Mini-couse on Dark Matter

Quantum Summer 2025

Iriri, ES, Brazil. April 07-11, 2025.

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Outline

- **Lecture 1:** Evidence for Dark Matter (DM) and its properties
- **Lecture 2:** DM inhomogeneities in the standard CDM model and others (WDM, SI-WDM)
- **Lecture 3:** Ultralight DM (spin 0 and 1). Can the spin of ULDM be distinguished on cosmological scales?

Lecture 2: DM inhomogeneities in the standard CDM model and others (WDM, SI-WDM).

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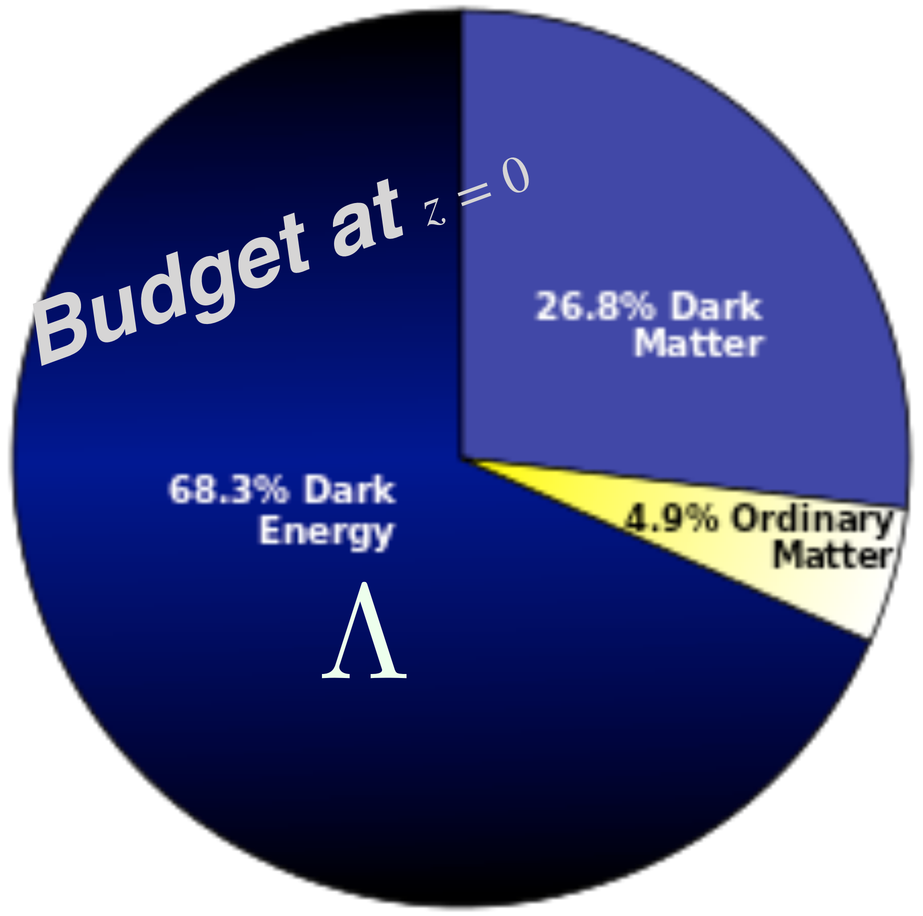
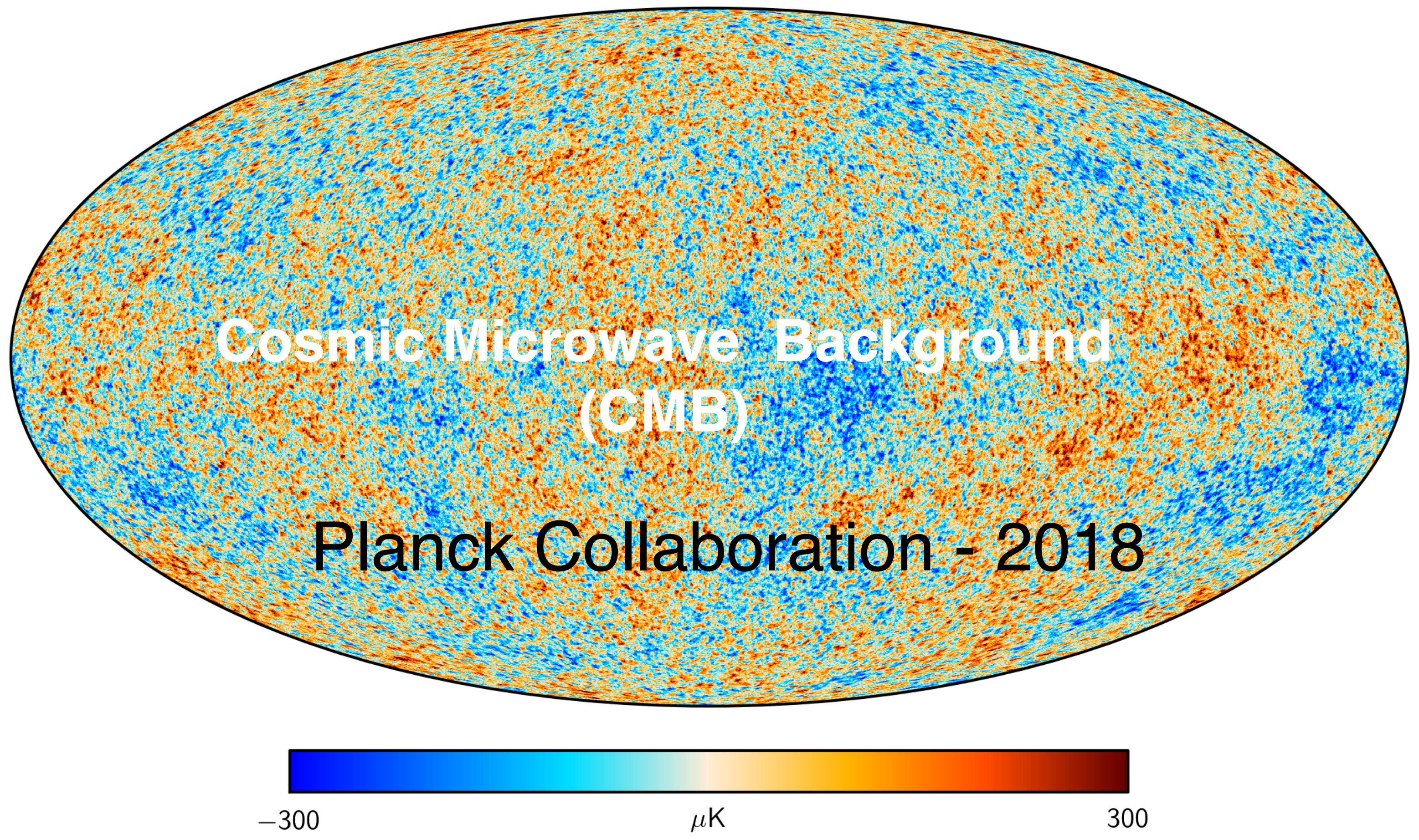
The standard cosmological model:

