}=\frac{1-\xi+\eta^2\pm\sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}$



The circular velocity of a ball source of mass M and radius R, with the potentials of Table I. We indicate case A by a green line, case B by a yellow line, case D by a red line, case C by a blue line, and the GR case by a magenta line. The black lines correspond to the Sanders model for $-0.95 < \alpha < -0.92$. The values of free parameters are $\omega(\varphi^{(0)})$.. . $-1/2$, $\Xi = -5$, $\eta = .3$, $m_Y = 1.5 * m_R$, $m_S = 1.5 * m_R$, $m_R = .1 * R^{-1}$.

Rotating sources and orbital parameters

Geodesic equations

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{tt} + 2\Gamma^i_{tj}\frac{dx^j}{ds} = 0,$$

in the coordinate system J = (0, 0, J) reads

$$\ddot{x} + \frac{GM}{r^3}x = -\frac{GM\Lambda(r)}{r^3}x + \frac{2GJ}{r^5}\left\{\zeta(r)\left[\left(x^2+y^2-2z^2\right)\dot{y} + 3yz\dot{z}\right] + 2\Sigma(r)L_x z\right\}$$

$$\ddot{y} + \frac{GM}{r^3}y = -\frac{GM\Lambda(r)}{r^3}y - \frac{2GJ}{r^5}\left\{\zeta(r)\left[\left(x^2+y^2-2z^2\right)\dot{x} + 3xz\dot{z}\right] - 2\Sigma(r)L_y z\right\},$$

$$\ddot{z} + \frac{GM}{r^3}z = -\frac{GM\Lambda(r)}{r^3}z + \frac{6GJ}{r^5}\left\{\zeta(r) + \frac{2}{3}\Sigma(r)\right\}L_z z,$$

where

$$\begin{aligned}\Lambda(r) \doteq\ & g(\xi,\eta)F(m_R\tilde{k}_R\mathcal{R})(1+m_R\tilde{k}_R r)e^{-m_R\tilde{k}_R r} \\ & + [1/3 - g(\xi,\eta)]F(m_R\tilde{k}_\phi\mathcal{R})(1+m_R\tilde{k}_\phi r)e^{-m_R\tilde{k}_\phi r} \\ & - \frac{4F(m_Y\mathcal{R})}{3}(1+m_Y r)e^{-m_Y r},\end{aligned}$$

$$\zeta(r) \doteq 1 - [1 + m_Y r + (m_Y r)^2]e^{-m_Y r},$$

$$\Sigma(r) \doteq (m_Y r)^2 e^{-m_Y r},$$

with L_x,L_y and L_z the components of the angular momentum

Rotating sources and orbital parameters

The first terms in the right-hand side of the above equation, depending on the three parameters m_R, m_Y and m_φ, represent the Extended Gravity contribution to the Newtonian acceleration.

The second terms in these equations, depending on the angular momentum J and the EG parameters m_R, m_Y and m_φ, correspond to DRAGGING CONTRIBUTIONS

The case $m_R \to \infty$, $m_Y \to \infty$ and $m_\varphi \to 0$ leads to $\Lambda(r) \to 0$, $\xi(r) \to 1$ and $\Sigma(r) \to 0$, and hence one recovers the familiar results of GR

These additional gravitational terms can be considered as perturbations of Newtonian gravity, and their effects on planetary motions can be calculated within the usual perturbation schemes assuming the Gauss equations

