Mathematical and Physical Foundations of Extended Gravity (I)

-Conceptual Aspects-

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Summary

- Foundation: gravity and space-time
- Shortcomings in General Relativity
- Alternatives, way out and extensions
- The key role of conformal transformations
- Metric or connections?
- The role of Equivalence Principle
- Testing EP at classical and quantum level
- Conclusions

Foundation: gravity and space-time

Einstein works hard at finding a new theory of Gravitation based on the following requirements:

- principle of equivalence → Gravity and Inertia are indistinguishable; there exist observers in free fall (inertial motion)
- principle of relativity → SR holds pointwise; the structure of the spacetime is pointwise Minkowskian
- principle of general covariance → “democracy” in Physics
- principle of causality → all physical phenomena propagate respecting the light cones
- Riemann’s teachings about the link between matter and curvature →

Foundation: gravity and space-time

Mathematical consequences:

- principle of equivalence → Inertial motion = geodesic motion
- principle of relativity → the spacetime M is endowed with a Lorentzian metric g
- principle of general covariance → tensoriality
- principle of causality → light cones structure generated by the metric g
- Riemann's teachings about the link between matter and curvature → the gravitational field is described by g → 10 equations
  Riem(g) has 20 (independent) components: too many!
  Ric(g) has 10 (independent) components: OK!

Foundation: gravity and space-time

Einstein releases GR, finally written in a fully coherent form, both on a physical (conservation of matter) and on a mathematical ground (Lagrangian formulation).

Gravity is identified by the (dynamical) metric structure *g* of a curved space-time *M*.

The simplest variational principle is assumed to be:

$$\mathcal{L}_H = \sqrt{g}\, R\, ds$$

Foundation: gravity and space-time

The distribution of matter influences Gravity through 10 second order equations, nowadays called Einstein equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

A linear concomitant of the Riemann tensor, nowadays called the Einstein tensor, equals the stress-energy tensor that reflects the properties of matter.

They have a structure that suitably reduces to Newtonian equations in the “weak field limit.”

So, is g the gravitational field?

Einstein knows that it is not, since g is a tensor, while the principle of equivalence holds true! Free fall is described by the geodesics of (*M*, *g*):

$$\ddot{x}^\lambda + \left\{ {\lambda \atop \mu\nu} \right\}_g \dot{x}^\mu \, \dot{x}^\nu = 0$$

This is the right object to represent the gravitational field: g is just the potential of the gravitational field... but being $\left\{ {\lambda \atop \mu\nu} \right\}_g$ constructed since g, the metric remains the fundamental variable: g gives rise to the gravitational field, to causality, to the principle of equivalence, to rods & clocks.

Foundation: gravity and space-time

Working on the theory of “parallelism” in manifolds, Tullio Levi-Civita understands that it is not a metric property of space, but rather a property of “affine” type, having to do with “congruences of privileged lines.”

Generalizing the case of Christoffel symbols $\left\{ {\lambda \atop \mu\nu} \right\}_g$

Levi-Civita introduces the notion of linear connection as the more general object $\Gamma^{\lambda}{}_{\mu\nu}$

such that the equation of geodesics

$$\ddot{x}^{\lambda} + \Gamma^{\lambda}{}_{\mu\nu}\, \dot{x}^{\mu}\, \dot{x}^{\nu} = 0$$

is generally covariant.

A connection in a 4D space has 64 components. Only 40 if it is symmetric.

Any linear connection defines a (different) covariant derivative.

Foundation: gravity and space-time

Specific and particular is the case of metric connections, i.e. linear symmetric connections that are directly generated by a metric structure in space (as in Riemannian manifolds).

When a connection Γ is “metric,” its components are specific functions of a metric g and its first derivatives (so called Christoffel symbols).

In this case we say that Γ is the Levi-Civita connection of g and we write $\Gamma=\Gamma_{LC}(g)$.

Foundation: gravity and space-time

If Γ is the Levi-Civita connection of g, then the covariant derivative of g vanishes:

$$\nabla_\nu \left(g^{\alpha\beta} \sqrt{|g|} \right) = 0$$

If has no torsion (i.e. it is symmetric) this is a characteristic property of $\Gamma_{L-C}(g)$:

$$\nabla_\nu \left(g^{\alpha\beta} \sqrt{|g|} \right) = 0 \quad \Rightarrow \quad \Gamma^\alpha_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\}_g$$

Its 40 components are function of 10 fields.