Homogeneous

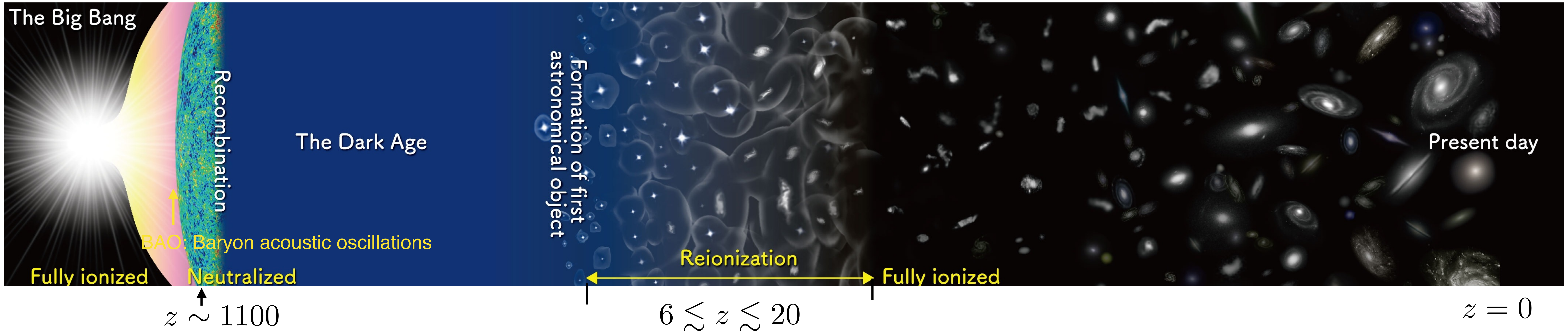


Hypothesis:
General Relativity,
standard physical laws,
statistically isotropic
and homogeneous

Perturbations



Evolution of primordial perturbations with characteristic properties: gaussian and adiabatic initial conditions



Linear evolution of the initial perturbations

- ☹ Only applies to large scales
- ☺ Theory well understood (precise numerical codes, such as:
- ☺ Preserve many properties of the initial state: eg. gaussianity

- CAMB: https://camb.info
- CLASS: https://lesgourg.github.io/class_public/class.html

Cosmological perturbations - Linear regime

$$dt = a(\tau)\, d\tau$$

- **FLRW background metric (in conformal time τ):**

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \delta_{ij}\, dx^i dx^j \right]$$

- **Metric perturbations: Scalar-Vector-Tensor decomposition**

$$g_{00} = -a^2(1 - h_{00}) = -a^2(1 + 2A)$$

$$g_{0i} = a^2 h_{0i} = -a^2[B_{,i} + B_i]$$

$$g_{ij} = a^2[\delta_{ij} + h_{ij}] = a^2[(1 + 2D)\delta_{ij} - 2E_{,ij} + V_{i,j} + V_{j,i} + h_{ij}^{TT}]$$

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10 components of $h_{\mu\nu}$ split according 3 dimensional rotations:

-4 scalars: A, B, D, E

-4 vectorial: B_i, V_i $B_i^{;i} = V_i^{;i} = 0$

-2 tensorial: h_{ij}^{TT}, $h_{ii}^{TT} = h_{ij}^{TT;i} = h_{ij}^{TT;j} = 0$

Cosmological perturbations - *Linear regime*

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10 components of $h_{\mu\nu}$ split according 3 dimensional rotations:

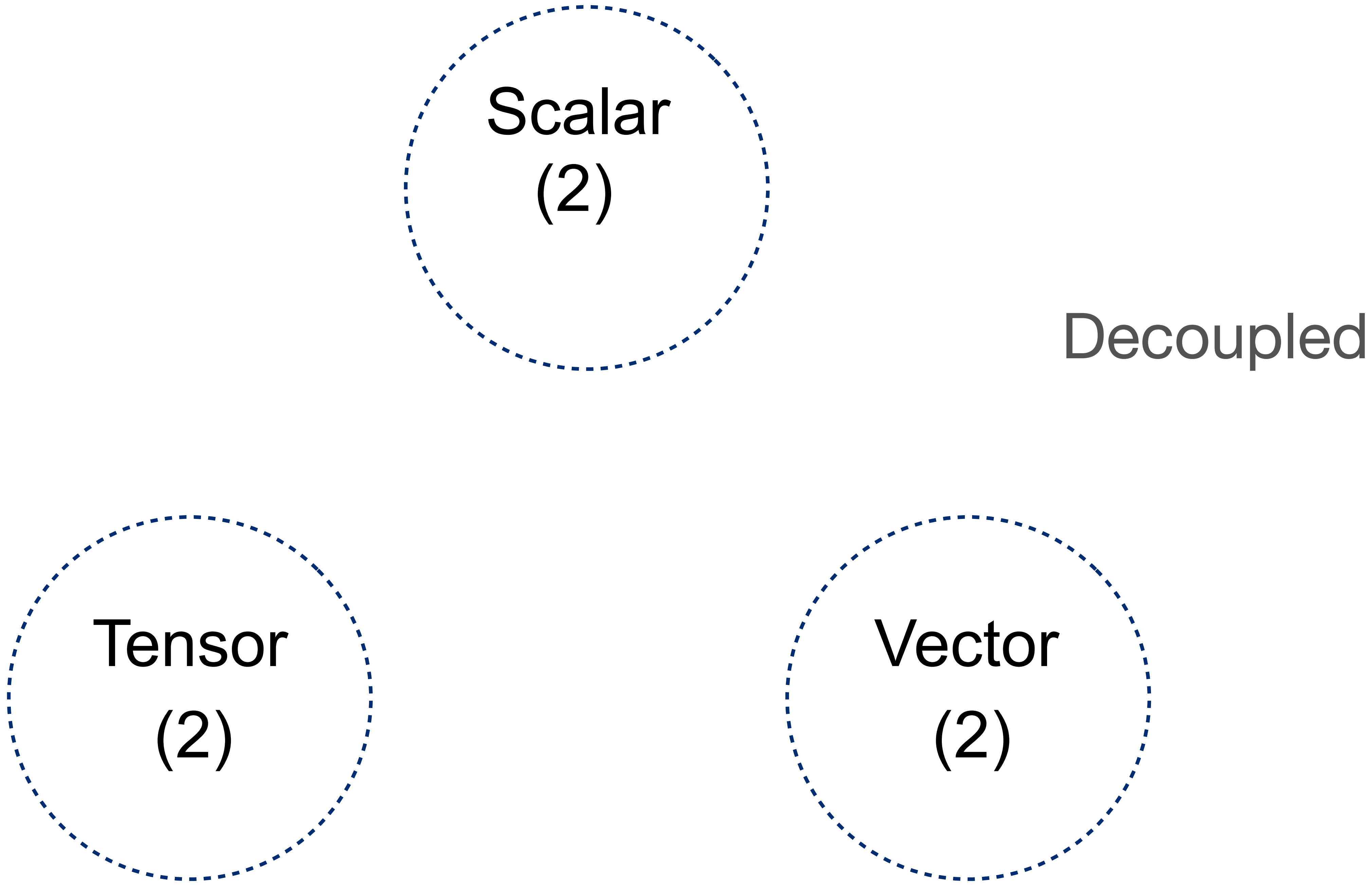
-4 scalars: A, B, D, E $\longrightarrow$ 2 scalars