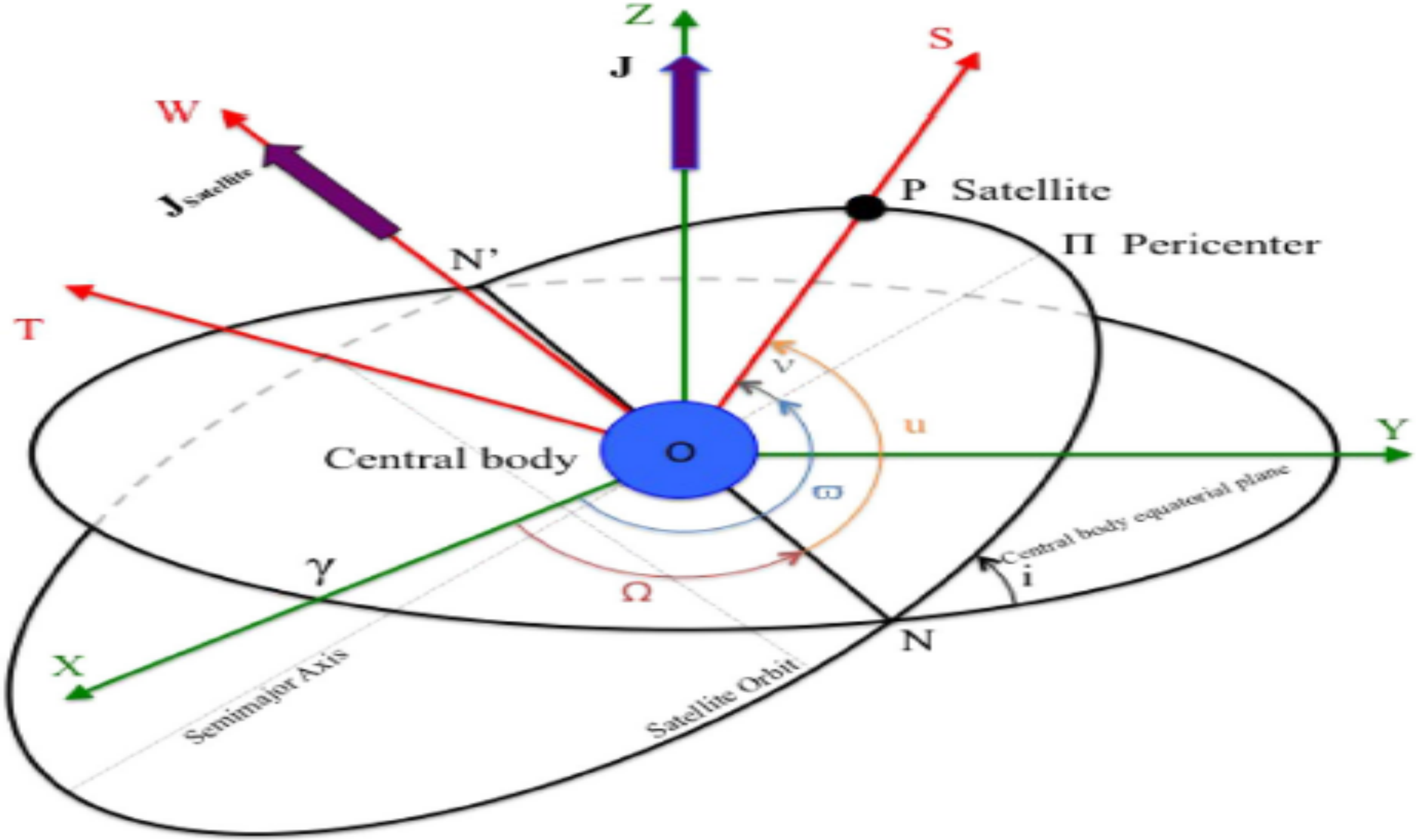
Rotating sources and orbital parameters

Let us consider the right-hand side of the above equations as the components (A_x , A_y , A_z) of the perturbing acceleration in the system (X, Y, Z) (see next Fig.), with X the axis passing through the vernal equinox γ, Y the transversal axis, and Z the orthogonal axis parallel to the angular momentum J of the central body

In the system (S,T,W), the three components can be expressed as (A_s , A_t , A_w), with S the radial axis, T the transversal axis, and W the orthogonal one

We will adopt the standard notation:

- *a is the semimajor axis;*
- *e is the eccentricity*
- *$p=a(1-e^2)$ is the semilatus rectum;*
- *i is the inclination;*
- *Ω is the longitude of the ascending node N;*
- *$\omega\sim$ is the longitude of the pericenter Π;*
- *M^0 is the longitude of the satellite at time t = 0;*
- *ν is the true anomaly;*
- *u is the argument of the latitude given by $u = \nu + \omega\sim - \Omega$;*
- *n is the mean daily motion equal to $n=(GM/a^3)^{1/2}$;*
- *and C is twice the velocity, namely* $C = r^2 \dot{\nu} a^2 (1-e^2)^{1/2}$



i = ∢ YN Π is the inclination; Ω = ∢ XON is the longitude of the ascending node N; ω~ = broken ∢ XO Π is the longitude of the pericenter Π; ν = ∢ ΠOP is the true anomaly; u = ∢ ΩOP = ν + ω~ − Ω is the argument of the latitude; J is the angular momentum of rotation of the central body; and $J_{Satellite}$ is the angular momentum of revolution of a satellite around the central body.