Foundation: gravity and space-time

Hermann Weyl makes a celebrated attempt to unify Gravity with Electromagnetism. He understands that Electromagnetism is a gauge field.

He introduces a scalar factor ϕ (a "gauge") that point by point calibrates the interaction.

The metric g (Gravitation) and the scalar factor ϕ (the "phase") determine in fact a linear connection in spacetime.

Weyl's idea fails. The Lagrangian is not appropriate and field equations describe a "massive photon" (it is in fact a Proca-Yukawa interaction in modern language).

Weyl's idea generates however a keypoint: connections may have an interesting dynamics. Fields may be gauge fields - i.e. fields with group properties coming from further principles and "internal symmetries".

Foundation: gravity and space-time

Paul Henri Cartan works on the theory of "parallelism" due to Levi-Civita and understands that the group of linear transformations might be changed to other groups

De facto he introduces a transport of "frames" that undergo gauge transformations.

He has in mind properties of matter (and in fact this has to do both with microstructures in continua and with the symmetries of elementary particles and fields).

Foundation: gravity and space-time

Einstein is not so happy with the fact that the gravitational field is not the fundamental object, but just a by-product of the metric. Using a method invented few years before by Attilio Palatini, he realizes that one can obtain field equations by working on a theory that depends on *two* variables, varied independently:

a metric g and a linear connection Γ assumed to be symmetric.

$$R \equiv R(g,\Gamma) = g^{\mu\nu} R_{\mu\nu}(\Gamma) \qquad \mathcal{L}_{PE} = g^{\mu\nu} R_{\mu\nu}(\Gamma, \partial\Gamma)$$

There are 10 + 40 independent variables and the equations are:

$$R_{(\mu\nu)} - \frac{1}{2} R\, g_{\mu\nu} = \kappa T_{\mu\nu}$$

$$\nabla_\alpha^\Gamma \left(\sqrt{g} g^{\mu\nu}\right) = 0$$

Foundation: gravity and space-time

Field equations for the 40 components of Γ ensure that Γ is the Levi-Civita connection of g (Levi-Civita theorem):

$$R_{(\mu\nu)} - \frac{1}{2} R\, g_{\mu\nu} = \kappa T_{\mu\nu}$$

$$\nabla^{\Gamma}_{\alpha} \left(\sqrt{g} g^{\mu\nu} \right) = 0$$

$$\Rightarrow \quad \Gamma^{\alpha}_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\}_g$$

Field equations for the remaining 10 variables transform directly into Einstein equations.

In Palatini formalism, the metric g determines rods & clocks, while the connection Γ the free fall.

Foundation: gravity and space-time

Einstein gets convinced that Weyl's idea to use a linear connection in order to construct a unified theory (of Gravity and Electromagnetism) was good.

He introduces purely affine theories (PAT), investigated by him and others (Eddington and Schrödinger) until the Fifties, and then abandoned (because of substantialfailure) after the birth of gauge theories.

In a PAT the *only* independent variable is a linear connection Γ.

$$\mathcal{L}_{EE}(\Gamma, \partial\Gamma) = \sqrt{|det R_{(\mu\nu)}|}$$

A metric is introduced only by a Legendre transformation, as a momentum canonically conjugated to the connection:

$$g^{\mu\nu}\sqrt{|g|} = \frac{\partial \mathcal{L}_{EE}}{\partial R_{(\mu\nu)}}$$

Eddington finds a PAT that is fully equivalent to GR (BUT THIS IS THE ONLY ONE!):

$$\nabla^{\Gamma}_{\alpha}\left(\sqrt{g}g^{\mu\nu}\right) = 0 \quad \Rightarrow \quad \Gamma^{\alpha}_{\mu\nu} = \left\{{}^{\alpha}_{\mu\nu}\right\}_g$$

Shortcomings in General Relativity

Is still *g* the fundamental object of Gravity?

Einstein tries to consider directly the connection as the fundamental object of Gravity, but he never completes the process of "dethronizing" *g* (he died before!).

