Cauchy horizons in General Relativity

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Overview

1. Causality problems in stationary Black Holes (BH) and Naked Singularities (NS)
    - timelike singularities
    - Closed Causal Curves (CCC)
    - Can GR save itself from causal pathologies?
2. Dynamics in spacetimes with a timelike singularity
    - Ambiguity of dynamics: a toy model
    - Instability of spacetimes with timelike singularities: static NS and the region beyond the RN-BH inner horizon.
3. Dynamics in spacetimes with Closed Causal Curves
    - CCC and self-consistent initial data: a toy model.
    - Instability of the Kerr NS, and the Kerr BH region beyond the inner horizon
4. Instability of Black Holes Cauchy Horizon
    - Causally pathological stationary BH regions lie beyond the Cauchy Horizon
    - Penrose's observation and Poisson-Israel-Hartle-Hawking "mass inflation"
    - Recent CH instability results.
    - A toy model exhibiting an unstable CH with CCC beyond it.

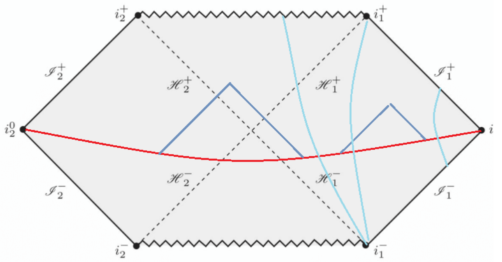
Basic causality concepts in General Relativity

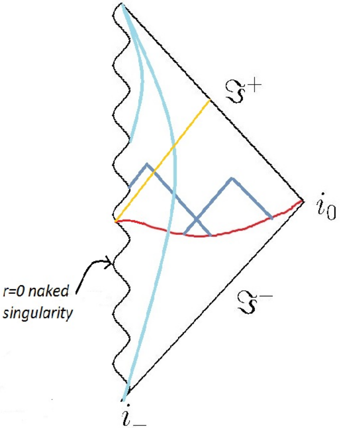
- A spacetime (M, g_{ab}), g_{ab} of signature $(- + ++)$ is time oriented if it admits a smooth vector field V^a that never vanishes and is causal everywhere: $V^a V_a \leq 0$
- In time oriented spacetimes it is possible to choose in a continuous way a future half-cone at every point (e.g., that where V^a belongs). That is, there is a notion of future and past directed timelike and null curves at every event.
- A closed causal (timelike or null) curve (CCC) γ is one for each $\gamma(\tau + \tau_o) = \gamma(\tau)$. Such a curve can be extended to $-\infty < \tau < \infty$ with period τ_o.

"Reasonable" spacetimes are expected to be time orientable and to have no CCC

- The *domain of dependence* of a spacelike hypersurface Σ is the region for which fields obeying wave-like equations are determined by their initial data on Σ. If this domain if the whole spacetime we say that Σ is a *Cauchy surface*
- A point p to the future (past) of Σ lies in its domain of dependence if all past (future) directed causal curves from p intersect Σ.

Basic causality concepts in General Relativity





Kruskal spacetime:
in red: a *Cauchy* Surface Σ. Every causal curve intersects Σ exactly once. Note that Σ is a complete 3-manifold.
in light blue: inextendible causal curves

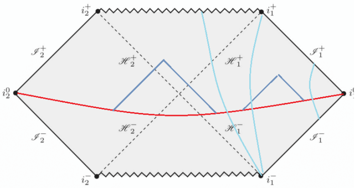
in dark blue: past cones of two points, one inside, one outside the BH.

Schwarzschild $\mathrm{M} < 0$ NS or RN $\mathrm{Q} > \mathrm{M}$ NS:
in red: partial Cauchy surface Σ: causal curves intersect it *at most* once.

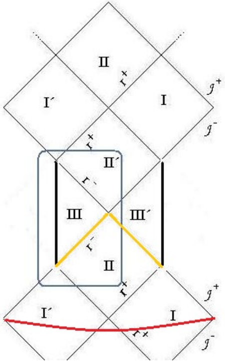
in yellow: boundary of Σ domain of dependence

in light blue: inextendible timelike curves

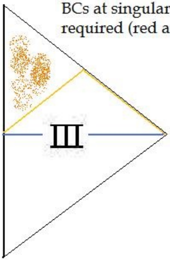
in dark blue: past cones of two points, one inside, one outside the domain of dependence of Σ



Kruskal spacetime:
in red: a Cauchy hypersurface
in light blue: inextendible timelike curves
in dark blue: past cones of two points, one inside, one outside the BH



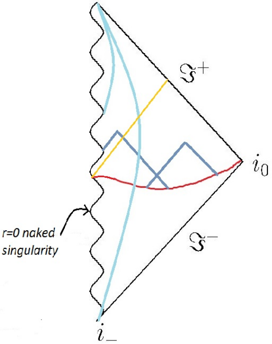
Region III: initial data surface (blue), region where BCs at singularity are required (red airbrush)



LEFT: RN BH with outer horizon r^+ and inner horizon r^-.
in red: complete Cauchy surface Σ for the region $r > r_-$
in yellow: inner horizon, a CH for Σ.
Dropping all regions above it (and similarly in the past) we get a globally hyperbolic spacetime
RIGHT: region III above the CH of the RN BH. Note the similarity with the Schwarzschild NS. The blue surface is partial Cauchy for this region, the airbrush region is outside its domain of dependence.

Timelike singularities and non-uniqueness of evolution

The $r = 0$ singularity is a timelike boundary of:
i) the Schwarzschild NS,
ii) the RN NS and
iii) the region RNIII of a RN BH, defined by $0 < r < r^-$.
Dynamics on these spaces requires, besides initial data, specification of boundary conditions at the timelike boundary. Note that any claim of instability is related to fields growing unbounded *at large times*, that is, for observers leaving the domain of dependence of the partial Cauchy surface, and so depends on chosen bc's.