Rotating sources and orbital parameters

The transformation rules between the coordinates frames (X, Y, Z) and (S, T,W) are

$$x = r(\cos u \cos\Omega - \sin u \sin\Omega \cos i),$$
$$y = r(\cos u \sin\Omega + \sin u \cos\Omega \cos i),$$
$$z = r \sin u \sin i$$
$$r = \frac{p}{1 + e\cos\nu},$$

and the components of the angular momentum obey the equations

$$L_x = y\dot{z} - z\dot{y} = C \sin i \sin\Omega,$$
$$L_y = z\dot{x} - x\dot{z} = -C \cos\Omega \sin i,$$
$$L_z = x\dot{y} - y\dot{x} = C \cos i.$$

The components of the perturbing acceleration in the (S, T,W) system read

$$A_s = -\frac{GM\Lambda(r)}{r^2} + \frac{2GJC\cos i}{r^4}\zeta(r),$$
$$A_t = -\frac{2GJCe\cos i \sin\nu}{pr^3}\zeta(r),$$
$$A_w = \frac{2GJC\sin i}{r^4}\left[\left(\frac{re\sin\nu\cos u}{p} + 2\sin u\right)\zeta(r) + 2\sin u\Sigma(r)\right].$$

The A_s component has two contributions: one from the modified Newtonian potential $\Phi_{ball}(x)$, another from the gravito-magnetic field
A_i is a higher order term.

The components A_t and A_w depend only on the gravito-magnetic field

Rotating sources and orbital parameters

The Gauss equations for the variations of the six orbital parameters, resulting from the perturbing acceleration with components A_x , A_y ,A_z are

$$\frac{da}{dt} = \dot{a}_{\rm EG} = \frac{2eGM\Lambda(r)\sin\nu}{n\sqrt{1-e^2}C}\dot{\nu},$$

$$\frac{de}{dt} = \dot{e}_{\rm GR} + \dot{e}_{\rm EG} = \frac{\sqrt{1-e^2}GM\Lambda(r)\sin\nu}{naC}\dot{\nu} + \dot{e}_{\rm GR}[1 - e^{-m_Y r}(1 + m_Y r + (m_Y r)^2)],$$

$$\frac{d\Omega}{dt} = \dot{\Omega}_{\rm GR} + \dot{\Omega}_{\rm EG} = \dot{\Omega}_{\rm GR}\{1 - e^{-m_Y r}[1 + m_Y r + (1 + f(\nu,u,e))(m_Y r)^2]\},$$

$$\frac{di}{dt} = \dot{i}_{\rm GR} + \dot{i}_{\rm EG} = \dot{i}_{\rm GR}\{1 - e^{-m_Y r}[1 + m_Y r + (1 + f(\nu,u,e))(m_Y r)^2]\},$$

$$\frac{d\tilde{\omega}}{dt} = \dot{\tilde{\omega}}_{\rm GR} + \dot{\tilde{\omega}}_{\rm EG} = -\frac{\sqrt{1-e^2}GM\Lambda(r)\cos\nu}{naeC}\dot{\nu} + \dot{\tilde{\omega}}_{\rm GR}[1 - e^{-m_Y r}(1 + m_Y r + (m_Y r)^2)] - 2\sin^2\frac{i}{2}\dot{\Omega}_{\rm GR} f(\nu,u,e)\Sigma(r),$$

$$\frac{dM^0}{dt} = \dot{M^0}_{\rm GR} + \dot{M^0}_{\rm EG} = -\frac{GM\Lambda(r)}{naC}\left[\frac{2r}{a} + \frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\cos\nu\right]\dot{\nu} + \dot{M^0}_{\rm GR}[1 - e^{-m_Y r}(1 + m_Y r + (m_Y r)^2)]$$
$$- 2\sin^2\frac{i}{2}\dot{\Omega}_{\rm GR} f(\nu,u,e)\Sigma(r),$$