Fixing coordinates

-4 vectorial: B_i, V_i $B_i^{;i} = V_i^{;i} = 0$ $\longrightarrow$ 2 vectorial

-2 tensorial: h_{ij}^{TT}, $h_{ii}^{TT} = h_{ij}^{TT;i} = h_{ij}^{TT;j} = 0$

Scalar-Vector-Tensor decomposition -Isotropic FLRW background-



The different sectors evolve independently in linear perturbation theory

Linear regime - Scalar sector

Metric perturbations in Fourier space

Synchronous gauge:

Two scalar perturbations

$$\delta g_{00} = \delta g_{0i} = 0 \qquad \delta g_{ij} = a^2 \left[-2\delta_{ij}\eta + \hat{k}_i \hat{k}_j (h + 6\eta) \right]$$

Newtonian gauge:

Two scalar perturbations

$$\delta g_{00} = -2a^2\phi, \quad \delta g_{0i} = 0, \quad \delta g_{ij} = -2a^2\delta_{ij}\psi$$

Linear regime - Scalar sector

Metric perturbations in Fourier space

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Newtonian potential ϕ

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Two scalar perturbations

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Newtonian potential ϕ

Curvature perturbations ψ: $a(t) \to a(\mathbf{x}, t) = a(t)\sqrt{1 + 2\psi(\mathbf{x}, t)}$

Fundamental equations for standard cosmological perturbations

$$G_{\mu\nu} = 8\pi G \sum_I T_{I_{\mu\nu}},$$

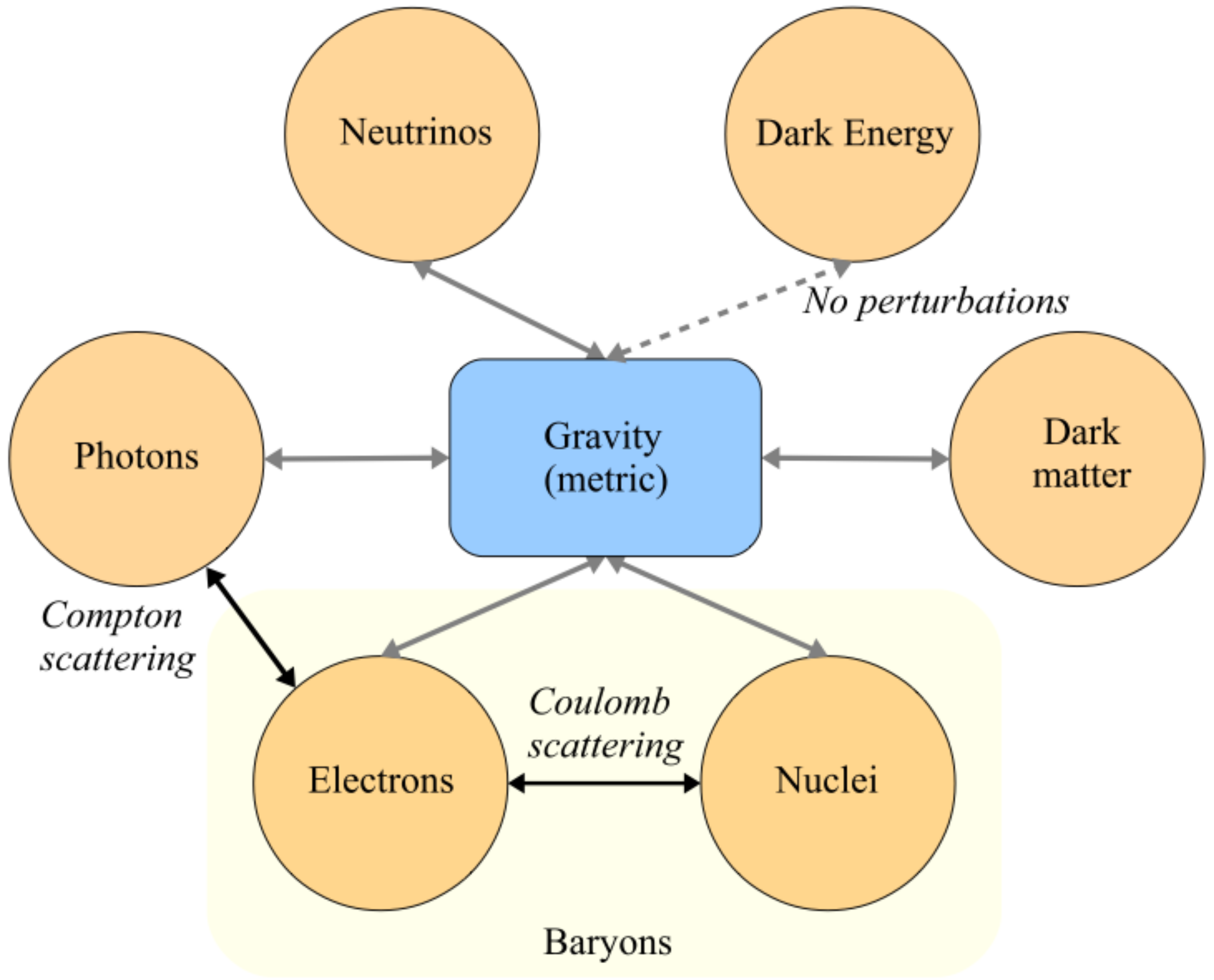
Gravity theory

Macroscopic models for matter and its interactions

Perturbed *Einstein's eqns*

$$\delta G^{\mu}{}_{\nu} = 8\pi G \sum_I \delta T^{\mu}_{I\ \nu}$$

$$\delta G^{\mu}{}_{\nu} = \delta G^{\mu}{}_{\nu}(\delta g_{00}, \delta g_{i0}, \delta g_{ij})$$