In tortoise radial coordinates x these regions get mapped onto $0 < x < \infty$ and the (scalar / Maxwell / linear gravity) perturbations decompose in harmonic modes satisfying a 1+1 wave equation

$$[\partial_t^2 \underbrace{-\partial_x^2 + V(x)}_{\mathcal{H}}]\Phi = 0, \quad x > 0 \qquad (1)$$

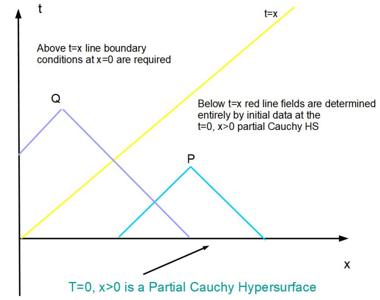
So we are led to analyze wave equations with a potential $V(x)$ on the $x > 0$ half of 1+1 Minkowski space

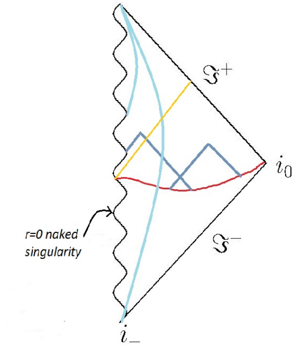
As a toy model we consider the $V = 0$ case on the $x > 0$ half of 1+1 Minkowski spacetime

$$ds^2 = -dt^2 + dx^2, \quad x > 0 \tag{2}$$

$$(\partial_t^2 - \partial_x^2)\Phi = 0, \quad \Phi(t=0,x) = \phi(x), \quad \dot{\Phi}(t=0,x) = \psi(x), \quad x > 0$$

(a semi-infinite string $x > 0$ pulled away from the $\Phi(t,x) = 0$ equilibrium position)
Recall that solution **on the entire Minkowski space** is $\Phi(t,x) = L(t+x) + R(t-x)$





The solution on the $x > 0$ half-space depends on the initial datum at $t = 0$:

$$\phi(x) = \Phi(t = 0, x), \quad \psi(x) = \dot{\Phi}(t = 0, x), \quad x > 0$$

AND on the boundary condition at $x = 0$:

$$\cos(\alpha)\ \Phi(t, x = 0) + \sin(\alpha)\ \Phi'(t, x = 0) = 0, \quad -\tfrac{\pi}{2} < \alpha \leq \tfrac{\pi}{2}$$

known as Dirichlet ($\alpha = 0$), Neumann ($\alpha = \pi/2$) or Robin (other α values)

Under Dirichlet boundary conditions we find that the left and right moving waves must be opposite to each other: $\Phi_D(t, x) = f(t - x) - f(t + x), \quad f : \mathbb{R} \to \mathbb{R},\ f(0) = 0$

$$f(z) = \begin{cases} -\frac{1}{2}(\phi(z) + \Psi(z)) & , z \geq 0 \\ \frac{1}{2}(\phi(-z) - \Psi(-z)) & , z < 0 \end{cases} \quad \text{where} \quad \Psi(z) = \int_0^z \psi(z')\, dz'$$

(note that $\phi(0) = \Phi(t = 0, x = 0) = 0$ and also $\Psi(0) = 0$)
For $t \geq 0$ this gives

$$\Phi_D(t, x) = \begin{cases} \frac{1}{2}\left[\phi(t + x) + \Psi(t + x) + \phi(-t + x) - \Psi(-t + x)\right] & , 0 \geq t < x \\ \frac{1}{2}\left[\phi(t + x) + \Psi(t + x) - \phi(t - x) - \Psi(t - x)\right] & , t \geq x \end{cases} \qquad (2)$$

Under Neumann boundary conditions ($\partial_x \Phi(t, x = 0) = 0$, the string is free at $x = 0$) the left and right moving waves must be equal:

$$\Phi_N(t,x) = g(t-x) + g(t+x), \quad g : \mathbb{R} \to \mathbb{R} \tag{2}$$

$$g(z) = \begin{cases} \frac{1}{2}(\phi(z) + \Psi(z)) & , z \geq 0 \\ \frac{1}{2}(\phi(-z) - \Psi(-z)) & , z < 0 \end{cases} \quad \text{where} \quad \Psi(z) = \int_0^z \psi(z')\, dz' \tag{3}$$

(note that $\phi(0) = \Phi(t = 0, x = 0) = 0$, also $\Psi(0) = 0$)
For $t \geq 0$ this gives

$$\Phi_N(t,x) = \begin{cases} \frac{1}{2}\left[\phi(t+x) + \Psi(t+x) + \phi(-t+x) - \Psi(-t+x)\right] & , 0 \leq t < x \\ \frac{1}{2}\left[\phi(t+x) + \Psi(t+x) + \phi(t-x) + \Psi(t-x)\right] & , t \geq x \end{cases} \tag{4}$$

The solution on the $x > 0$ half-space under Dirichlet boundary conditions ($\alpha = 0$)

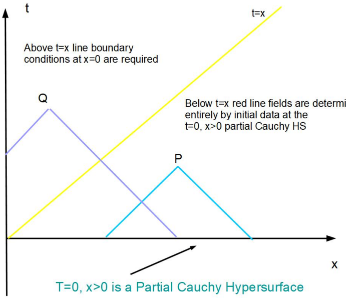
$$\Phi_D(t,x) = \begin{cases} \frac{1}{2}\left[\phi(t+x) + \Psi(t+x) + \phi(-t+x) - \Psi(-t+x)\right] & , 0 \leq t < x \\ \frac{1}{2}\left[\phi(t+x) + \Psi(t+x) - \phi(t-x) - \Psi(t-x)\right] & , t \geq x \end{cases} \tag{2}$$