where

$$\dot{e}_{\rm GR} = \frac{2GJ\cos i\sin\nu}{aC}\dot{\nu},$$

$$\dot{\Omega}_{\rm GR} = \frac{2GJ\sin u}{pC}[e\sin\nu\cos u + 2(1+e\cos\nu)\sin u]\dot{\nu},$$

$$\dot{i}_{\rm GR} = \frac{2GJ\cos u\sin i}{Cp}[e\sin\nu\cos u + 2(1+e\cos\nu)\sin u]\dot{\nu},$$

$$\dot{\tilde{\omega}}_{\rm GR} = -\frac{2GJ\cos i}{aC}\left(2 + \frac{1+e^2}{e}\cos\nu\right)\dot{\nu} + 2\sin^2\frac{i}{2}\dot{\Omega}_{\rm GR},$$

$$\dot{\mathcal{M}^0}_{\rm GR} = -\frac{4GJ\cos i}{na^2p}(1+e\cos\nu)\dot{\nu} + \frac{e^2}{1+\sqrt{1-e^2}}\dot{\tilde{\omega}}_{\rm GR}$$
$$+ 2\sqrt{1-e^2}\sin^2\frac{i}{2}\dot{\Omega}_{\rm GR},$$

$$f(\nu,u,e) = \frac{1+e\cos\nu}{1+e(\frac{\sin\nu\cot u}{2}+\cos\nu)}. \quad 0)$$

Corresponding equations of the six orbital parameters for ETGs, with the dynamics of a; e; ω~ ; L^0 depending on terms related to the modifications of Newtonian potential. Dynamics of Ω and i depend on the dragging terms.

Rotating sources and orbital parameters

Considering an almost circular orbit ($e \ll 1$), we integrate the Gauss equations with respect to the only anomaly ν, from 0 to $\nu(t) = nt$, since all other parameters have a slower evolution than ν, hence they can be considered as constraints with respect to ν At first order we get

$$\Delta a(t) = 0,$$

$$\Delta e(t) = 0,$$

$$\Delta i(t) = \frac{GJe^2 \sin i}{na^3} e^{-m_Y p} (m_Y p)^2 \left[1 + \frac{(m_Y p)^2}{2} (m_Y p - 4)\right.$$
$$\times \sin(\tilde{\omega}(t) - \Omega(t)) \nu(t) + \mathcal{O}(e^4),$$

$$\Delta \Omega(t) = \frac{2GJ}{na^3} [1 - e^{-m_Y p} (1 + m_Y p + 2(m_Y p)^2)] \nu(t)$$
$$+ \mathcal{O}(e^2),$$

$$\Delta \tilde{\omega}(t) = \left\{ \frac{\tilde{\Lambda}(p)}{2} - \frac{2GJ}{na^3} [3 \cos i - 1 \right.$$
$$+ e^{-m_Y p} (1 + m_Y p + \frac{3}{2} (m_Y p)^2$$
$$- (3 + 3 m_Y p + 3 (m_Y p)^2$$
$$\left. + \frac{1}{12} (m_Y p)^3) \cos i)] \right\} \nu(t) + \mathcal{O}(e^2),$$

$$\Delta \mathcal{M}^0(t) = \left\{ 2\Lambda(p) - \frac{2GJ}{na^3} [3 \cos i - 1 \right.$$
$$\left. - e^{-m_Y p} (1 + m_Y p + 2(m_Y p)^2)(\cos i - 1)] \right\} \nu(t)$$
$$+ \mathcal{O}(e^2),$$

where

$$\tilde{\Lambda}(p) \doteq g(\xi, \eta) F(m_R \tilde{k}_R \mathcal{R}) (m_R \tilde{k}_R p)^2 e^{-m_R \tilde{k}_R p}$$
$$+ [1/3 - g(\xi, \eta)] F(m_R \tilde{k}_\phi \mathcal{R}) (m_R \tilde{k}_\phi p)^2 e^{-m_R \tilde{k}_\phi p}$$
$$- \frac{4F(m_Y \mathcal{R})}{3} (m_Y p)^2 e^{-m_Y p}.$$