See textbook Dodelson & Schmidt or Ma & Bertschinger Astrop. J. 1995

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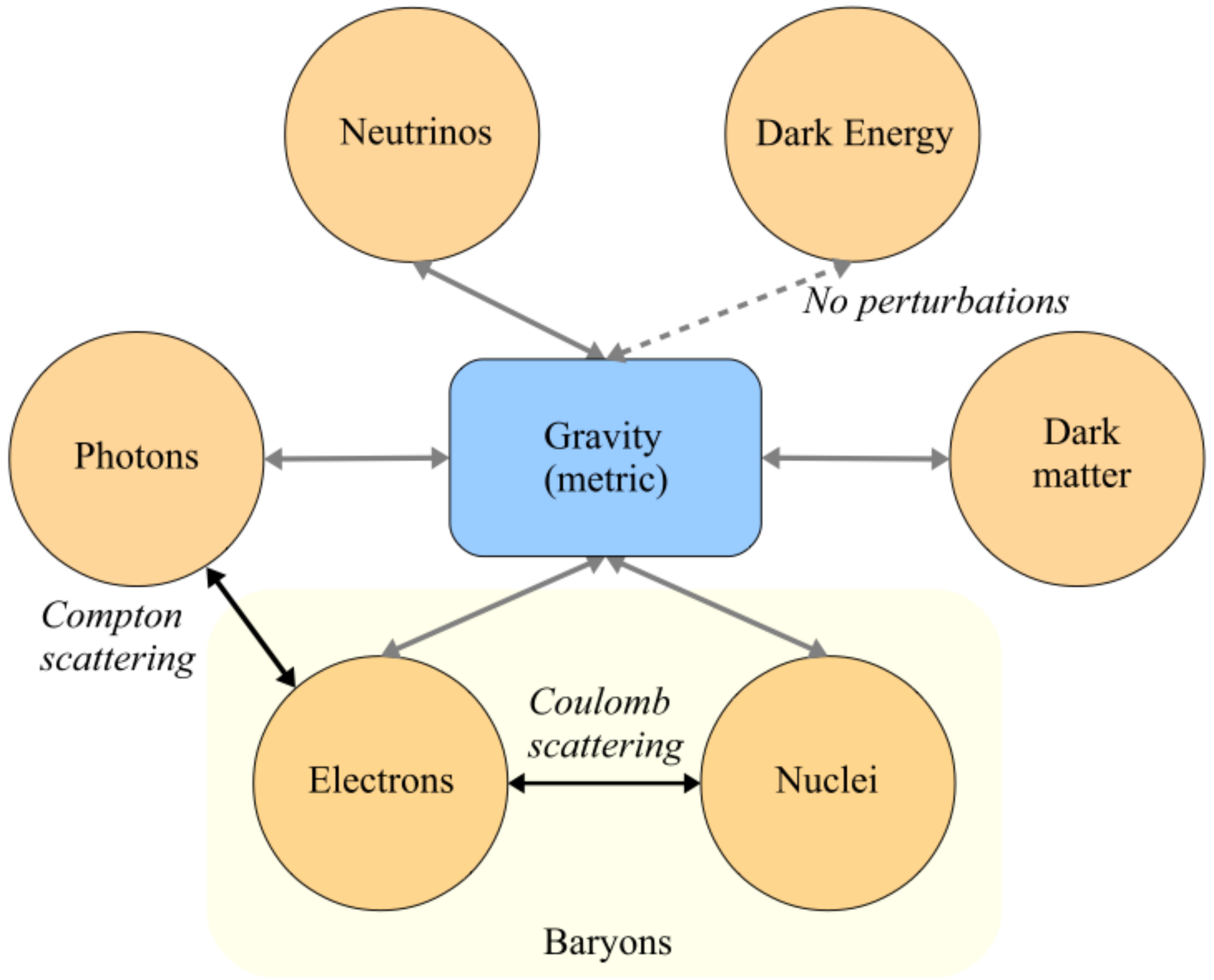

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Eg. Boltzmann's eqns. (statistic model) (*)

$$\frac{df}{dt} = C[f]$$

$$T^{\mu}{}_{\nu}(\boldsymbol{x}, t) = \frac{g}{\sqrt{-\det[g_{\alpha\beta}]}} \int \frac{dP_1 dP_2 dP_3}{(2\pi)^3} \frac{P^{\mu} P_{\nu}}{P^0} f(\boldsymbol{x}, \boldsymbol{p}, t)$$

$P^{\mu} =$ Comoving momenta

(*)For a formal derivation see for instance The Relativistic Bolzmann Equation: Theory and Applications, C. Cercignani, G, M. Kremer, Birkhäuser Basel, 2002.

- **Boltzmann Eq. in coordinates:**

Geodesic Eq.

Comoving momenta $P^\mu = \frac{dx^\mu}{d\lambda}$ $\qquad$ $\frac{dP^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} P^\alpha P^\beta = 0,$ with $g_{\mu\nu} P^\mu P^\nu = -m^2$

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$$\frac{Df}{D\lambda} = \frac{dx^\alpha}{d\lambda}\frac{\partial f}{\partial x^\alpha} + \frac{dP^\alpha}{d\lambda}\frac{\partial f}{\partial P^\alpha} = P^\alpha \frac{\partial f}{\partial x^\alpha} - \Gamma_{\alpha\beta} P^\alpha P^\beta \frac{\partial f}{\partial P^\alpha} = C[f]$$

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In newtonian gauge:

$$g_{\mu\nu} P^\mu P^\nu = -(1+2\phi)a^2(P^0)^2 + p^2 = -m^2 \rightarrow P^0 = \frac{\sqrt{p^2+m^2}}{a\sqrt{1+2\phi}} = \frac{E}{a}(1-\psi)$$

$$p^2 = g_{ij} P^i P^j$$

$$E = \sqrt{p^2+m^2}$$

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$p^2 = g_{ij}P^iP^j$ $\qquad E = \sqrt{p^2+m^2}$

$$P^\mu = \left[\frac{E}{a}(1-\psi), p^i\frac{1-\psi}{a}\right],$$ with $p^i = p\hat{n}^i$ and $\hat{n}^i = \hat{n}_i$.

- **Boltzmann Eq. in coordinates:**

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$$g_{\mu\nu} P^\mu P^\nu = -(1+2\phi)a^2 (P^0)^2 + p^2 = -m^2 \to P^0 = \frac{\sqrt{p^2+m^2}}{a\sqrt{1+2\phi}} = \frac{E}{a}(1-\psi)$$

$$p^2 = g_{ij} P^i P^j \qquad E = \sqrt{p^2+m^2}$$

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$$P^0 = \frac{d\tau}{d\lambda} \qquad P^i = \frac{dx^i}{d\lambda} = \frac{d\tau}{d\lambda}\frac{dx^i}{d\tau} = P^0 \frac{dx^i}{d\tau} \to \quad \frac{dx^i}{d\tau} = \frac{P^i}{P^0}$$

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...calculations...

$$\boxed{\frac{\partial f}{\partial \tau} + \frac{p}{E}\hat{n}^i \frac{\partial f}{\partial x^i} - \left[\mathcal{H} - \psi' + \frac{E}{p}\hat{n}^i \frac{\partial \phi}{\partial x^i}\right] p \frac{\partial f}{\partial p} = \frac{C[f]}{P^0}}$$

Energy-momentum perturbations

Fourier space

$$\frac{\partial}{\partial x^i} \to ik^i$$

- **Perturbed DF**

$$f(k^i, p_j, \tau) = f^{(0)}(p, \tau) + f^{(1)}(k^i, p, n_j, \tau)$$

$$\vec{p} = p\,\hat{n}$$

- **Boltzmann Eq.**

$$P^\alpha \frac{\partial f}{\partial x^\alpha} - \Gamma_{\alpha\beta} P^\alpha P^\beta \frac{\partial f}{\partial P^\alpha} = C[f]$$

$$T^0{}_0 = -\int p^2 dp d\Omega\, E\,(f^{(0)} + f^{(1)}) := -\bar{\rho} - \delta\rho$$

$$T^0{}_i = \int p^2 dp d\Omega\, p\, n_i\, f^{(1)} := (\bar{\rho} + \bar{p})v_i$$

$$T^i{}_j = \int p^2 dp\, d\Omega\, \frac{p^2 n_i n_j}{E} (f^{(0)} + f^{(1)}) := (\bar{p} + \delta p)\delta^i_j + \Sigma^i{}_j \ \ (\Sigma^i{}_i = 0)$$