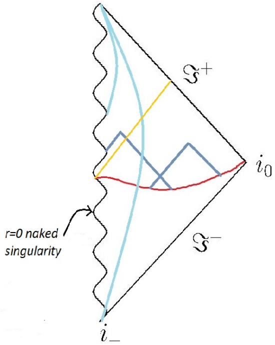
agrees with the solution under Neumann boundary conditions ($\alpha = \pi/2$)

$$\Phi_N(t,x) = \begin{cases} \frac{1}{2}\left[\phi(t+x) + \Psi(t+x) + \phi(-t+x) - \Psi(-t+x)\right] & , 0 \leq t < x \\ \frac{1}{2}\left[\phi(t+x) + \Psi(t+x) + \phi(t-x) + \Psi(t-x)\right] & , t \geq x \end{cases} \tag{3}$$

and with those with Robin boundary condition (for which the relation between left and right moving modes is more complicated) only for $t < x$

Timelike singularities and non-uniqueness of evolution: a toy model





Link

- Schwarzschild NS: Schwarzschild's solution with $M < 0$
- RN NS: Reissner Nordström solution with $|Q| > M > 0$
- RNIII: region III (beyond Cauchy horizon, i.e., $0 < r < r_-$) of a Reissner Nordström BH ($|Q| \leq M$)

Electro-gravitational waves decompose into even/odd harmonic modes satisfying a wave equation

$$[\partial_t^2 \underbrace{-\partial_x^2 - V(x)}_{\mathcal{H}}]\Phi = 0, \quad x > 0$$

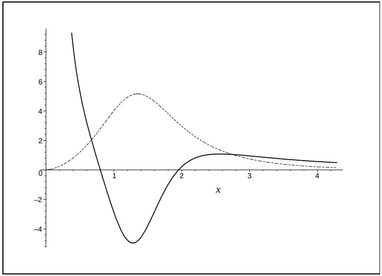
Solution: choose a self-adjoint extension $\mathcal{H}_\alpha$ of $\mathcal{H}$ (that is, bc at $x = 0$), then $\mathcal{H}_\alpha$ has a complete set of eigenfunctions $\phi_E(x)$, and

$$\Phi(t,x) = \int dE a_E(t)\phi_E(x), \quad a_E(t) := \int \phi_E(x)^* \Phi(t,x) dx$$

$$\ddot{a}_E = -E a_E \quad \dot{a}_E(0) = \dot{a}_E^0 := \int \psi_E^* \dot{\Psi}_z^0 \, dx, \quad a_E(0) = a_E^0 := \int \psi_E^* \Psi_z^0 \, dx$$

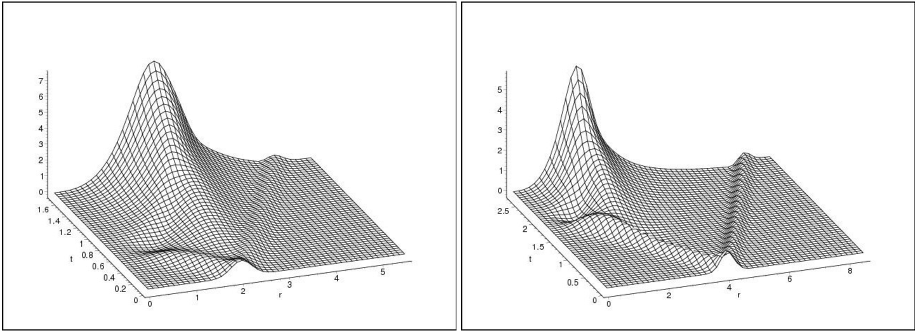
$$a_E(t) = \begin{cases} a_E^0 \; \cos(\sqrt{E}\,t) + \dot{a}_E^0 \; E^{-1/2} \; \sin(\sqrt{E}\,t) & , E > 0 \\ a_E^0 + t \; \dot{a}_E^0 & , E = 0 \\ a_E^0 \; \cosh(\sqrt{-E}\,t) + \dot{a}_E^0 \; (-E)^{-1/2} \; \sinh(\sqrt{-E}\,t) & , E < 0 \end{cases}$$

An α value selecting a bc must be given. The spectrum of $\mathcal{H}_\alpha$ depends on α. A unique physically consistent bc at the singularity α_o was found for RNIII, Schwa-NS and RN-NS This bc allows bound, unstable states of $\mathcal{H}_{\alpha_o}$

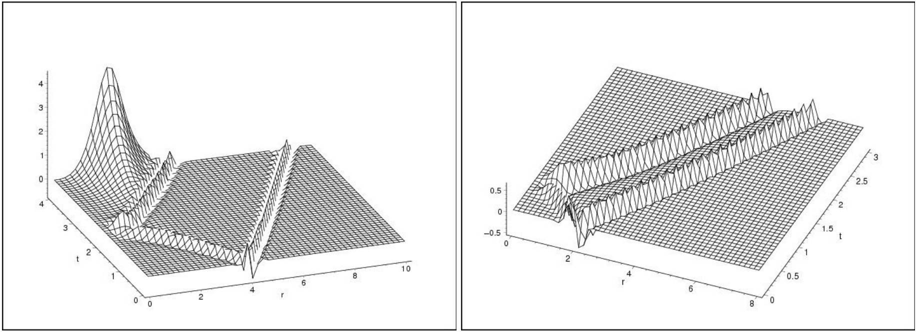


$V(x)$ for the Schwarzschild NS with $M = -2, \ell = 2$ (solid line) and the bound state.

Timelike singularities: Instability of the RN and Schwarzschild NS, and of RNIII



Evolution of an $\ell = 2$ even gravitational wave on a Schwarzschild NS with initial condition having a strong (left) moderate (right) projection on the unstable mode.



Evolution of an $\ell = 2$ even gravitational wave on a Schwarzschild NS with initial condition having a mild (left) null

(right, requires fight tuning) projection on the unstable mode.