We hence notice that the contributions to the semimajor axis a and eccentricity e vanish, as in GR, while there are nonzero contributions to i, Ω, ω~ and M^0. In particular, the contributions to the inclination i and the longitude of the ascending node Ω depend only on the drag effects of the rotating central body, while the contributions to the pericenter longitude ω~ and mean longitude at M^0 depend also on the modified Newtonian potential

Rotating sources and orbital parameters

In the considered ETG models, the inclination i has a nonzero contribution, in contrast to the results in GR, and also $\Delta\omega(t) \neq \Delta M^0(t)$, given by

$$\Delta\tilde{\omega}(t) - \Delta\mathcal{M}^0(t) \simeq \left\{ \frac{\tilde{\Lambda}(p) - 4\Lambda(p)}{2} + \frac{2GJ}{na^3} e^{-m_Y p} \left[\frac{(m_Y p)^2}{2} + \left(2 + 2m_Y p + (m_Y p)^2 + \frac{(m_Y p)^3}{12} \right) \cos i \right] \right\} \nu(t) + \mathcal{O}(e^2).$$

In the limit $m_R \to \infty$; $m_Y \to \infty$ and $m_\varphi \to 0$, we obtain results of GR.

Experimental constrains

Experimental constrains

The orbiting gyroscope precession can be split into a part generated by the metric potentials, Φ and Ψ, and one generated by the vector potential A

The equation of motion for the gyrospin three-vector S is

$$\frac{d\mathbf{S}}{dt} = \left.\frac{d\mathbf{S}}{dt}\right|_{\mathrm{G}} + \left.\frac{d\mathbf{S}}{dt}\right|_{\mathrm{LT}}$$

where the geodesic and Lense-Thirring precessions are

$$\left.\frac{d\mathbf{S}}{dt}\right|_{\mathrm{G}} = \Omega_{\mathrm{G}} \times \mathbf{S} \quad \text{with} \quad \Omega_{\mathrm{G}} = \frac{\nabla(\Phi + 2\Psi)}{2} \times \mathbf{v}$$

$$\left.\frac{d\mathbf{S}}{dt}\right|_{\mathrm{LT}} = \Omega_{\mathrm{LT}} \times \mathbf{S} \quad \text{with} \quad \Omega_{\mathrm{LT}} = \frac{\nabla \times \mathbf{A}}{2}$$

The geodesic precession, Ω_G can be written as the sum of two terms, one obtained with GR and the other being the extended gravity contribution

Then we have

$$\Omega_{\mathrm{G}} = \Omega_{\mathrm{G}}^{(\mathrm{GR})} + \Omega_{\mathrm{G}}^{(\mathrm{EG})}$$

where

$$\Omega_{\mathrm{G}}^{(\mathrm{GR})} = \frac{3GM}{2|\mathbf{x}|^3}\mathbf{x} \times \mathbf{v},$$

$$\Omega_{\mathrm{G}}^{(\mathrm{EG})} = -\Big[g(\xi,\eta)(m_R\tilde{k}_R r + 1)F(m_R\tilde{k}_R\mathcal{R})e^{-m_R\tilde{k}_R r} + \frac{8}{3}(m_Y r + 1)F(m_Y\mathcal{R})e^{-m_Y r} + \left[\frac{1}{3} - g(\xi,\eta)\right](m_R\tilde{k}_\phi r + 1)F(m_R\tilde{k}_\phi\mathcal{R})e^{-m_R\tilde{k}_\phi r}\Big]\frac{\Omega_{\mathrm{G}}^{(\mathrm{GR})}}{3},$$

Where $|x| = r$

Experimental constrains

Similarly one has $\Omega_{\mathrm{LT}} = \Omega_{\mathrm{LT}}^{(\mathrm{GR})} + \Omega_{\mathrm{LT}}^{(\mathrm{EG})}$

with $\Omega_{\mathrm{LT}}^{(\mathrm{GR})} = \frac{G}{2r^3}\mathbf{J}$ *and* $\Omega_{\mathrm{LT}}^{(\mathrm{EG})} = -e^{-m_Y r}(1 + m_Y r + m_Y{}^2 r^2)\Omega_{\mathrm{LT}}^{(\mathrm{GR})}$

where we have assumed that, on the average, $<(J \cdot x).x_i>$.