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- **Perturbed Boltzmann Eq.**

- Background:

$$\frac{\partial f^{(0)}}{\partial \tau} - \mathscr{H} p \frac{\partial f_c^{(0)}}{\partial p} = a \frac{C[f]^{(0)}}{E}$$

$$\left(\mathscr{H} = \frac{a'}{a} = aH \right)$$

Hubble rate

$$T^0{}_0 = -\int p^2 dp d\Omega\, E\, (f^{(0)} + f^{(1)}) := -\bar{\rho} - \delta\rho$$

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- **Perturbed Boltzmann Eq.**

- Background:

$$\frac{\partial f^{(0)}}{\partial \tau} - \mathscr{H} p \frac{\partial f_c^{(0)}}{\partial p} = a \frac{C[f]^{(0)}}{E}$$

$$\left(\mathscr{H} = \frac{a'}{a} = aH\right)$$

Hubble rate

- Linear:

In newtonian gauge:

$$\frac{\partial f^{(1)}}{\partial \tau} + i\frac{pk}{E}(\hat{k}\cdot\hat{n}) f^{(1)} - \mathscr{H} p \frac{\partial f^{(1)}}{\partial p} + \left[\psi' - i\frac{Ek}{p}(\hat{k}\cdot\hat{n})\phi\right] p \frac{\partial f^{(0)}}{\partial p} = \left(\frac{C[f]}{P^0}\right)^{(1)}$$

In synchronous gauge:

$$\frac{\partial f^{(1)}}{\partial \tau} + i\frac{pk}{E}(\hat{k}\cdot\hat{n}) f^{(1)} - \mathscr{H} p \frac{\partial f^{(1)}}{\partial p} + \left[\eta' - \frac{h' + 6\eta'}{6}(\hat{k}\cdot\hat{n})^2\right] p \frac{\partial f^{(0)}}{\partial p} = \left(\frac{C[f]}{P^0}\right)^{(1)}$$

Boltzmann equation for DM

For collisionless massive particles (to linear order):

$$f(k^i, p_j, \tau) = f^{(0)}(p, \tau) + f^{(1)}(k^i, p, n_j, \tau)$$

Zeroth-order + perturbation

Boltzmann equation for DM

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Zeroth-order + perturbation

- Background:

$$\frac{\partial f^{(0)}}{\partial \tau} - \mathscr{H} p \frac{\partial f^{(0)}}{\partial p} = 0 \;\rightarrow f^{(0)}(\tau, p) = f_0(q),\; q = pa\,.$$

since $\quad \dfrac{\partial f^{(0)}}{\partial \tau} = a' p \dfrac{df^{(0)}}{dq} \quad$ and $\quad \mathscr{H} p \dfrac{\partial f^{(0)}}{\partial p} = \mathscr{H} pa \dfrac{df_0}{dq} = a' p \dfrac{df_0}{dq}$

Boltzmann equation for DM

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since $\frac{\partial f^{(0)}}{\partial \tau} = a' p \frac{d f^{(0)}}{dq}$ and $\mathscr{H} p \frac{\partial f^{(0)}}{\partial p} = \mathscr{H} p a \frac{d f_0}{dq} = a' p \frac{d f_0}{dq}$

It is convenient to change variables:

$q = ap$ $\quad \epsilon = aE = \sqrt{q^2 + m^2 a^2}$ $\quad \left(\bar{n} = a^{-3} \int q^2 dq d\Omega f_0(q)\right)$

$$\bar{\rho} = -T^0{}_0 = a^{-4} 4\pi \int q^2 dq\, \epsilon f_0(q) \simeq a^{-3} 4\pi m \int q^2 dq\, f_0(q) = m\bar{n}$$

Non relativistic:

$$v_p \ll 1 \rightarrow \; p \ll E \simeq m \rightarrow q \ll \epsilon \simeq ma$$

Cold (small velocity dispersion)

$$v_p := \frac{p}{E} = \frac{q}{\epsilon}$$

$$\bar{p} = \frac{1}{3} T^i{}_i = a^{-4} \frac{4\pi}{3} \int q^2 dq \frac{q^2}{\epsilon} f^{(0)}(q) = a^{-4} \frac{1}{3} \int q^2 dq \epsilon v_p^2 f^{(0)} \simeq a^{-3} \frac{4\pi}{3} m \int q^2 dq\, v_p^2 f^{(0)}(q) \ll \bar{\rho}$$

$$v_p \simeq a^{-1} q/m$$

Boltzmann equation for DM (Newtonian gauge)

- Linear:

$\mu = \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}$

$$f(k^i, p_j, \tau) = f_0(q) + F(k, q, \mu, \tau)$$

$q = ap \qquad \vec{q} = q\,\hat{n}$

$$\frac{\partial F}{\partial \tau} + i\frac{qk}{\epsilon}(\hat{k}\cdot\hat{n})F + \left[\psi' - i\frac{\epsilon k}{q}(\hat{k}\cdot\hat{n})\phi\right] p\frac{\partial f^{(0)}}{\partial p} = 0$$

$$v_p := \frac{p}{E} = \frac{q}{\epsilon}$$

Legendre polynomials

$$F(k, q, \mu, \tau) = \sum_{l=0}^{\infty} (-i)^l (2l+1) F_l(k, q, \tau) P_l(\mu)$$

$$F_l(\mathbf{k}, q, \tau) = \frac{i^l}{2}\int_{-1}^{1} d\mu P_l(\mu) F(k, q, \mu, \tau)$$

Useful formulas:

$$(l+1)\mathcal{P}_{l+1}(\mu) = (2l+1)\mu\mathcal{P}_l(\mu) - l\mathcal{P}_{l-1}(\mu)$$

$$\int_{-1}^{1} d\mu \mathcal{P}_l(\mu)\mathcal{P}_{l'}(\mu) = \delta_{ll'}\frac{2}{2l+1}$$

Boltzmann equation for DM (Newtonian gauge)

- Linear:

$\mu = \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}$

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$$q = ap \qquad \vec{q} = q\,\hat{n}$$

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$$v_p := \frac{p}{E} = \frac{q}{\epsilon}$$

$$F_0' + v_p k F_1 = -\psi' \frac{df_0}{d\log q}$$

$$F_1' + v_p \frac{k}{3}[2F_2 - F_0] = -\frac{k}{3v_p}\phi\frac{df_0}{d\log q}$$

$$F_l' = \frac{v_p k}{(2l+1)}[lF_{l-1} - (l+1)F_{l+1}],\ l > 2.$$

Legendre polynomials

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$$\int_{-1}^{1} d\mu \mathcal{P}_l(\mu)\mathcal{P}_{l'}(\mu) = \delta_{ll'}\frac{2}{2l+1}$$

Boltzmann equation for DM (Newtonian gauge)

- Linear:

