Perturbations in spherically symmetric solutions

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- *On Horizons and Wormholes in k-Essence Theories* (Bronnikov et al., 2016; arXiv:1511.08036)
- *Static, spherically symmetric solutions with a scalar field in Rastall gravity* (Bronnikov et al., 2016; arXiv:1606.06242)
- *Duality between k-essence and Rastall gravity* (Bronnikov et al., 2017; arXiv:1701.06662)

The k-essence theories can be defined by the following most general Lagrangian:

$$\mathcal{L} = \sqrt{-g}[R - F(X, \phi)]$$

Starting with the model

$$F(X) = F_0 X^n - 2\Lambda, \qquad F_0, \Lambda = const,$$

where

$$X = \eta \phi_{;\rho} \phi^{;\rho},$$

where $\eta = \pm 1$ is set to avoid ill-defined power-law.
Now, consider a general static, spherically symmetric metric

$$ds^2 = e^{2\gamma} dt^2 - e^{2\alpha} du^2 - e^{2\beta}(d\theta^2 + \sin^2\theta d\phi^2),$$

where γ, α and β functions of u only.

- Solution 1: $n = \frac{1}{3}$ e $\Lambda = 0$

$$ds^2 = \frac{B(x)}{k^2 x} dt^2 - \frac{k^2 x}{B(x)} dx^2 - \frac{1}{k^2 x} d\Omega^2, \qquad B(x) \;\; = B_0 - \frac{1}{2} k^2 x^4,$$

B_0 and k are constants.

- Solution 2: $n = \frac{1}{2}$

$$ds^2 = \frac{(9b^2 - x^2)^2}{9b^4} dt^2 - \frac{9b^4}{(9b^2 - x^2)^2} dx^2 - \frac{3b^4}{9b^2 - x^2} d\Omega^2,$$

with $b = \sqrt{\frac{3}{\Lambda}}$

Rastall gravity

Rastall gravity field equations

$$R^{\nu}_{\mu} - \frac{\lambda}{2}\delta^{\nu}_{\mu}R = -T^{\nu}_{\mu},$$

$$\nabla_{\nu}T^{\nu}_{\mu} = \frac{\lambda - 1}{2}\partial_{\mu}R,$$

where RG is recovered with $\lambda = 1$.

$$G^{\nu}_{\mu} = -\left\{T^{\nu}_{\mu} - \frac{b-1}{2}\delta^{\nu}_{\mu}T\right\} \equiv -\widetilde{T}^{\nu}_{\mu},$$

$$\nabla_{\nu}T^{\nu}_{\mu} = \frac{b-1}{2}\partial_{\mu}T, \quad b := \frac{3\lambda - 2}{2\lambda - 1}, \quad T = T^{\alpha}_{\alpha}.$$

In this parameterization, RG is recovered with $b = 1$.

- Solution for $b = -1$:

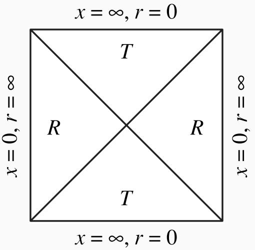
$$ds^2 = A(x)dt^2 - A(x)^{-1}dx^2 - \sqrt{\frac{3}{2C^2}}\frac{1}{x}d\Omega^2,$$

with $A = \frac{K}{x} - \frac{C}{\sqrt{6}}x^3$.

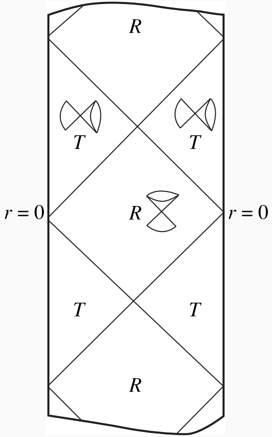
- Solution for $b = 0$:

$$ds^2 = \frac{9b^4}{\cosh^4(bu)}dt^2 - du^2 - \frac{\cosh^2(bu)}{3b^2}d\Omega^2,$$

with $b = \sqrt{\frac{\Lambda}{3}}$



Solution for $n = 1/3$ / $b = -1$.



Solution for $n = 1/2$ / $b = 0$.

Perturbation in spherically symmetric solutions

Adopting, now, a spherically symmetric metric

$$ds^2 = e^{2\gamma(u,t)}dt^2 - e^{2\alpha(u,t)}du^2 - e^{2\beta(u,t)}d\Omega^2,$$