The Gravity Probe B (GPB) satellite contains a set of four gyroscopes and has tested two predictions of GR: the geodetic effect and frame-dragging (Lense-Thirring effect)

The changes in the direction of spin gyroscopes, contained in the satellite orbiting at h = 650 km of altitude and crossing directly over the poles, have been measured with extreme precision

The geodesic precession and the Lense-Thirring precession, measured by the Gravity Probe B satellite and those predicted by GR, are

Effect	Measured (mas/y)	Predicted (mas/y)
Geodesic precession	6602 ± 18	6606
Lense-Thirring precession	37.2 ± 7.2	39.2

Experimental constrains

Imposing $|\Omega^{(EG)}{}_{G}| \lesssim \delta\Omega_G$ *and* $|\Omega^{(EG)}{}_{LT}| \lesssim \delta\Omega_{LT}$, *with* $r^* = R_\oplus + h$ *where* $R_\oplus$ *is the radius of the Earth and h = 650 km is the altitude of the satellite, we get*

$$g(\xi,\eta)(m_R\tilde{k}_R r^* + 1)F(m_R\tilde{k}_R R_\oplus)e^{-m_R\tilde{k}_R r^*} + [1/3 - g(\xi,\eta)](m_R\tilde{k}_\phi r^* + 1)F(m_R\tilde{k}_\phi R_\oplus)e^{-m_R\tilde{k}_\phi r^*}$$

$$+\frac{8}{3}(m_Y r^* + 1)F(m_Y R_\oplus)e^{-m_Y r^*} \lesssim \frac{3\delta|\Omega_{\mathrm{G}}|}{|\Omega_{\mathrm{G}}^{(\mathrm{GR})}|} \simeq 0.008,$$

$$(1 + m_Y r^* + m_Y{}^2 r^{*2})e^{-m_Y r^*} \lesssim \frac{\delta|\Omega_{\mathrm{LT}}|}{|\Omega_{\mathrm{LT}}^{(\mathrm{GR})}|} \simeq 0.19,$$

From the experiments, we have $|\Omega^{(GR)}{}_{G}|$ *= 6606 mas and* $\delta|\Omega_G|$*=18 mas,* $|\Omega^{(GR)}{}_{LT}|$ *= 37.2 mas and* $\delta|\Omega_{LT}|$ *= 7.2 mas*

We obtain that $m_Y \geq 7.3 \times 10^{-7} m^{-1}$

Experimental constrains

The Laser Relativity Satellite (LARES) mission of the Italian Space Agency is designed to test the frame dragging and the Lense-Thirring effect, to within 1% of the value predicted in the framework of GR

The body of this satellite has a diameter of about 36.4 cm and weights about 400 kg

It was inserted in an orbit with 1450 km of perigee, an inclination of 69.5 ± 1 degrees and eccentricity 9.54×10^{-4}

It allows to obtain a stronger constraint for m_Y:

$$(1 + m_Y r^* + {m_Y}^2 r^{*2}) e^{-m_Y r^*} \lesssim \frac{\delta |\Omega_{\rm LT}|}{|\Omega_{\rm LT}^{\rm (GR)}|} \simeq 0.01$$

From which we obtain $m_Y \geq 1.2 \times 10^{-6} m^{-1}$

Experimental constrains

In the specific case of the Non-Commutative Spectral Geometry, the above quantities become for $m_R \to \infty$,

$$m_Y = \sqrt{\frac{5\pi^2(k_0^2\mathbf{H}^{(0)}-6)}{36 f_0 k_0^2}}$$

and $m_\phi = 0$ *implying that* $\xi = \frac{a f_0 (\mathbf{H}^{(0)})^2}{12\pi^2}$

$$\eta = 0, \qquad g(\xi,\eta) = \frac{a f_0(\mathbf{H}^{(0)})^2 + 12\pi^2}{6|a f_0(\mathbf{H}^{(0)})^2 - 12\pi^2|} + \frac{1}{6} \qquad \text{and} \qquad \tilde{k}^2_{R,\phi} = 1 - \frac{a f_0(\mathbf{H}^{(0)})^2}{12\pi^2}$$

The first relation $\frac{8}{3}(m_Y r^* + 1) F(m_Y R_\oplus) e^{-m_Y r^*} \lesssim 0.008;$

hence the constraint on m_Y *imposed from GPB is* $m_Y > 7.1 \times 10^{-5}\ \mathrm{m}^{-1}$

whereas the LARES experiment implies $m_Y > 1.2 \times 10^{-6}\ \mathrm{m}^{-1}$

A bound similar to the one obtained earlier by using binary pulsars, or the GPB data.