Shortcomings in General Relativity

But after all, what is the problem with GR?

GR is *simple*, beautiful... and (seems to be) wrong:

- cosmological constant Λ
- inflation
- anomalous acceleration
- the quantum gravity problem
- consistency of EP at classical and quantum level

Today observations say that there is way too few matter in the Universe! Thence the need, in order to save GR, for dark energy and dark matter:

$$G_{\mu\nu} = T_{\mu\nu} + T_{\mu\nu}^{dark}$$

Is there any way out to these shortcomings?

Alternatives, way out and extensions

Non-linear Theories of Gravitation (NLTG) are field theories in which the (gravitational part of the) Lagrangian is an arbitrary function of a metric together with its first and second order derivatives.

Covariance then imposes that the Lagrangian should be a scalar function of *g* and of the Riemann tensor of *g*.

Of course particular cases are those in which the Lagrangian depends only on the Ricci tensor or on the scalar curvature of the metric (see later).

$$\mathcal{L}_{NL} = f\left(g^{\mu\nu}\, R_{\mu\nu}\left(\Gamma, \partial\Gamma\right)\right)$$

NLTG complicate the mathematics of the problem

These genuine second order Lagrangians do in fact produce field equations that are fourth order in the metric, something that cannot be accepted if one believes that physical laws should be governed by second order equations.

A NLTG gives second order field equations if and only if the Lagrangian is degenerated; e.g. if one chooses a "topological Lagrangian"…

Alternatives, way out and extensions

A particular family of NLTG is f(R)-gravity in metric formalism, in which the Hilbert Lagrangian is replaced by any non-linear density depending on R. GR is retrieved in (and only in) the particular case f (R)=R.

In these theories there is a second order part that resembles Einstein tensor (and reduces to it if and only if f(R) = R) and a fourth order "curvature part" (that reduces to zero if and only if f (R) = R):

$$f'(R(g))\, R_{\mu\nu}(g) - \frac{1}{2} f(R(g)) g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R(g)) + g_{\mu\nu} \Box f'(R(g)) = \kappa T_{\mu\nu}$$

Higher order Gravity (4^{th})!

Pushing the 4th order part to the r.h.s. lets interpret it as an "extra gravitational stress" $T_{\mu\nu}^{\text{curv}}$, much in the spirit of Riemann.

In any case the fourth order character of these equations makes them very unsuitable under several aspects, so that they are eventually abandoned.

Alternatives, way out and extensions

From f(R) theories GR is retrieved in (and only in) the particular case f(R)=R.

$f'(R) = 1$

$f''(R) = 0$

Second Order Field Equations

degenerate theory

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 4\pi T_{\mu\nu}$$

Gravitational contribution

Matter contribution

Alternatives, way out and extensions

Discovery that for non-linear theories of Gravitation written under the Palatini form holds a universality property.

Taking the trace of the first equation

$$\begin{cases} f'(R)R_{(\mu\nu)}(\Gamma) - \frac{1}{2}f(R)g_{\mu\nu} = \kappa T_{\mu\nu} \\ \nabla_\alpha^\Gamma(\sqrt{g}f'(R)g^{\mu\nu}) = 0 \end{cases}$$

one gets (in 4D) the so-called master equation:

$$f'(R)R - 2f(R) = \kappa\tau$$

If the trace τ of the stress tensor T vanishes (e.g. if there is no matter at all….) then the master equation forces the scalar curvature R to take specific values that depend on the analiticity properties of *f*.

The only degenerate cases are the linear case $f(R) = R$ and the quadratic case $f(R) = R^2$.

Alternatives, way out and extensions

In the linear case the conformal factor is a constant, in geometrical units equal to 1.

In the quadratic case the master equation is empty and field equations are in fact conformally invariant.

The fundamental field is the linear connection Γ, that has a dynamics since it enters the Lagrangian together with its first derivatives.

The metric g is no longer a Lagrange multiplier, but still has no dynamics since it enters algebraically the Lagrangian.

The key role of conformal transformations

A conformal transformation is a transformation of a metric structure that does not change the angles but changes the scale.

In linear Euclidean Geometry this is just a dilatation.