References

Instability of the Schwarzschild NS:

G. Gibbons, S. Hartnoll and A. Ishibashi, Prog.Theor.Phys. 113 (2005) 963-978; G.D. and R.J. Gleiser: Class.Quant.Grav. 23 (2006) 5063-5078 and Class.Quant.Grav. 26 (2009) 215002;

Instability of RN NS and RNIII:

Cardoso and Cavaglia, Phys.Rev. D74 (2006) 024027, G. D. , R.J. Gleiser and J. Pullin: Phys.Lett. B644 (2007) 289-293; G. D. and R.J.Gleiser: Class.Quant.Grav. 27 (2010) 185007

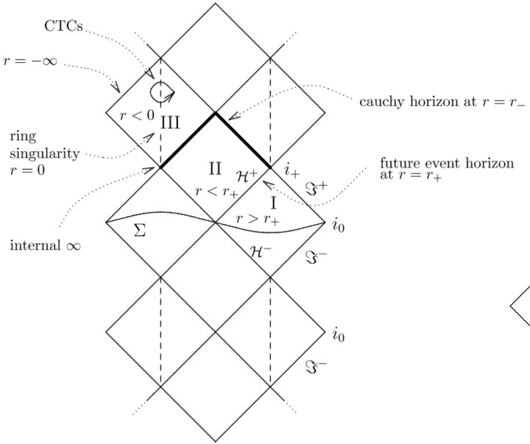
Stationary Black holes in General Relativity: the Kerr family

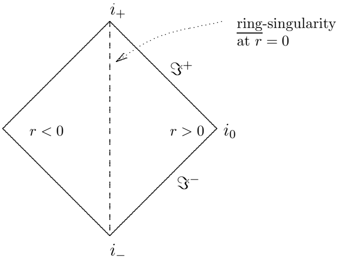
Kerr Newman metric (gives most general asymptotically flat electro-vac stationary BH)

$$ds^2 = -\frac{\Delta_r^2}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta_r^2}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left(a dt - \frac{(r^2+a^2)}{\Sigma}d\phi\right)^2$$

$$\Delta_r^2 = (r^2+a^2) - 2Mr + Q^2, \rho^2 = r^2 + a^2\cos^2\theta,\ A_\mu = -\frac{Qr}{\rho^2}(\delta_\mu^t - \frac{a\sin^2\theta}{\Sigma}\delta_\mu^\phi)$$

- These are Boyer-Lindquist coordinates, and whenever $a \neq 0$ the range of r is $-\infty < r < \infty$!!
- $\Delta_r^2 = 0$ are *coordinate* singularities, they signal locations of horizons.
- $\rho^2 = 0$, that is, $r = 0$ *and* $\theta = \pi/2$ is a codimension 2 curvature singularity (ring singularity). It can be avoided and transverse to an AF $r < 0$ region!
- $g_{\phi\phi} < 0$ in a region near the singularity (“Time Machine”). This allows CCC





- The Kerr metric exhibits horizons at $r^{\pm} = M \pm \sqrt{M^2 - a^2}$
- r^- (bold in the left figure) is a Cauchy horizon for Σ.
- If $a > M$ ("super-extreme") there is no horizon (Naked Singularity)

Left: $a < M$ Kerr BH, Right: Kerr NS (figures from Townsend's BH Lecture Notes)

CCC within Kerr BH and Kerr NS

- KIII: region beyond the Cauchy horizon of a Kerr BH ($-\infty < r < r_i$)
- KNS: Kerr solution with $a > M$, which is a naked singularity (standard Boyer-Lindquist (t, r, θ, ϕ) coordinates are global in this case, with $-\infty < t, r < \infty$ and $(\theta, \phi) \in S^2$. For both of these $r^2 + 2Mr + a^2 > 0$
- These spaces are totally vicious: given *any two points* p and q in either of these spaces, there is a *future oriented timelike curve* from p to q, therefore one from q to p, and also a CTC through p (take $q = p$)
- This implies that there exists no even a partial Cauchy surface.

How can we possibly claim that KIII and the KNS are linearly unstable?

Einstein's equations are local, they don't tell you what the spacetime global structure and topology are.

Example:

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{2}$$

is a solution of Einstein's vacuum field equations for $(x^0 = ct, x^1, x^2, x^3) \in \mathbb{R}^4$ (Minkowski spacetime $\mathcal{M}$

Problems associated to CCC: a toy model

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...but we may assume that x^3 is periodic: $x^3 \sim x^3 + a$ (i.e. $\frac{2\pi}{a}x^3$ is an angular variable), then (2) is a solution of Einstein's equation on the spacetime manifold $S^1 \times \mathbb{R}^3$

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...or that $x^i \sim x^i + a^i, i = 1, 2, 3$: the is spacetime $\mathbb{R}_{\{x^0\}} \times \mathbb{T}^3$, which has compact time slices.

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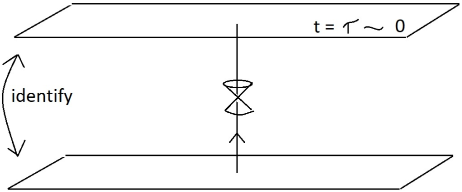
Even more interesting is the possibility of assuming that $x^0 \sim x^0 + c\tau$ to construct a *flat* spacetime with closed timelike curves (CCC)



Self-consistent initial data for toy spacetime with CCC

Consider again a massless scalar field $\Box\Phi = 0$ in 1+1 Minkoski spacetime

$$ds^2 = -du\,dv, \quad u = x^0 - x^1, v = x^0 + x^1, \quad x^0 = ct \quad \boxed{\Phi = L(v) + R(u)}$$

If we identify $x^0 \sim x^0 + c\tau$ then $(u, v) \sim (u + c\tau, v + c\tau)$ and we need that

$$\begin{aligned} L(v + c\tau) + R(u + c\tau) &= L(v) + R(u), \\ L(v + c\tau) - L(v) &= -[R(u + c\tau) - R(u)] = k \end{aligned}$$

This implies

$$L(v) = \hat{l}(v) + \frac{k}{c\tau} v, \quad R(u) = \hat{r}(u) - \frac{k}{c\tau} u$$

where all hatted functions are periodic with period $c\tau$

$$\boxed{\Phi = \hat{l}(v) + \hat{r}(u) + \frac{k}{c\tau}(v - u)}$$

Self-consistent initial data for toy spacetime with CCC

$$\boxed{\Phi = \hat{l}(v) + \hat{r}(u) + \cancel{\frac{k}{c\tau}(v-u)}}$$

Note that initial data has to be periodic with period $c\tau$

$$\Phi(x^0 = 0, x^1) = \hat{l}(x^1) + \hat{r}(-x^1), \quad \frac{\partial \Phi}{\partial x^0}(x^0 = 0, x^1) = \hat{l}'(-x^1) + \hat{r}'(-x^1)$$

The reason is that, say, a right moving pulse at $t = 0$ multiplicates to its right at $t = c\tau$, etc.