Considering the perturbed quantities

$$\alpha(u,t) = \alpha(u) + \delta\alpha(u,t),$$

$$\beta(u,t) = \beta(u) + \delta\beta(u,t),$$

$$\gamma(u,t) = \gamma(u) + \delta\gamma(u,t),$$

$$\phi(u,t) = \phi(u) + \delta\phi(u,t)$$

$$F_X e^{\alpha+2\beta-\gamma}\ddot{\phi} - \left(F_X e^{-\alpha+2\beta+\gamma}\phi'\right)' = 0,$$

$$-e^{2\alpha-\beta} + \beta'(\beta' + 2\gamma') = (\frac{1}{2}F - XF_X)e^{2\alpha},$$

$$-e^{2\alpha-2\beta} + 2\beta'' + 3\beta'^2 - 2\alpha'\beta' = \frac{1}{2}Fe^{2\alpha},$$

$$e^{2\alpha-2\beta} + e^{2\alpha-2\gamma}\ddot{\beta} - \beta''\beta'(\gamma' - \alpha' + 2\beta') = -\frac{1}{2}e^{2\alpha}(F - XF_X),$$

$$\dot{\beta}' + \dot{\beta}\beta' - \dot{\alpha}\beta' - \dot{\beta}\gamma' = \frac{1}{2}F_X\dot{\phi}\phi'.$$

and perturbed quantities for X and $F(X,\phi)$ are given by

$$\delta X = 2e^{-2\alpha}(\phi'\delta\phi' - \phi'^2\delta\alpha), \quad F(X,\phi) = F(X) + F_X\delta X + F_\phi\delta\phi,$$

$$F_X(X,\phi) = F_X(X) + F_{XX}\delta X + F_{X\phi}\delta\phi.$$

Perturbed scalar field equation

$$-e^{2\alpha-2\gamma}\delta\ddot{\phi}+\delta\phi''+\phi'(2\delta\beta'+\delta\gamma'-\delta\alpha')$$

$$(2\beta'+\gamma'-\alpha')\delta\phi'+\frac{F_X'}{F_X}\delta\phi'+\phi'\delta\left(\frac{F_X'}{F_X}\right)=0,$$

Adopting $\delta\beta=0$

$$-e^{2\alpha-2\gamma}\delta\ddot{\phi}+\delta\phi''+\phi'(\delta\gamma'-\delta\alpha')$$

$$+(2\beta'+\gamma'-\alpha')\delta\phi'+\frac{F_X'}{F_X}\delta\phi'+\phi'\delta\left(\frac{F_X'}{F_X}\right)=0,$$

or

$$-e^{2\alpha-2\gamma}\delta\ddot{\phi}+(2n{-}1)\delta\phi''+\delta\phi'[2\beta'+\gamma'-(2n-1)\alpha']$$

$$+\phi'[\delta\gamma'-(2n-1)\delta\alpha']=0.$$

Schrödinger-like equation

We can rewrite $\delta\phi$ equation

$$- e^{2\alpha-2\gamma}\delta\ddot{\phi} + (2n-1)\delta\phi'' + \delta\phi'[\sigma' - 2(n-1)\alpha'] - \frac{n^2}{\beta'^2}(e^{-2\beta} - \Lambda)e^{4\alpha}F_0 X^n \delta\phi = 0.$$

Using transformations

$$\frac{du}{dz} = e^{\gamma-\alpha}, \qquad \delta\phi = \Psi(z)e^{i\omega t}, \qquad \Psi(z) = f(z)\psi(z),$$

with

$$f(z) = \exp\left[-\frac{\beta + (1-n)\gamma}{2n-1}\right].$$

We found the Schrödinger-like equation

$$(2n-1)\frac{d^2\psi}{dz^2} + [\omega^2 - V(x)]\psi(z) = 0,$$

Solution for $n = 1/3$

Replacing the solution for $n = 1/3$, we obtain

$$\psi_{zz} - \left[3\omega^2 + V(z)\right]\psi = 0,$$

$$\text{with} \quad V(x) = -F_1 B(x) + \frac{5B_0^2}{4k^4x^4} + \frac{31B_0}{4k^2} - \frac{3x^4}{16}.$$

The potential, near $x = 0$

$$z \propto x^2 \;\; \text{as} \;\; x \to 0, \quad V(z) \approx \frac{5B_0}{4k^4x^4} \propto .\frac{1}{z^2}.$$

On the other hand, near $x = x_h$,

$$z \propto -\ln(x_h - x) \to \infty, \qquad V(z) \to -\frac{8B_0}{k^2}.$$

What about ψ?

$$\psi \sim e^z \;\; \text{as} \;\; z \to \infty, \quad \psi \sim z^{-1/4} \;\; \text{as} \;\; z \to 0.$$

Solution for $n = 1/3$

- We see that these requirements are much milder than could be imposed on a quantum-mechanical wave function.
- Therefore, ω^2 can take negative values arbitrarily large in absolute value, $e^{|\omega|t}$, and thus our static configuration is catastrophically unstable.

Also, the expression of sound speed for k-essence is

$$c_s{}^2 = \frac{F_X}{F_X + 2XF_{XX}},$$

or, in our model,

$$c_s{}^2 = \frac{1}{2n-1},$$

where c_s is imaginary for $n < 1/2$.