In Riemannian Geometry it is a relation between two metrics that requires them to be proportional to each other by a conformal factor, assumed to be positive for obvious signature reasons.

$$\tilde{g} = \phi g \qquad \sqrt{\tilde{g}} = \phi^2 \sqrt{g}$$

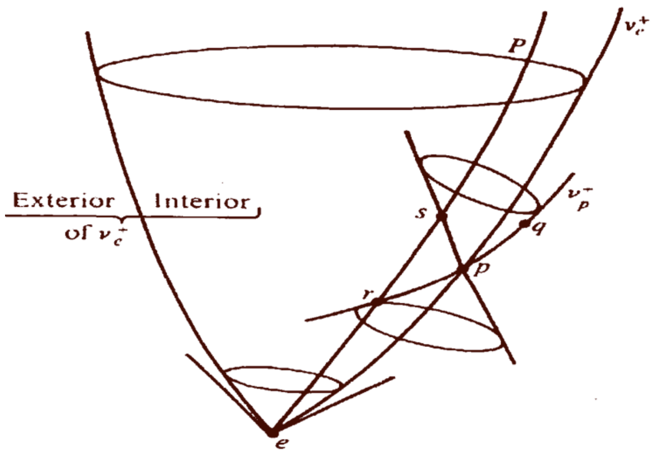
$$\tilde{R} = \phi^{-1} \left(R + (d-2) \triangle_g \phi + (d-1)(d-2) |\nabla_\phi|_g^2 \right)$$

The key role of conformal transformations

The principle of causality determines a conformal structure in the spacetime, i.e. it selects not a metric but a whole family of conformally related metrics.

In fact the distributions of light cones of two conformally related (Lorentzian) metrics are necessarily the same, so that photons of the two metrics are the same (a photon is a particle that travels in such a way that its velocity lies in the light cone at each point of the trajectory of the particle).

Null geodesics are left unchanged by conformal transformations



The key role of conformal transformations

What happens to the Hilbert action if the metric undergoes a conformal transformation?

Or, in other words, is GR conformally invariant?

The answer is: NO!. If g is replaced by a conformally related metric, the action is multiplied by a factor and an additional term appears.

Taking away a divergence (that does not affect field equations) it reduces to a conformal Hilbert action plus a second order action in the conformal factor ϕ.

In a sense the dynamics of g is driven by this factor, that obeys fourth order equations. The theory is not conformally invariant, since the leading equation that would impose conformal invariance has only particular solutions (e.g. $\phi = 0$).

$$\mathcal{A}(\tilde{R}) = \int \tilde{R}\sqrt{\tilde{g}} = \int \phi\left(R + 2\triangle_g \phi + 6|\nabla_\phi|_g^2\right)\sqrt{g}$$

$$\Downarrow$$

$$\nabla_g \phi + 3|\nabla_\phi|_g^2 = 0$$

Metric or connections?

Let's return to our Questions:

Was Einstein right in assuming the metric g of the space-time as the fundamental object to describe Gravity?

Metric or connections?

When Einstein formulated GR, the only geometrical field he could use was a (Lorentzian) metric *g*, the structure that Gauss (1830) and Riemann (1856) introduced in surfaces and higher-dimensional manifolds to define curvature.
At that time he had no other choice.

In GR $\Gamma^{\alpha}_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\}_g$ are not equations. They express a founding issue.

Assumption on space-time structure: there is a connection Γ ; this connection has no dynamics; it is in fact *a priori* the Levi-Civita connection of the metric *g*. Only *g* has dynamics. So the single object *g* determines at the same time the causal structure (light cones), the measurements (rods and clocks) and the free fall of test particles (geodesics). Spacetime is a couple (M,g).

Even if it was clear to Einstein that Gravity induces “freely falling observers” and that the principle of equivalence selects an object that cannot be a tensor - since it is capable of being “switched off” and set to zero at least in a point - he was obliged to choose it as being determined by the metric structure itself.

Metric or connections?

When in 1919 Levi-Civita introduced connections, Einstein had another choice. But he didn't really take it. Why?