$\mu = \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}$

$$f(k^i, p_j, \tau) = f_0(q) + F(k, q, \mu, \tau)$$

$q = ap \qquad \vec{q} = q\,\hat{n}$

$$\frac{\partial F}{\partial \tau} + i\frac{qk}{\epsilon}(\hat{k}\cdot\hat{n})F + \left[\psi' - i\frac{\epsilon k}{q}(\hat{k}\cdot\hat{n})\phi\right] p\frac{\partial f^{(0)}}{\partial p} = 0$$

$$v_p := \frac{p}{E} = \frac{q}{\epsilon}$$

$$F_0' + v_p k F_1 = -\psi' \frac{df_0}{d\log q}$$

$$F_1' + v_p\frac{k}{3}[2F_2 - F_0] = -\frac{k}{3v_p}\phi\frac{df_0}{d\log q}$$

$$F_l' = \frac{v_p k}{(2l+1)}[lF_{l-1} - (l+1)F_{l+1}],\ l > 2.$$

$F_l' \sim \mathscr{H} F_l \qquad \dfrac{kv_p}{\mathscr{H}} = \dfrac{kv_p}{aH} \ll 1 \rightarrow$ **Hierarchy**

Legendre polynomials

$$F(k, q, \mu, \tau) = \sum_{l=0}^{\infty}(-i)^l(2l+1)F_l(k, q, \tau)P_l(\mu)$$

$$F_l(\mathbf{k}, q, \tau) = \frac{i^l}{2}\int_{-1}^{1} d\mu P_l(\mu)F(k, q, \mu, \tau)$$

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$F_l' \sim \mathcal{H} F_l$

$$\frac{kv_p}{\mathcal{H}} = \frac{kv_p}{aH} \ll 1 \rightarrow$$ **Hierarchy** $\rightarrow$ **Only the first moments are relevant**

$$F_l \sim \left(\frac{kv_p}{aH}\right)^{l-2} F_2$$

Legendre polynomials

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Note: $$\frac{kv_p}{Ha} = 2\pi\frac{v_p \delta t_H}{\lambda_{phys}}$$

$\delta t_H = 1/H$

$\lambda_{phys} = a\lambda$

Effective fluid equations for DM

Keeping only the first two moments F_0 & F_1:

$$F_l(\mathbf{k}, q, \tau) = \frac{i^l}{2}\int_{-1}^{1} d\mu P_l(\mu) F(k, q, \mu, \tau)$$

$q \ll ma$ $\qquad$ $m/\bar{\rho} = 1/\bar{n}$

$$\delta = \frac{\delta\rho}{\bar{\rho}} \simeq \frac{m}{\bar{\rho}} a^{-3} \int q^2 dq d\Omega\, F = \frac{m}{\bar{\rho}} 4\pi a^{-3} \int q^2 dq\ F_0$$

$$\theta := ik^j v_j = \frac{ik}{\bar{\rho}} a^{-4} 2\pi \int q^3 dq \int_{-1}^{1} d\mu\, \mu\, F = \frac{4\pi k}{\bar{\rho}} a^{-4} \int q^3 dq\ F_1$$

$P_1(\mu) = \mu$

$v_p \ll 1 \rightarrow\ c_s^2 := \frac{\delta p}{\delta \rho} \ll 1$

$$\delta p := \frac{\delta T^i{}_i}{3} = \frac{1}{3} a^{-4} \int q^2 dq \int_{-1}^{1} d\mu\, v_p\, q\, F = \frac{4\pi}{3} a^{-4} \int q^2 dq\ v_p^2 \epsilon\, F_0 \simeq \frac{4\pi m}{3} a^{-3} \int q^2 dq\ v_p^2\, F_0$$

$$F_0' + v_p k F_1 = -\psi' \frac{df_0}{d\log q} \quad \rightarrow \quad \boxed{\delta' + \theta = -3\psi'}$$

$$F_1' - v_p \frac{k}{3} F_0 = -\frac{k}{3 v_p} \phi \frac{df_0}{d\log q} \quad \rightarrow \quad \boxed{\theta' + \mathscr{H}\theta - k^2 \frac{\delta p}{\bar{\rho}} = k^2 \phi} \longleftrightarrow \quad \theta' + \mathscr{H}\theta - k^2 c_s^2 \delta = k^2 \phi$$

Effective fluid eqns. for DM

Growth of structure and Jeans scale

Let's focus on "subhorizon" scales: $k \gg \mathscr{H}$, $k\phi \gg \psi'$ and define $c_s^2 := \dfrac{\delta p}{\delta \rho}$

$$\boxed{\begin{aligned} &\delta' + \theta = -3\psi' \\ &\theta' + \mathscr{H}\theta - k^2 c_s^2 \delta = k^2 \phi \end{aligned}}$$

These two first order eqns. can be combined to yield $\boxed{\delta'' + \mathscr{H}\delta' + k^2 c_s^2 \delta = -k^2 \phi}$

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During matter dominated (MD) era $\dfrac{\mathcal{H}^2}{a^2} = H^2 = \dfrac{8\pi G}{3}\bar{\rho}_{DM}$ (neglecting Baryons), and using Poisson's equations for ϕ,

$$\frac{\nabla^2 \phi}{a^2} = 4\pi G \bar{\rho}_{DM} \delta,$$

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$k > k_J$ Oscillations

$k < k_J$ Gravitational growth

The Jeans wavenumber:

$$k_J = \frac{\sqrt{4\pi G \bar{\rho}_{DM}}}{c_s}$$

Growth of structure in CDM model

$k_J \simeq 0$

Changing variables $\tau \to a$: $$a^2 \frac{\partial^2 \delta_c}{\partial a^2} + \frac{3}{2} a \frac{\partial \delta_c}{\partial a} - \frac{3}{2}\delta_c = 0$$

$$\delta_c = \frac{C_1}{a^{3/2}} + C_2 a$$ (MD era)

Decaying mode

Growing mode (attractor solution)

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During RD dominated era $$H^2 = \frac{8\pi G}{3}(\rho_r + \rho_{DM}) \sim \frac{8\pi G}{3}\rho_r \propto a^{-4}$$

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On "superhorizon" scales, $k/\mathscr{H} \ll 1$, it can be shown that $\delta_c \propto cte$ for adiabatic initial conditions.

How can DM be produced? What is the prediction for $f_0(q)$?

Traditional production of (WIMP) DM: E.g. $X_{DM} + \bar{X}_{DM} \leftrightarrow \psi_{SM} + \bar{\psi}_{SM}$

$T \qquad T_f \qquad T_{BB} \qquad T_0$

In equil. with SM radiation

$\Gamma(T) \gg H(T)$

$$f_{eq}(\vec{p}, \tau) = \frac{1}{\exp\left[\frac{E(p)-\mu(\tau)}{T(\tau)}\right] \pm 1}$$

Decoupling

$\Gamma(T_f) = H(T_f)$

"freeze-out" mechanism

$na^3 = \text{cte}$

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$f_{eq}(\vec{p},\tau) = \dfrac{1}{\exp\left[\frac{E(p)-\mu(\tau)}{T(\tau)}\right] \pm 1}$	"freeze-out" mechanism $\longrightarrow$	$f(\vec{p},\tau) \equiv f(\vec{p}\dfrac{a}{a_f}, t_f)$	$q = ap$

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Note: the macroscopic distribution of DM is sensitive to both the fundamental properties of DM (such as mass and spin) and its production mechanism in the early Universe.