Self-consistent initial data for toy spacetime with CCC

Is compactifying along a spatial direction a way out of the infinite energy problem?
Say $x^1 \sim x^1 + a$, i.e., $(u, v) \sim (u - a, v + a) \sim (u - na, v + na), n \in \mathbb{Z}$

$$\hat{l}(v + na) - \hat{l}(v) + n\frac{2ka}{c\tau} = -[\hat{r}(u - na) - \hat{r}(u)],\ n \in \mathbb{Z}$$

Since $\hat{r}$ and $\hat{l}$ are bounded, last equation above implies $k = 0$:
$\Phi = \hat{l}(v) + \hat{r}(u)$ where $\hat{l}$ and $\hat{r}$ are periodic with period $c\tau$ and also with period a

If $c\tau$ and a are commensurable ($c\tau/a = z_1/z_2, z_i \in \mathbb{N}$), then

$$p := \frac{c\tau}{z_1} = \frac{a}{z_2},\ \ c\tau = z_1 p,\ a = z_2 p$$

and any periodic function with period p fulfills the double periodicity condition.
If $c\tau/a$ is not a rational number then the set $n_1\tau + n_2 a$ is dense in $\mathbb{R}$ and the only continuous double-period functions are the constants!

What do we mean when we say that KIII and the KNS are linearly unstable?

- Both KIII and KNS are time oriented: in Boyer-Lindquist coordinates (t, r, θ, ϕ) $V = (r^2 + a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial \phi}$ is never zero and everywhere timelike.
- This fields gives a time orientation and defines a congruence of (accelerated) observers with unit speed $u^c = V^c/\sqrt{-V^b V_b}$
- The integral curves of these fields extend infinitely in proper time towards the past and the future: besides there being CTC, there are “eternal observers” and the question of stability (things blowing up at large times) does make sense.
- Will a family of observers like these see gravitational or other perturbations blow up in the future?
- The timelike singularity of KIII and KNS (“ring singularity”) has codimension 2, it is milder, can be avoided and crossed to $r < 0$. The singularity doesn’t show up in the perturbation equations (Teukolsky equations).
- So in many ways the complicated KNS and KIII are simpler and better behaved than our toy model.

Instability results for spacetimes with CCC: KIII and KNS

Gravitational perturbations, Maxwell and scalar fields, as well as Weyl spinors were found that are independent of ϕ and:
i) decay exponentially with $|r|$ for the KNS
ii) decay exponentially as $r \to -\infty$ and also decay as $r \to r_-{}^-$ for KIII)
iii) grow exponentially with t.

The Teukolsky functions of these fields are of the form

$$\Psi_s(t,r,\theta,\phi) = e^{kt} R_s(r) S_s(\theta), \quad s = 0, 1/2, 1, 2; \quad k > 0$$

This implies that the bundle of observers u^c measure an exponential growth of the relevant fields (e.g. $u^c \nabla_c \Psi_{s=0} \propto e^{kt}$)

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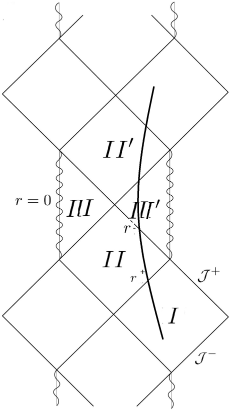
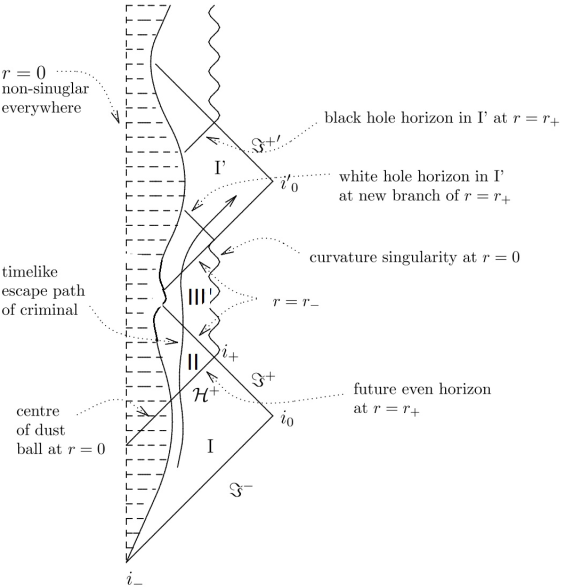
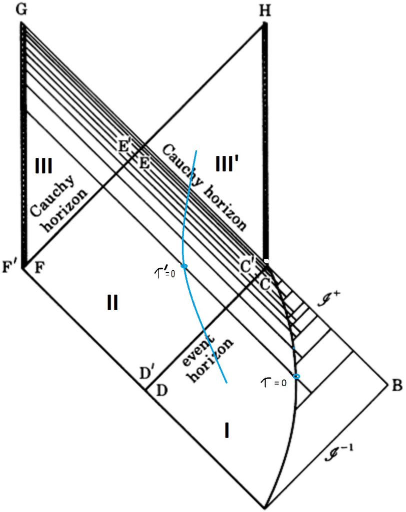
G.D., R.J..Gleiser, I.Sandoval and H.Vucetich, Class.Quant.Grav. 25 (2008) 245012; G.D., R.J..Gleiser and I. Sandoval, Class.Quant.Grav. 29 (2012) 095017