In Palatini's formalism a (symmetric) connection Γ and a metric g are given and varied independently. Spacetime is a triple (M, g, Γ) where the metric determines rods and clocks (i.e., it sets the fundamental measurements of spacetime) while Γ determines the free fall

The second equation tells us *a posteriori* that Γ is the Levi-Civita connection of g. The first equation is then turned into the standard Einstein equations. This is why Einstein considered the metric as the fundamental object of Gravity

.. But this coincidence (between Γ and the Levi-Civita's connection of g) is entirely due to the particular Lagrangian considered by Einstein, which is the *simplest*… but not the only one possible!

Metric or connections?

Furthermore Einstein didn't recognize that Palatini method privileges the affine structure towards the metric structure

In fact the Einstein-Palatini Lagrangian contains only derivatives of Γ, that is the real dynamical field. The metric g has no dynamics since it enters the Lagrangian as a "Lagrange multiplier"

The metric g gains a dynamics from that of Γ

The dynamics of Γ tells us that a sort of Einstein's equation holds for the Ricci's tensor of Γ

This dynamics is obtained by varying the Lagrangian with respect to the metric.These are 10 equations. Other 40 equations come out when varying the Lagrangian with respect to the connection Γ. These additional equations govern the form of Γ and impose it to be the Levi-Civita connection of the metric. The first equation then transforms into Einstein Equations.

In Palatini's formalism $\Gamma^{\alpha}_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\}_g$ are now equations.

The fact that Γ is the Levi-Civita connection of g is no longer an assumption but becomes an outcome of field equations!

Metric or connections?

Among the different Theories of Gravity, we really should prefer the simplest (in the sense of the one with the simplest Lagrangian)?

Metric or connections?

In Geometry we are forced by Nature to consider variational principles different from the *simplest*. Why should be so strange that the same happens in Gravity?

Metric or connections?

The universality properties for non-linear theories of gravity, written in the Palatini form, tell us that the true dynamical field is Γ and not the metric g .

The metric g is no longer a Lagrange multiplier, but still has no dynamics since it enters algebraically the Lagrangian. However g gains dynamics from the dynamics of the connection Γ.

The connection is the Gravitational Field and, as such, it is the fundamental field in the Lagrangian. The metric g enters the Lagrangian with an “ancillary role.”

It reflects the fundamental need we have to define lenght and distances, as well as areas and volumes. It defines rods & clocks, that we use to make experiments. It defines the causal structure of spacetime.

But it has no dynamical role!

There is no reason to assume g to be the potential for Γ !

Nor that it has to be a true field just because it appears in the action !

Metric or connections?

Newtonian and Galilean Physics are based on the Geometry of Euclidean space.

Both the affine and the metric structure enter the game.

These two structures are separated, one is not obtained from the other.

The structure of Newtonian Physics is a subtle mix-up of both structures…… but the principle of inertia selects the family of parallel lines, that become "more fundamental" than the metric structure!

The Physics of Gravitation is based on the Geometry of a curved 4D spacetime. An affine and a metric structure enter the game.

These two structures are separated. One is not obtained from the other.
The structure of Gravitational Physics is a subtle mix-up of both structures.

… but the principle of equivalence selects the family of geodesics of Γ, that become "more fundamental" than the metric structure of g ! The principle of equivalence selects the true dynamical field; rods & clocks follow up.

Metric or connections?

There is another way to half the order of the equations: choosing a connection Γ as the fundamental variable, as Einstein did in 1923 in the case f (R) = R

But what happens in the generic case f(R)?

Universality properties state that in vacuum (i.e. if the stress tensor vanishes) field equations for the 10 + 40 variables still reduce to Einstein equations.

This time, however, the metric g that satisfies Einstein equations is no longer the metric given a priori in the Lagrangian.

In fact it is a conformally related metric. The conformal factor is the derivative f '(R): h = f '(R) g.

$$f'(R)R_{(\mu\nu)}(\Gamma) - \frac{1}{2}f(R)g_{\mu\nu} = \kappa T_{\mu\nu}$$

$$\nabla^{\Gamma}_{\alpha}\left(\sqrt{g}f'(R)G^{\mu\nu}\right) = 0$$

$$h_{\mu\nu} = f'(R)g_{\mu\nu} \Longleftrightarrow \Gamma = \Gamma_{LC}(h)$$

Metric or connections?

And now we propose a new interpretation.