Field equations
Rastall

$$-e^{2(\alpha-\gamma)}(\ddot{\alpha}+2\ddot{\beta})+\gamma''+\gamma'(\gamma'-\alpha'+2\beta') = -\frac{\epsilon}{2}(1-a)\phi'^2-(3-2a)e^{2\alpha}V,$$

$$-e^{2(\alpha-\gamma)}\ddot{\alpha}+\gamma''+2\beta''-\alpha'(\gamma'+2\beta')+\gamma'^2+2\beta'^2 = -\frac{\epsilon}{2}(3-a)\phi'^2-(3-2a)e^{2\alpha}V,$$

$$-e^{2(\alpha-\gamma)}\ddot{\beta}+\beta''+\beta'(\gamma'-\alpha'+2\beta')-e^{2(\alpha-\beta)} = -\frac{\epsilon}{2}(1-a)\phi'^2-(3-2a)e^{2\alpha}V,$$

$$\dot{\beta}'+(\beta'-\gamma')\dot{\beta}-\beta'\dot{\alpha} = -\frac{\epsilon}{2}\phi'\dot{\phi},$$

$$a\phi''+[\gamma'-a\alpha'+2\beta']\phi'-e^{2(\alpha-\gamma)}\ddot{\phi} = \epsilon(3-2a)e^{2\alpha}V_\phi$$

$$-e^{2(\alpha-\gamma)}(\delta\ddot{\alpha}+2\ddot{\delta}\beta)+\delta\gamma''+\delta\gamma'(\gamma'-\alpha'+2\beta')+\gamma'(\delta\gamma'-\delta\alpha'+2\delta\beta')$$
$$=-\epsilon(1-a)\phi'\delta\phi'-e^{2\alpha}(2\delta\alpha W+W_\phi\delta\phi),$$

$$-e^{2(\alpha-\gamma)}\delta\ddot{\alpha}+\delta\gamma''+2\delta\beta''-\delta\alpha'(\gamma'+2\beta')-\alpha'(\delta\gamma'+2\delta\beta')+2\delta\gamma\gamma'+4\beta'\delta\beta'$$
$$=-\epsilon(3-a)\phi'\delta\phi'-e^{2\alpha}(2\delta\alpha W+W_\phi\delta\phi),$$

$$-e^{2(\alpha-\gamma)}\delta\ddot{\beta}+\delta\beta''+\delta\beta'(\gamma'-\alpha'+2\beta')+\beta'(\delta\gamma'-\delta\alpha'+2\delta\beta')-2e^{2(\alpha-\beta)}(\delta\alpha-\delta\beta)$$
$$=-\epsilon(1-a)\phi'\delta\phi'-e^{2\alpha}(2\delta\alpha W+W_\phi\delta\phi),$$

$$\delta\dot{\beta}'+(\beta'-\gamma')\delta\dot{\beta}-\beta'\delta\dot{\alpha}=-\frac{\epsilon}{2}\phi'\delta\dot{\phi},$$

$$-e^{2(\alpha-\gamma)}\delta\ddot{\phi}+a\delta\phi''+[\gamma'-a\alpha'+2\beta']\delta\phi'+[\delta\gamma'-a\delta\alpha'+2\delta\beta']\phi'$$
$$=\epsilon e^{2\alpha}(2\delta\alpha W_\phi+W_{\phi\phi}\delta\phi),$$

where $W(\phi)=(3-2a)V(\phi)$.

From the previous equations we can find the expression

$$\delta\gamma' = \eta\delta\phi' - \eta'\delta\phi,$$

with $\eta = \epsilon\phi'/(2\beta')$.
However, $\delta\gamma'$ may be alternatively calculated, which gives

$$2\beta'\delta\gamma' = \epsilon a\phi'\delta\phi' + e^{2\alpha}\Big[2(e^{-2\beta} - W)\delta\alpha - W_\phi\delta\phi\Big],$$

The comparison between the two expressions for $\delta\gamma'$ gives us

$$(1-a)\phi'\delta\phi' = (1-a)[\phi'' - \alpha'\phi' + \eta\phi'^2]\delta\phi$$

and its integration with respect to u give us

$$\delta\phi = F(t)\phi' e^{\alpha+Q(u)}, \qquad Q(u) = \int \eta(u)\phi'(u)du,$$

Rastall perturbation: $b = -1$, $V = 0$

For this solution, we have the following master equation

$$e^{-4\gamma}\delta\ddot{\phi} + \delta\phi'' + \frac{2}{x}\delta\phi' = 0.$$

We obtain two expressions for $\delta\gamma'$

$$\delta\gamma' = \sqrt{3/2}\delta\phi' = -\sqrt{3/2}\delta\phi' - \sqrt{6}e^{2\alpha-2\beta}\delta\phi.$$

Which is integrated in relation to x, we obtemos

$$\delta\phi = F(t)\psi(x), \qquad \psi(x) = (K - (C/\sqrt{6})x^4)^{-1/2}.$$

By separation of variables, we also obtain

$$-\frac{\ddot{F}}{F} = \frac{\psi''}{\psi} + \frac{2}{x}\frac{\psi'}{\psi} = \mathrm{constant},$$

which is not satisfied by background solution.

Conclusion

- K-essence

  Both solutions, for $n = 1/3$ and $n = 1/2$, are unstable.

  On the instability of some k-essence space-times

  (Bronnikov et al., 2019; arXiv:1908.09126)

- Rastall gravity

  Stability analysis in the Rastall case is inconsistent: linear time-dependent spherically symmetric perturbations simply do not exist.

  Rastall's theory of gravity: Spherically symmetric solutions and the stability problem

  (Bronnikov et al., 2020; arXiv:2007.01945)

Obrigado!