Timelike curve crossing the CH into copies of region II and III: this could be the surface of a spherical collapsing charged star: the metric to its right is RN's, to the left has to be replaced by inner non vacuum star regular metric, see here $\Longrightarrow$

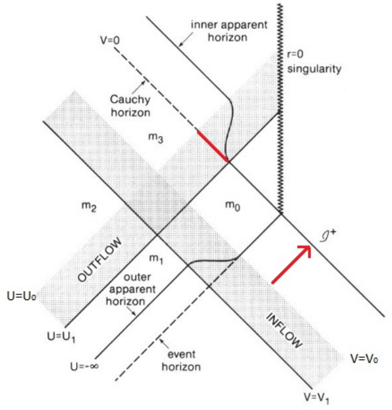




Consider an eternal source (staying outside the BH) of radial out and ingoing radiation in region I (black worldline ending at i^{+}). There is an accumulation of ingoing wavefronts as we approach the E'-C' piece of the Cauchy Horizon (but not along the E-F piece!).
More precisely: define $\tau'(\tau)$ as the proper time of the receiver when receiving the signal sent by the emisor at its proper time τ. Assume the receiver attempts to cross from II to III', then we find that $d\tau/d\tau' \to \infty$ as the Cauchy Horizon is approached.

References:
Penrose and Simpson, Int.Jour.Theor.Phys. 7 (1973), 183; J. M. McNamara, Proc. R. Soc. London A358, 449 (1978); A364, 121 (1978); Y. Gursel, V. D. Sandberg, I. D. Novikov, and A. A. Starobinsky, Phys. Rev. D 19, 413 (1979); R. A. Matzner, N. Zamorano, and V. D. Sandberg, ibid. 19, 2821 (1979);**S. Chandrasekhar and J. B. Hartle, Proc. R. Soc. London A284, 301 (1982)**; N. Zamorano, Phys. Rev. D 26, 2564 (1982)

Cauchy Horizon Instability: Poisson and Israel “mass inflation”



Poisson and Israel models the crossing of out and inflowing radiation by two flows of null dust:

$$G_{ab} = 8\pi[T^{EM}_{ab} + \rho_{in} l_a l_b + \rho_{out} n_a n_b]$$

where T^{EM}_{ab} corresponds to the background RN electromagnetic field. The solution contains four pieces of RN spacetimes (different values of Q and M), two regions of charged Vaydia spacetimes and a new region where the flow crossing. AS we take the limit where the outflow approaches the CH (see red arrow), a curvature divergence arises as $r \to CH$ at the crossing flow region.

Reference:
E. Poisson and W. Israel, Phys. Rev. D 15 (1990), 1796

For a proof of divergence of a gauge invariant combination of perturbation of curvature scalars see G.D. and J.M.

Fernández .Tío,Phys.Rev. D95 (2017) 124041, arXiv:1607.00975

Cauchy Horizon Instability: recent results for extremal BH

For extremal BH $r_- = r_+ \equiv r_o$ is the Cauchy Horizon, the linearly unstable $r < r_o$ region lies adjacent to the exterior, stable region $r > r_o$, so is to be expected that transverse derivatives of linear fields at $r = r_o$, which probe the inner region, should reveal the inner instability.

- For extreme RN: $ds^2 = -(r - r_o)^2 dv^2 + \frac{dr^2}{(r-r_o)^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$, if $\Box\Phi = 0$, write $\Phi = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \Phi_{(\ell,m)} =: \sum_\ell \Phi_\ell$, then

$$\boxed{\partial_r^{\ell+m+k} \partial_v^m \Phi_\ell \sim v^{k-1} \quad \text{as } v \to \infty, \quad k \geq 2, m \geq 0}$$

  The decay is milder if the initial data for Φ has compact support away from the horizon, e.g.: $\partial_r^2 \Phi_0 \sim v^{2/5}$ in the same limit.
- For extreme Kerr it was proved that transverse derivatives at the horizon of Teukolsky functions of any spin blow up along generators. This shows instability under scalar, spinors and gravitational perturbations

References: Aretakis, S Ann. Henri Poincare 12 (2011) (arXiv:1110.2009),
S. Dain and G. D. , Class.Quant.Grav 30 (2013) 055011 (arXiv:1209.0213),
J.Lucietti and H. Reall, Phys.Rev.D86 (2012) (arXiv:1208.1437)

Misner spacetime: a flat spacetime with CH and CCC

Misner spacetime is constructed from Minkowski space by identifying points in the null surface (here $u = x^0 - x^1, v = x^0 + x^1$)

$$\ell = \{(u, v_o, y, z)\}, \quad v_o < 0$$

with their boosted images in

$$\mathcal{B}\ell = \{(\exp(\gamma)u, \exp(-\gamma)v_o, y, z)\}$$

Introduce

$$v = v_o e^{-\psi}, \; u = -\frac{t}{v_o} e^{\psi}$$

then

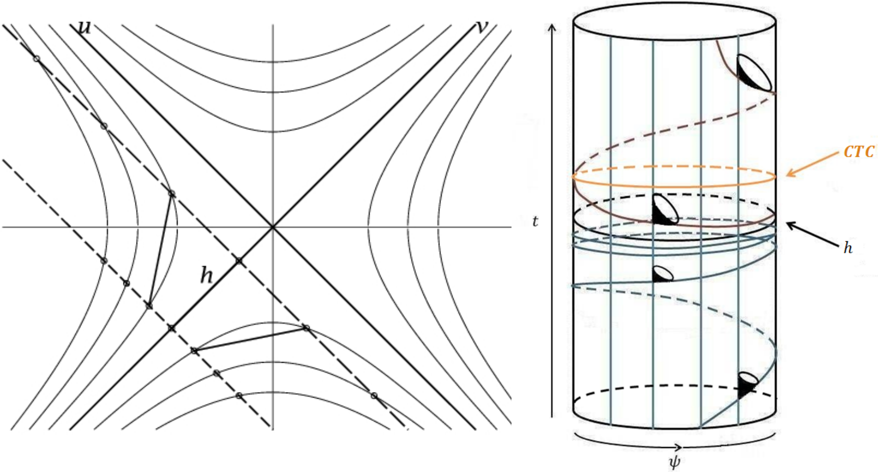
$$ds^2 = -du\, dv + dy^2 + dz^2 = \boxed{-d\psi dt - t d\psi^2 + dy^2 + dz^2}$$

and

$$(\exp(\gamma)u, \exp(-\gamma)v), \gamma > 0 \Rightarrow \boxed{(\psi, t) \to (\psi + \gamma, t)}$$

so in the metric above we must identify $\psi \sim \psi + \gamma$ (that is, we may think of $2\pi\psi/\gamma$ as an angular variable)

Misner space is a <u>flat</u> spacetime exhibiting a Cauchy Horizon with CCC beyond it!