The new metric (that preserves the causal structure of spacetime) introduces new rods & clocks, by means of which

redefine the correct $T_{\mu\nu}$

This way both $T_{\mu\nu}^{\text{dark}}$ and $T_{\mu\nu}^{\text{curv}}$ may be interpreted as errors due to the wrong rods & clocks.

One has to set a different "gauge" in each point… and this "gauge" depends on the curvature at that point.

Field equations tell us that rods & clocks we are using (defined by a metric *g* that we like to use) measure different lenghts and different time intervals according to the curvature. The true "gravitational metric" is not *g* but the conformally related metric *h*. Therefore curvature effects are not measured exactly by *g* but rather by *h*. This new metric is a fundamental field, determined by dynamics. The original metric *g* is "frozen."

Metric or connections?

What can we say about conformal invariance of non-linear theories under Palatini form? They are conformally invariant!

If one perfors a conformal transformation, then the Lagrangian changes.

What happens if we impose variations with respect to the conformal factor?

The new equation turns out to be equivalent to the master equation, so that only transformations with constant factor are allowed.

There are no conformal degrees of freedom.

$$\tilde{g} = \phi g \qquad \tilde{\Gamma} = \Gamma \qquad \tilde{R} = \phi^{-1} R$$

$$\nabla^{\Gamma}\left(\tilde{g}^{\mu\nu}\phi^{-2}\sqrt{g}\right) = 0 \qquad \tilde{\mathcal{L}} = f(\tilde{R})\sqrt{\tilde{g}} = f(\phi^{-1}R)\phi^{2}\sqrt{\tilde{g}}$$

$$\phi^{-1} f'(\phi^{-1}R)R^{\mu\nu} - \frac{1}{2} f(\phi^{-1}R)g^{\mu\nu} = 0$$

$$\phi^{-1} f'(\phi^{-1}R)R - 2f(\phi^{-1}R) = 0$$

We must suitably discriminate among theories of gravity!!!

$$G_{\mu\nu} = (8\pi G)\, T_{\mu\nu}$$

$G_{\mu\nu} \Rightarrow \tilde{G}_{\mu\nu}$

$T_{\mu\nu} \Rightarrow \tilde{T}_{\mu\nu}$

Extended theories of Gravity

WHY?

- QFT on curved spacetimes
- String/M-theory corrections
- Brane-world models

$$R,\ R^{\mu\nu}R_{\mu\nu},\ R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta},\ R\Box^{l}R,$$

Curvature invariants should be taken into account

Dark Energy and Λ

- Cosmological constant (Λ)
- Time varing Λ
- Scalar field theories
- Phantom fields
- Phenomenological Theories
- Exotic matter

The role of Equivalence Principle

We need to investigate the EP:

- discriminating among theories of gravity
- its validity at classical and quantum level
- Investigating geodesic and causal structure

The role of Equivalence Principle

EP is the physical foundation of any metric theory of gravity

The first formulation of EP comes out from the theory of gravitation formulates by **Galileo** and **Newton**

Weak Equivalence Principle (WEP)

the "inertial mass" m_i and the "gravitational mass" m_g of any object are equivalent

The role of Equivalence Principle

Einstein Equivalence Principle states:

- Weak Equivalence Principle is valid;
- the outcome of any local non-gravitational test experiment is independent of velocity of free-falling apparatus;
- the outcome of any local non-gravitational test experiment is independent of where and when in the Universe it is performed.

The role of Equivalence Principle

One defines as “local non-gravitational experiment” an experiment performed in a small-size & freely falling laboratory

One gets that the gravitational interaction depends on the curvature of space-time, i.e. the postulates of any metric theory of gravity have to be satisfied

- space-time is endowed with a metric $g_{\mu\nu}$;
- the world lines of test bodies are geodesics of the metric;
- in local freely falling frames, called local Lorentz frames, the non-gravitational laws of physics are those of Special Relativity;

The role of Equivalence Principle

One of the predictions of this principle is the gravitational red-shift, experimentally verified by Pound and Rebka in 1960

Gravitational interactions are excluded from WEP and Einstein EP

In order to classify alternative theories of gravity, the Gravitational WEP and the Strong Equivalence Principle (SEP) has to be introduced

The role of Equivalence Principle

The SEP extends the Einstein EP by including all the laws of physics in its terms:

- WEP is valid for self-gravitating bodies as well as for test bodies (Gravitational Weak Equivalence Principle);
- the outcome of any local test experiment is independent of the velocity of the free-falling apparatus;
- the outcome of any local test experiment is independent of where and when in the Universe it is performed.

