Fluid and scalar field representation in the Brans-Dicke theory

Cosmological scenarios

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Index

Index

The original Brans-Dicke theory is defined by the Lagrangian

$$\mathcal{L} = \sqrt{-g}\left\{\Phi R - \omega \frac{\Phi_{;\rho}\Phi^{;\rho}}{\Phi}\right\} + \sqrt{-g}\mathcal{L}_m, \tag{1}$$

and a flat FLRW metric

$$ds^2 = dt^2 - a(t)dx^i dx^j \tag{2}$$

For the Matter component $\mathcal{L}_m$ two models are analyzed

- A perfect fluid

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} - g^{\mu\nu}p, \tag{3}$$

with state equation $p = \alpha\rho$.

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- A scalar field with self interaction

$$T_{\mu\nu} = \epsilon\left(\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\rho}\phi^{;\rho}\right) + g_{\mu\nu}V(\phi). \tag{4}$$

We can obtain a correspondence between the two models

$$
\begin{aligned}
8\pi\rho_\phi &= \epsilon\frac{\dot{\phi}^2}{2} + V(\phi), && (5)\\
8\pi p_\phi &= \epsilon\frac{\dot{\phi}^2}{2} - V(\phi). && (6)
\end{aligned}
$$

or,

$$
\begin{aligned}
\epsilon\dot{\phi}^2 &= 8\pi(\rho_\phi + p_\phi), && (7)\\
V(\phi) &= 8\pi(\rho_\phi - p_\phi), && (8)
\end{aligned}
$$

Index

The general equations of motion with a flat FLRW metric for the fluid

$$3H^2 = 8\pi\frac{\rho}{\Phi} + \frac{\omega}{2}\frac{\dot{\Phi}^2}{\Phi^2} - 3H\frac{\dot{\Phi}}{\Phi}, \tag{9}$$

$$\ddot{\Phi} + 3H\dot{\Phi} = \frac{8\pi}{3+2\omega}(\rho - 3p), \tag{10}$$

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{11}$$

For the scalar field

$$3H^2 = \frac{1}{\Phi}\left\{\epsilon\frac{\dot{\phi}^2}{2} + V(\phi)\right\} + \frac{\omega}{2}\frac{\dot{\Phi}^2}{\Phi^2} - 3H\frac{\dot{\Phi}}{\Phi}, \tag{12}$$

$$\ddot{\Phi} + 3H\dot{\Phi} = -\frac{8\pi}{3+2\omega}\left\{\epsilon\dot{\phi}^2 - 4V(\phi)\right\}, \tag{13}$$

$$\ddot{\phi} + 3H\dot{\phi} = -V_{\phi}(\phi). \tag{14}$$

Solutions Gurevich[a]

[a] L. E. Gurevich, A. M. Finkelstein e V. A. Ruban. "On the problem of the initial state in the isotropic scalar-tensor cosmology of Brans-Dicke". Em: *Astrophysics and Space Science* 22.2 (1973). ISSN: 0004640X. DOI: 10.1007/BF00647424.

$$a(t) = \bar{a}_0 t^{\bar{r}}, \tag{15}$$

$$\Phi(t) = \bar{\Phi}_0 t^{\bar{s}}, \tag{16}$$

$$\bar{r} = \frac{2[1+\omega(1-\alpha)]}{4+3\omega(1-\alpha^2)}, \tag{17}$$

$$\bar{s} = \frac{2(1-3\alpha)}{4+3\omega(1-\alpha^2)}. \tag{18}$$

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$$a(t) = \bar{a}_0 t^{\bar{r}}, \tag{15}$$

$$\Phi(t) = \bar{\Phi}_0 t^{\bar{s}}, \tag{16}$$

Einstein's Solution

$$a(t) = a_0 t^{A}, \tag{17}$$

$$\Phi(t) = \Phi_0 + B \ln \frac{t}{t_0}, \tag{18}$$

The initial conditions of the fluid are derived

$$8\pi\frac{\rho_0}{\Phi_0} = 2\left(3+2\omega\right)\frac{[4-6\alpha+3\omega(1-\alpha)^2]}{[4+3\omega(1-\alpha^2)]^2} \tag{19}$$

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For the scalar field we found the solutions in terms of the energy density

$$\begin{aligned}\phi &= \pm\phi_0 t^{\frac{z\bar{s}}{2}}, \quad \phi_0=\frac{\sqrt{32\pi(1+\alpha)\epsilon\rho_0}}{\bar{s}}, & (20)\\ V(t) &= V_0 t^{(\bar{s}-2)}, \quad V_0=4\pi(1-\alpha)\rho_0, & (21)\end{aligned}$$

and the potential in terms of the scalar field

$$V(\phi)=\bar{V}_0\phi^{2(\bar{s}-2)/\bar{s}}, \quad \bar{V}_0=\frac{V_0}{\phi_0^{2/\bar{s}}}. \tag{22}$$