Left: Minkowski space and a few boost orbits. The strip between $\ell = \{(u, v_o) \mid u \in \mathbb{R}\}$ (dashed line) and the long-dashed line $\mathcal{B}\ell = \{(u, \exp(-\gamma v_o)) \mid u \in \mathbb{R}\}$ with boosted points identified (marked circles) gives Misner space on the right.

CH instabilities in GR + scalar field

On the $t < 0$ piece of (y, z)-compactified Misner consider a monoparametric family of solutions $(\Phi_\epsilon,\ (g_\epsilon)_{ab})$ of

$$G_{ab} = 8\pi\ \left[(\partial_a \Phi)(\partial_b \Phi) - \tfrac{1}{2} g_{ab}\ g^{cd}(\partial_c \Phi)(\partial_d \Phi)\right] \qquad 0 = g^{cd}\ \nabla_d \nabla_c \Phi,$$

of the form (overdots denote derivatives w.r.t. ϵ)

$$\Phi_\epsilon = \epsilon \dot{\Phi} + \tfrac{1}{2}\epsilon^2 \ddot{\Phi} + \ldots \tag{3}$$

$$(g_\epsilon)_{ab} = \eta_{ab} + \epsilon \dot{g}_{ab} + \tfrac{1}{2}\epsilon^2 \ddot{g}_{ab} + \ldots \tag{4}$$

For the the Ricci scalar $\mathcal{R}$ we obtain

$$\mathcal{R}_\epsilon = \tfrac{1}{2}\epsilon^2 \ddot{\mathcal{R}} + \mathcal{O}(\epsilon^3) \tag{5}$$

where

$$\ddot{\mathcal{R}} = 16\pi\ \eta^{ab}(\partial_a \dot{\Phi})(\partial_b \dot{\Phi}), \tag{6}$$

and

$$0 = \eta^{ab} \partial_a \partial_b \dot{\Phi}. \tag{7}$$

Misner CH instability: the case of GR + scalar field

$$\Phi = \phi_{(0)}(\psi, t) + \phi_{(1)}(\psi, t, y, z) \tag{8}$$

$$\phi_{(0)}(\psi, t) = \frac{1}{ab} \int_0^a dy \int_0^b dz\ \Phi(\psi, t, y, z) \tag{9}$$

$$\phi_{(1)} = \sum_{\substack{k,l,n=-\infty \\ (l,n)\neq(0,0)}}^{\infty} C_{(k,l,n)}(t) e^{2\pi i k\psi/\gamma} e^{2\pi i l y/a} e^{2\pi i n z/b}, \tag{10}$$

(the $C_{(k,l,n)}(t)$ obey a Bessel equation)

$$\phi_{(0)} = R(u) + L(v), \quad R(e^{\gamma} u) - R(u) = L(v) - L(e^{-\gamma} v) = c$$

$$\Phi = \begin{cases} \hat{l}_<(\psi) + \hat{r}_<(\psi + \ln(|t|/v_o^2)) + \cancel{\frac{c_<}{\gamma}\ \ln(|t|/v_o^2)}, & t < 0 \\ \hat{l}_>(\psi) + \hat{r}_>(\psi + \ln(|t|/v_o^2)) + \frac{c_>}{\gamma}\ \ln(|t|/v_o^2), & t > 0 \end{cases} \tag{11}$$

(hatted functions are periodic with period γ)

$$\mathcal{R}_\lambda = \tfrac{1}{2}\epsilon^2 \ddot{\mathcal{R}} + \mathcal{O}(\epsilon^3), \quad \ddot{\mathcal{R}} = \ddot{\mathcal{R}}_{(0)(0)} + \ddot{\mathcal{R}}_{(1)(1)} + 2\ddot{\mathcal{R}}_{(0)(1)}$$

where

$$\ddot{\mathcal{R}}_{(i)(j)} = -4\frac{\partial\phi_{(i)}}{\partial\psi}\frac{\partial\phi_{(j)}}{\partial t} - 4\frac{\partial\phi_{(i)}}{\partial t}\frac{\partial\phi_{(j)}}{\partial\psi} + 8\frac{\partial\phi_{(i)}}{\partial t}\frac{\partial\phi_{(j)}}{\partial t} + 2\frac{\partial\phi_{(i)}}{\partial y}\frac{\partial\phi_{(j)}}{\partial y} + 2\frac{\partial\phi_{(i)}}{\partial z}\frac{\partial\phi_{(j)}}{\partial z}$$

$$\ddot{\mathcal{R}}_{(0)(0)} = -\frac{4}{\gamma t}\left[\hat{r}'_{<}(\psi + \ln(|t|/v_o^2))\right]\hat{l}'_{<}(\psi)$$

Note that $\mathcal{R}$ decays along past oriented causal curves and diverges as the future Cauchy horizon is approached.