The SEP contains the Einstein Equivalence Principle, when gravitational forces are neglected.

The role of Equivalence Principle

Many authors claim that the only theory coherent with the Strong Equivalence Principle is GR

An extremely important issue is related to the consistency of Equivalence Principle with respect to the Quantum Mechanics.

Some phenomena, like neutrino oscillations could violate it if induced by the gravitational field.

GR is not the only theory of gravitation and, several alternative theories of gravity have been investigated from the 60's, considering the space-time to be "special relativistic" at a background level and treating gravitation as a Lorentz-invariant field on the background

The role of Equivalence Principle

Two different classes of experiments have to be considered:

- the first ones testing the foundations of gravitation theory (among them the EP)
- the second ones testing the metric theories of gravity where space-time is endowed with a metric tensor (EP could be violated at quantum level).

For several fundamental reasons extra fields might be necessary to describe the gravitation, e.g. scalar fields or higher-order corrections in curvature invariants.

The role of Equivalence Principle

Two sets of equations can be distinguished

- The first ones couple the gravitational fields to the non-gravitational contents of the Universe, i.e. the matter distribution, the electromagnetic fields, etc...
- The second set of equations gives the evolution of non-gravitational fields.

Within the framework of metric theories, these laws depend only on the metric: this is a consequence of the EEP and the so-called ”minimal coupling”.

Several theories are characterized by the fact that a scalar field (or more than one scalar field) is coupled or not to gravity and ordinary matter

There are several reasons to introduce a scalar field:

* Scalar fields are unavoidable for theories aimed to unify gravity with the other fundamental forces: e.g. Superstring, Supergravity (SUGRA), M-theories.

* Scalar fields appear both in particle physics and cosmology:
  - the Higgs boson in the Standard Model
  - the dilaton entering the supermultiplet of higher dimensional gravity
  - the super-partner of spin ½ in SUGRA.

- The introduction of a scalar field gives rise typically to a possible "violation" of the Einstein Equivalence Principle (EEP).

In order to distinguish competing theories, a possibility is related to the so-called “fifth force” approach.
For example, the case of f(R)-gravity:

$$\mathcal{A} = \int d^4x \sqrt{-g}\Big[f(R) + \mathcal{X}\mathcal{L}_m\Big] \,, \qquad \mathcal{X} = \frac{16\pi G}{c^4}$$

The variation with respect to the metric tensor gives

$$f' R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - f'_{;\mu\nu} + g_{\mu\nu}\Box f' = \frac{\mathcal{X}}{2} T_{\mu\nu}$$

$$3\Box f' + f' R - 2f = \frac{\mathcal{X}}{2} T$$

Trace equation

$$f(R) = \sum_n \frac{f^n(R_0)}{n!}(R - R_0)^n \simeq f_0 + f_1 R + f_2 R^2 + f_3 R^3 + \ldots \,.$$

In the Newtonian limit,
let us consider the perturbation of the metric with respect to the Minkowskin background

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

The metric entries can be developed as

$$\begin{cases} g_{tt}(t,r) \simeq 1 + g_{tt}^{(2)}(t,r) + g_{tt}^{(4)}(t,r) \\ g_{rr}(t,r) \simeq -1 + g_{rr}^{(2)}(t,r) \\ g_{\theta\theta}(t,r) = -r^2 \\ g_{\phi\phi}(t,r) = -r^2 \sin^2\theta \end{cases},$$

As general solution:

$$\Phi_{grav} = -\left(\frac{GM}{f_1 r} + \frac{\delta_1(t)e^{-r\sqrt{-\xi}}}{6\xi r}\right)$$