Evolution of $f_0(q)$ and Self interaction Decoupling (*)

T_f $\qquad$ T_{SID} $\qquad$ T_0

Decoupling from the SM	Local equilibrium	SI Decoupling
	$\Gamma_{SI}(T) \gg H(T)$	$\Gamma_{SI}(T_{SID}) = H(T_{SID})$

(*) Based on:

- Yunis, Argüelles & DLN, JCAP 09(2020)041.
- Yunis, Argüelles, Scóccola, DLN & Giordano , JCAP 02(2022)024, Astron.Rep.65 (2021)1068.

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$$T_f \longleftarrow T_{SID} \longrightarrow T_0$$

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$\mathscr{C}_R = 1$ if the production is thermal.

$\mathscr{C}_R, \mathscr{C}_{NR}$ are phenomenological constants to enforce the proper abundance today Ω_{DM}

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The SI may affect the shape of the $f_0(q)$ but not the *collisional invariants* (that is, the total number, energy and momentum of the DM particles) and this fix the new coefficients and effective temperatures

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Characteristic scales

$$\langle v_p \rangle = \frac{\int q^2 dq\, d\Omega\, v_p\ f_0(q)}{\int q^2\, dq\, d\Omega f_0(q)}$$

The comoving free-streaming wavenumber (*)

(equiv. to the Jeans wavenumber for a collisionless system)

$$k_{FS} = \sqrt{4\pi G\bar{\rho}}\frac{a}{\langle v_p \rangle} = \sqrt{\frac{3}{2}}\frac{\mathscr{H}}{\langle v_p \rangle}$$

(*) See [Boyarsky et al. JCAP 05 (2009)012, arXiv:0812.0010] for details and further discussion on the relevant scales.

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E. g. When $v_p = q/\epsilon \sim q/(ma)$

$$f_0(q) = \frac{\mathscr{C}_R}{\exp\left[\frac{q}{T_{0,R}}\right] + 1} \to \langle v_p \rangle = \frac{7T_{0,R}\pi^4}{180\zeta[3]a\, m} \simeq \frac{3T_{0,R}}{a\, m} \qquad f_0(q) = \mathscr{C}_{NR} \exp\left[-\frac{q^2}{2mT_{0,NR}}\right] \to \langle v_p \rangle = \frac{2}{a}\sqrt{\frac{2T_{0,NR}}{\pi m}}$$

$\mathscr{C}_R, \mathscr{C}_{NR}$ are determined so that we obtain the correct abundance Ω_{DM}

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$$k_{FS} \propto \begin{cases} \sim \text{constant} & \text{RD } (a < a_{eq}) \\ \sim a^{1/2} & \text{MD } (a > a_{eq}) \end{cases}$$

Suppressed perturbations on scales with $k \gtrsim k_{FS}$

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Suppressed perturbations on scales with $k \gtrsim k_{FS}$

E. g. $$k^R_{FS} = \sqrt{4\pi G\bar{\rho}_{R,0}}\frac{1}{a\langle v_p \rangle} = \sqrt{\frac{\Omega_R}{g}}H_0\frac{m}{T_{0,R}} \simeq 7\left(\frac{m}{1keV}\right)\left(\frac{0.7}{h}\right)\left(\frac{T_{0,\nu}}{T_{0,R}}\right)\frac{h}{Mpc} \quad \text{(RD)}$$

Neutrino temperature today

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Structure formation: lineal theory

Total matter

$$\delta_m = \mathscr{R}(k)D(k,a)$$

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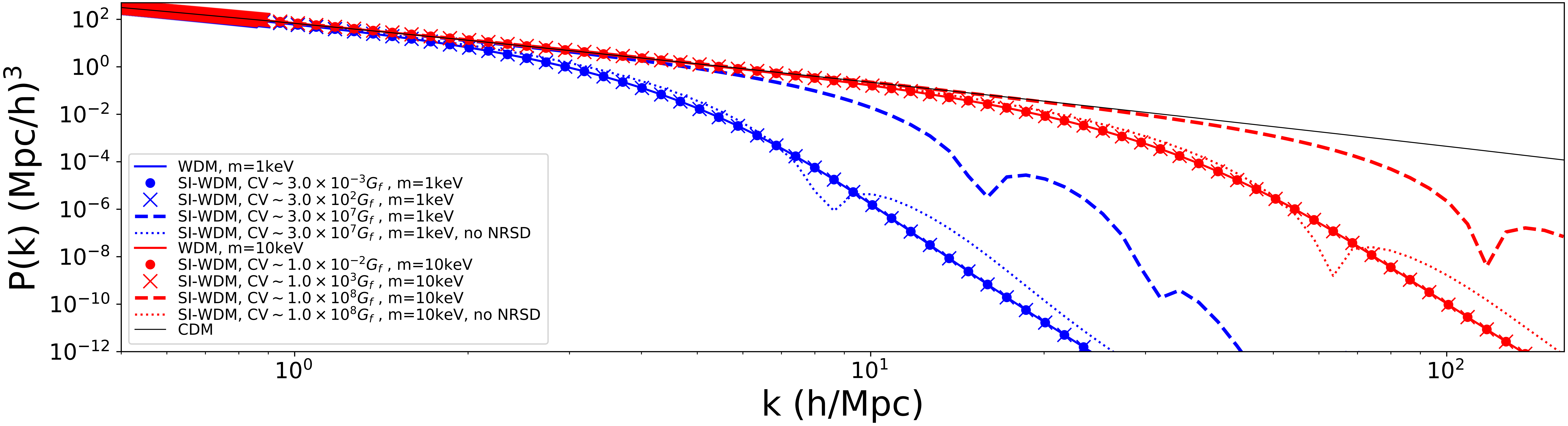
Effects on the Matter Power Spectrum (*)



Power Spectrum for a vector field mediated νMSM SI-WDM model (considering the effects of non-relativistic self decoupling (NRSD) when appropriate) with $T \sim (4/11)^{1/3} T_\gamma$. All WDM and SI-WDM models consider a non-resonant production scenario. Cases where the interaction is high enough to be in the NRSD regime but these effects are ignored, instead using a relativistic DF (dotted lines).