CH instabilities in Einstein-Maxwell theory

Now let $((F_\epsilon)_{ab},\ (g_\epsilon)_{ab})$ be a family of solutions for the Einstein-Maxwell theory with

$$(F_\epsilon)_{ab} = \epsilon \dot{F}_{ab} + \tfrac{1}{2}\epsilon^2 \ddot{F}_{ab} + \dots \quad (8)$$

$$(g_\epsilon)_{ab} = \eta_{ab} + \epsilon \dot{g}_{ab} + \tfrac{1}{2}\epsilon^2 \ddot{g}_{ab} + \dots \quad (9)$$

$\mathcal{R} = 0$, the simplest nontrivial Ricci invariant is $\mathcal{Q} = R_{ab}R_{cd}g^{ac}g^{bd}$, for which

$$\mathcal{Q}_\epsilon = \tfrac{1}{4!}\,\epsilon^4\,\ddddot{\mathcal{Q}} + \mathcal{O}(\epsilon^5) \quad (10)$$

$$\ddddot{\mathcal{Q}} = \ddot{R}_{ab}\ddot{R}_{cd}\eta^{ac}\eta^{bd} \quad (11)$$

$$\ddot{R}_{ab} = 2\dot{F}_{ac}\dot{F}_{bd}\eta^{cd} - \tfrac{1}{2}\eta_{ab}\dot{F}_{cd}\dot{F}_{ef}\eta^{ec}\eta^{df}\,. \quad (12)$$

Note that $\dot{F}$ satisfies the equations of a test Maxwell field *on the Misner background*, that is

$$\nabla_{[a}\dot{F}_{bc]} = 0, \quad \eta^{bc}\nabla_c\dot{F}_{ab} = 0, \quad (13)$$

Pure gravitational CH instabilities

Now consider a mono-parametric family of $R_{ab} = 0$ metrics through η_{ab}:

$$(g_\epsilon)_{ab} = \eta_{ab} + \epsilon \dot{g}_{ab} + \tfrac{1}{2}\epsilon^2 \ddot{g}_{ab} + \dots \tag{14}$$

Any algebraic curvature scalar is a polynomial on $K := R_{abcd}R^{abcd}$ and $L := \epsilon_{abpq}R^{pq}{}_{cd}R^{abcd}$, and

$$K_\epsilon = \epsilon^2 \ddot{K} = \epsilon^2 \dot{R}_{abcd}\dot{R}_{efgh}\eta^{ae}\eta^{bf}\eta^{cg}\eta^{dh} + \dots \tag{15}$$

Knowledge of a linearized solution $\dot{g}_{ab}$ of Einstein's equation provides information on the dominant contributions to K and L, which are second order in ϵ.
The simplest metric perturbations are the (y, z)-independent tensor modes:

$$\dot{g}_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{H}(\psi, t) & \widehat{P}(\psi, t) \\ 0 & 0 & \widehat{P}(\psi, t) & -\widehat{H}(\psi, t) \end{pmatrix} \tag{16}$$

Einstein's linearized equation reduce to $\Box \widehat{H} = \Box \widehat{P} = 0$

$$H = \widehat{H}_l(\psi) + \widehat{H}_r(\psi + \ln(|t|/v_o^2)) \tag{17}$$

$$P = \widehat{P}_l(\psi) + \widehat{P}_r(\psi + \ln(|t|/v_o^2)) \tag{18}$$

Set $C_H = C_P = 0$ to keep the perturbation bounded as $t \to -\infty$ (H and P are gauge invariant fields in the linearized gravity theory), we obtain

$$\ddot{K} = \dot{R}_{ab}{}^{cd}\dot{R}^{cdab} = \frac{32}{t^2}\,[\widehat{H}_r''\,\widehat{H}_l'' + \widehat{P}_r''\,\widehat{P}_l'' + \widehat{H}_r''\widehat{H}_l' + \widehat{P}_r''\widehat{P}_l' - \widehat{H}_l''\widehat{H}_r' - \widehat{P}_l''\widehat{P}_r' - \widehat{H}_r'\widehat{H}_l' - \widehat{P}_r'\widehat{P}_l'],$$

which decays along past oriented causal curves and diverges as the future Cauchy horizon is approached.

Reference: P. Denaro and G.D. Phys. Rev. D **92**, no. 2, 024017 (2015) [arXiv:1505.01092 [gr-qc]].

Conclusions

- The timelike singularity within the RN BH lies in the gravitationally unstable region III of the spacetime, which can be discarded within GR by appealing to stability arguments.
- The Schwarzschild and RN Naked Singularities are also gravitationally unstable, then unphysical.
- There is a notion of classical instability even for the “totally vicious" KIII and KNS, which contain CCC and the ring timelike singularity, these can also be discarded within GR.
- Cauchy Horizon instabilities indicate that realistic BHs approaching a stationary Kerr-Newman BH develop a null/spacelike curvature singularity, the unstable and causally problematic inner regions are cut out leaving a causally “healthy", globally hyperbolic spacetime.
  The singularity lies within the BH horizon, as in the Schwarzschild case, its treatment requires going beyond classical GR.

Thank you for your attention

List of invited speakers includes:
Iván Agullo, Louisiana State University, USA.
Parameswaran Ajith, International Centre for Theoretical Sciences, India.
Lars Anderson, Albert Einstein Institute, Germany.
Florian Beyer*, University of Otago, New Zealand.
Fabrizio Canfora, CECS, Chile.
Federico Carrasco, Universitat de les Illes Balears, España.
Ana Laura García Perciante, Universidad Autónoma Metropolitana, Mexico.
José Jaramillo, Institut de Mathematiques de Bourgogne , France.
Helmut Friedrich*, Max Planck institute for gravitational Physics, Germany.
Luis Lehner, Perimeter Institute for Theoretical Physics, Canada.
Steven Liebling, Long Island University, USA.
Carlos Palenzuela*, Universitat de les Illes Balears, España.
Alejandro Pérez, Universitè de Marseille, France.
Frans Pretorius, Princeton University, USA.
Jorge Pullin, Louisiana State University, USA.
István Rácz, Wigner RCP, Hungary.
Martín Reiris, Centro de Matemática, Udelar, Montevideo, Uruguay.
Adam Rogers, Brandon University, Canada.
Olivier Sarbach*, Universidad Michoacana de San Nicolás de Hidalgo, Mexico.
Daniel Siegel, Columbia University, USA.

* to be confirmed