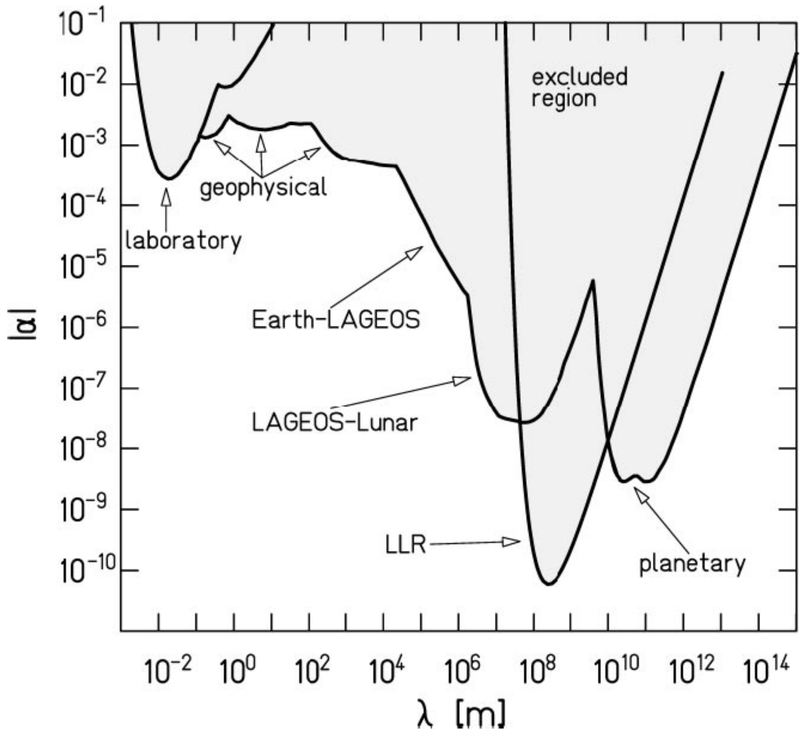
Fifth force

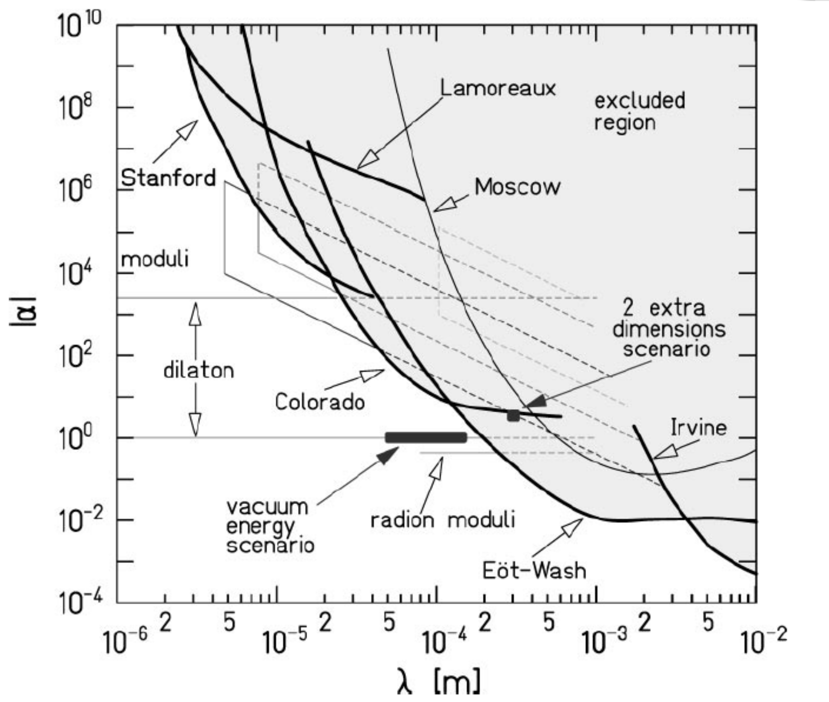
$$V(r) = -G\frac{m_1 m_2}{r}[1 + \alpha e^{-r/\lambda}]$$

α is a dimensionless strength parameter

λ is a length scale or range

Experimental bounds





Conclusions

- Several shortcoming in standard General Relativity
- We have to test the fundamental gravitational fields. Is it g or Γ ?
- Coincidence of geodesic and causal structures strictly depends on the validity of the Equivalence Principle (Levi-Civita)
- Discrimination of gravitational theories at quantum level
- Tools: atomic clocks, free fall, space-craft (STE-QUEST) see Altschul et al. Adv. Space. Res. 55, 501 (2015).

WORK IN PROGRESS!!!