(*) See details in: - Yunis, Argüelles & DLN, JCAP 09(2020)041.
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Code available at: https://github.com/yunis121/siwdm-class

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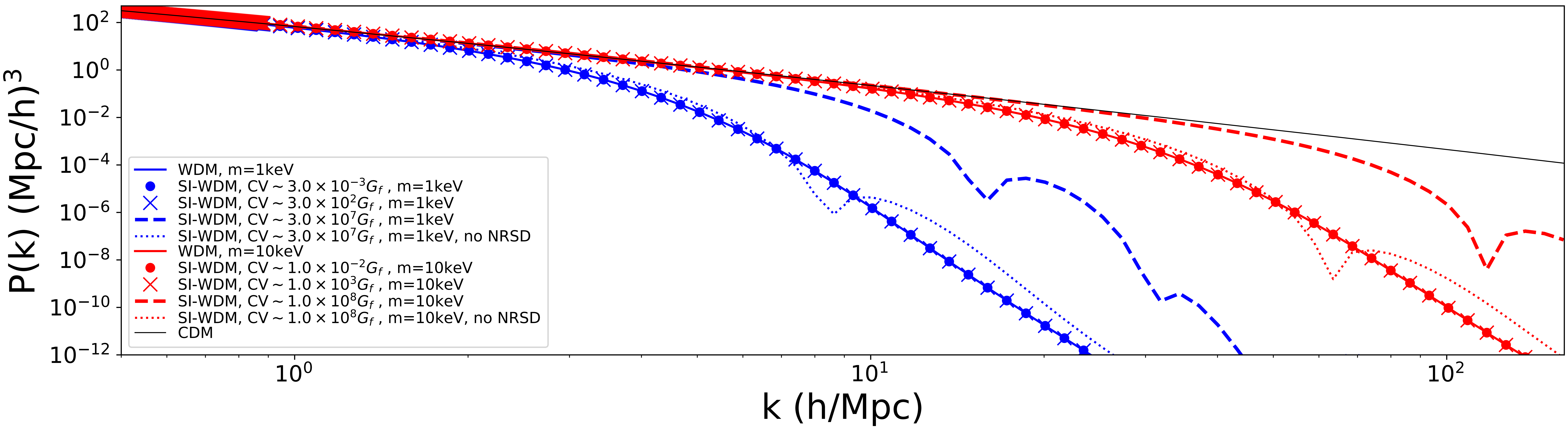

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A high interaction constant might make a SI-WDM model "colder" than its non-SI counterpart (as if corresponding to a higher particle mass), but the SI model show a distinctive oscillatory pattern at smaller k values.

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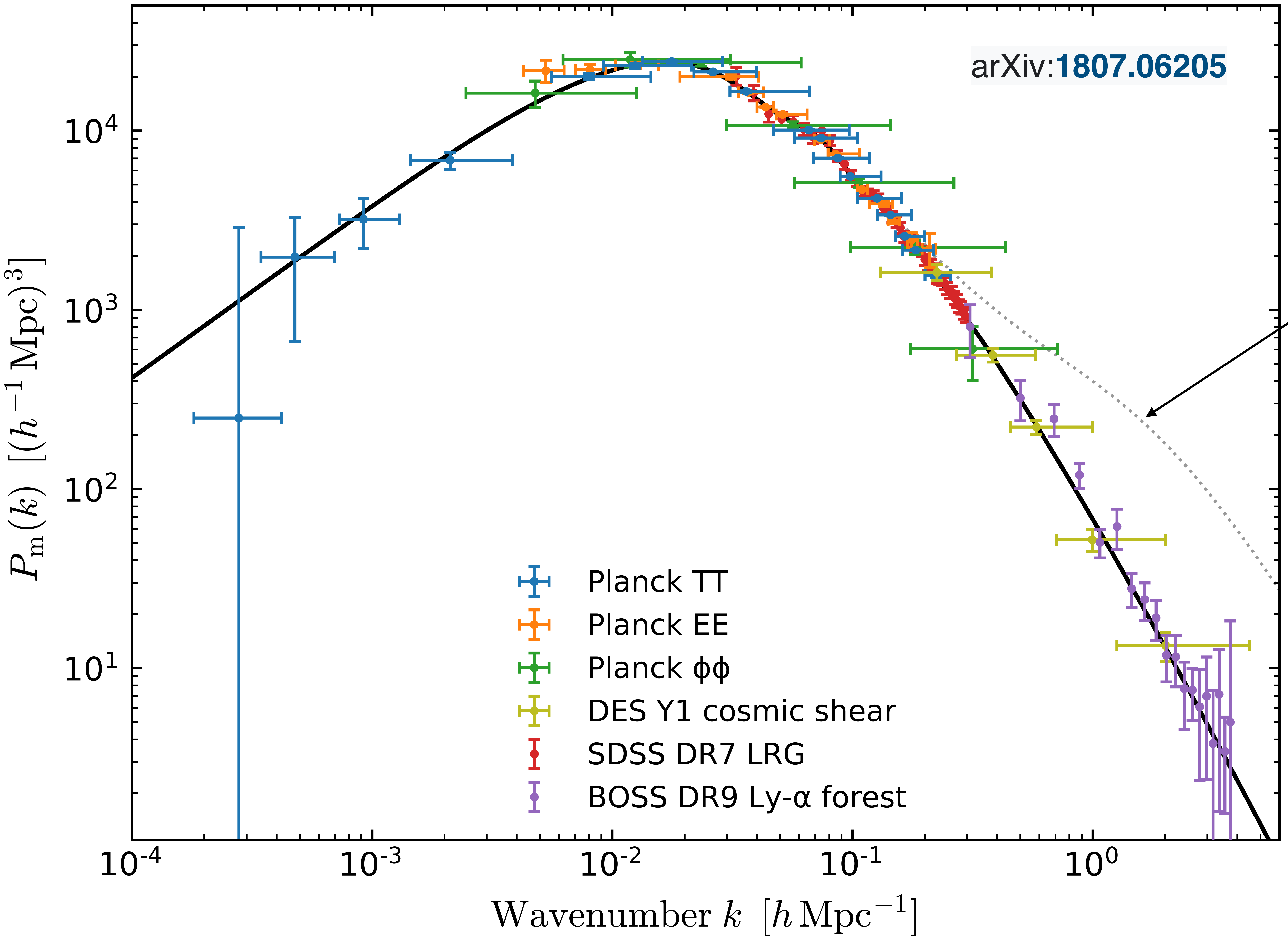
Matter Power Spectrum (linear theory-ΛCDM) at z=0 inferred from various observational data (the dotted line shows the impact of nonlinear effects)



$$P_L(k,a) = 2\pi^2 \mathscr{A}_s k^{-3} \left(\frac{k}{k_p}\right)^{n_s-1} |D(k,a)|^2$$

Impact of non-linear effects

The variance of linear density perturbations

$$\Delta_{\mathrm{L}}^2(k) = \frac{k^3 P_{\mathrm{L}}(k,a)}{2\pi^2}$$

Larger k's become nonlinear first

Structure formation beyond the linear regime

- Write macroscopic nonlinear equations for dark matter, such as the Vlasov-Poisson equations:

$$\frac{\partial f_m}{\partial \tau} + \frac{\partial f_m}{\partial x^i}\frac{p^i}{m} - \frac{\partial f_m}{\partial p^i}\left[\mathscr{H} p^j + m\frac{\partial \phi}{\partial x^j}\right] = 0, \qquad \nabla^2 \phi = 4\pi G a^2 \left[\int \frac{d^3p}{(2\pi)^3} f_m(x^i, p^j, t) - \rho_m(\tau)\right]$$

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- N body simulations.

- Implements a discretization of the phase space whose elements are characterized by $\mathbf{x}$ and $\mathbf{p}$

Eg. A simple model:

$$\frac{dx^i}{d\tau} = \frac{p^i}{m}, \qquad \frac{dp^i}{d\tau} = -\mathscr{H} p^i - \frac{m}{a}\frac{\partial \phi}{\partial x^i}$$

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Progress in this direction represents a great challenge!

End of Lecture 2: Thanks for your attention!!

Quantum Summer 2025

Iriri, ES, Brazil. April 07-11, 2025.

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