A more stringent constraint is obtained using torsion balance experiments

Results from laboratory experiments designed to test the fifth force gives the constraint $m_Y > 10^4\ m^{-1}$

Experimental constrains

In conclusion, using data from the Gravity Probe B and LARES missions, we obtain constraints on m_Y.

Using the stronger constraint for m_Y, namely $m_Y > 10^{-4}\ m^{-1}$, we observe that the modifications to the orbital parameters induced by Non-Commutative Spectral Geometry are indeed small, confirming the consistency between the predictions of NCSG, as a gravitational theory beyond GR, and Gravity Probe B and LARES measurements *Phys.Rev. D91 (2015) 044012*

This results show that space-based experiments can be used to test extensively parameters of fundamental theories

Conclusions

Conclusions

- ***In the context of ETGs, we have studied the linearized field equations in the limit of weak gravitational fields and small velocities generated by rotating gravitational sources, aimed to constrain the free parameters, which can be seen as effective masses (or lengths).***

- ***The precession of spin of a gyroscope orbiting around a rotating gravitational source can be studied.***

- ***Gravitational field gives rise, according to GR predictions, to geodesic and Lense-Thirring processions, the latter being strictly related to the off-diagonal terms of the metric tensor generated by the rotation of the source***

- ***The gravitational field generated by the Earth can be tested by Gravity Probe B and LARES satellites. These experiments tested the geodesic and Lense-Thirring spin precessions with high precision.***

- ***The corrections on the precession induced by scalar, tensor and curvature corrections can be measured and confronted with data.***

Conclusions

- *Considering an almost circular orbit, the Gauss equations can be integrated. The variation of the parameters at first order with respect to the eccentricity can be obtained.*

- *It is possible to show that the induced EG effects depend on the effective masses m_R, m_Y and m_φ, while the non validity of the Gauss theorem implies that these effects also depend on the geometric form and size of the rotating source.*

- *Requiring that the corrections be within the experimental errors, we then imposed constraints on the free parameters of the considered EG model. Merging the experimental results of Gravity Probe B and LARES, our results can be summarized as follows:*

$$
\begin{aligned}
& g(\xi,\eta)(m_R\tilde{k}_R r^* + 1)F(m_R\tilde{k}_R R_\oplus)e^{-m_R\tilde{k}_R r^*} \\
& \quad + [1/3 - g(\xi,\eta)](m_R\tilde{k}_\phi r^* + 1)F(m_R\tilde{k}_\phi R_\oplus)e^{-m_R\tilde{k}_\phi r^*} \\
& \quad + \frac{8}{3}(m_Y r^* + 1)F(m_Y R_\oplus)e^{-m_Y r^*} \lesssim 0.008,
\end{aligned}
$$

and $m_Y \geq 1.2 \times 10^{-6} m^{-1}$

Conclusions

- *The field equation for the potential A_i, is time independent provided the potential Φ is time independent.*
- *This aspect guarantees that the solution does not depend on the masses m_R and m_φ and, in the case of $f(R, \varphi)$. gravity, the solutions the same as in GR*
- *In the case of spherical symmetry, the hypothesis of a radially static source is no longer considered, and the obtained solutions depend on the choice of $f(R, \varphi)$ ETG model, since the geometric factor $F(x)$ is time dependent.*
- *Hence in this case, gravitomagnetic corrections to GR emerge with time-dependent sources*
- *The case of Non-commutative Spectral Geometry deserves some remarks:*
- *This model descends from a fundamental theory and can be considered as a particular case of ETGs;*
- ***Its parameters can be probed in the weak-field limit and at local scales, opening new perspectives for fundamental physics and astronomy by satellites.***