Index

With the following conformal transformation

$$g_{\mu\nu} = \Phi^{-1}\tilde{g}_{\mu\nu}. \tag{23}$$

we obtain the Lagrangian in Einstein Frame

$$\mathcal{L} = \sqrt{-\tilde{g}}\left\{\tilde{R} - \eta\sigma_{;\rho}\sigma^{;\rho}\right\} + \mathcal{L}_m\left(e^{-\frac{\sigma}{\sqrt{\tilde{\omega}}}}\tilde{g}_{\mu\nu}, \Psi\right) \tag{24}$$

using this time a metric FRLW like

$$ds^2 = d\tau^2 - b(\tau)dx^i dx^j \tag{25}$$

The conformal changes for perfect fluid

$$\tilde{T}_{\mu\nu} = \frac{T_{\mu\nu}}{\Phi^2} = e^{-2\gamma\sigma} T_{\mu\nu} = (\tilde{\rho} + \tilde{p})\tilde{u}_\mu \tilde{u}_\nu - \tilde{p}\tilde{g}_{\mu\nu}. \tag{26}$$

with,

$$\tilde{\rho} = \rho e^{-2\gamma\sigma}, \quad \tilde{p} = p e^{-2\gamma\sigma}, \tag{27}$$

Solutions are:

$$b = b_0 \tau^{\tilde{r}}, \tag{28}$$

$$\sigma = \sigma_0 + \tilde{s} \ln \tau, \tag{29}$$

with,

$$\tilde{r} = \frac{4(1-\alpha)\eta}{6(1-\alpha^2)\eta + (1-3\alpha)^2\gamma^2}, \tag{30}$$

$$\tilde{s} = \frac{4(1-3\alpha)\gamma}{6(1-\alpha^2)\eta + (1-3\alpha)^2\gamma^2}. \tag{31}$$

The initial energy density

$$8\pi G\tilde{\rho}_0 = 2(3+2\omega)\frac{[4-6\alpha+3\omega(1-\alpha)^2]}{[5-3\alpha+3\omega(1-\alpha^2)]^2}. \tag{32}$$

The initial energy density

$$8\pi G\tilde{\rho}_0 = 2(3+2\omega)\frac{[4-6\alpha+3\omega(1-\alpha)^2]}{[5-3\alpha+3\omega(1-\alpha^2)]^2}. \tag{32}$$

Comparing with Jordan's frame the numerator is the same

$$8\pi\frac{\rho_0}{\Phi_0} = 2\left(3+2\omega\right)\frac{[4-6\alpha+3\omega(1-\alpha)^2]}{[4+3\omega(1-\alpha^2)]^2} \tag{33}$$

Again for matter componet like a scalar field

$$8\pi G\tilde{T}_{\mu\nu} = \tilde{\epsilon}\left(\psi_{;\mu}\psi_{\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\psi_{\rho}\psi^{\rho}\right)+\tilde{g}_{\mu\nu}U(\psi), \tag{34}$$

n comparing with perfect fluid we obtain

$$\tilde{\epsilon}\dot{\psi}^2 = 8\pi G(1+\alpha)\tilde{\rho}, \tag{35}$$

$$U(\psi) = 4\pi G(1-\alpha)\tilde{\rho}. \tag{36}$$

Solutions for matter field

$$\psi = \pm\sqrt{8\pi G\tilde{\rho}_0(1+\alpha)\tilde{\epsilon}}\,\ln\tau, \tag{37}$$

$$U = 4\pi G(1-\alpha)\tilde{\rho}_0 \exp\left\{\mp 2\frac{\psi}{\sqrt{8\pi G\tilde{\rho}_0(1+\alpha)\tilde{\epsilon}}}\right\}. \tag{38}$$

Index

Two main conditions are analyzed. For both, Jordan and Einstein, Frame. For 3 kinds of parameter in the state equation, pressureless fluid ($\alpha = 0$), cosmological constant ($\alpha = -1$), and phantom fluid ($\alpha = -2$).

1 - For a universe in attractive gravity

Jordan Frame	Einstein Frame
$8\pi \frac{\rho_0}{\Phi_0} = 2(3+2\omega)\frac{[4-6\alpha+3\omega(1-\alpha)^2]}{[4+3\omega(1-\alpha^2)]^2} > 0$	$8\pi G\tilde{\rho}_0 = 2(3+2\omega)\frac{[4-6\alpha+3\omega(1-\alpha)^2]}{[5-3\alpha+3\omega(1-\alpha^2)]^2} > 0$

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1 - For a universe in attractive gravity

Jordan Frame	Einstein Frame
$8\pi \frac{\rho_0}{\Phi_0} = 2(3+2\omega)\frac{[4-6\alpha+3\omega(1-\alpha)^2]}{[4+3\omega(1-\alpha^2)]^2} > 0$	$8\pi G\tilde{\rho}_0 = 2(3+2\omega)\frac{[4-6\alpha+3\omega(1-\alpha)^2]}{[5-3\alpha+3\omega(1-\alpha^2)]^2} > 0$

2 - For a universe in accelerated expansion

Jordan Frame	Einstein Frame
$\frac{\ddot{a}}{a} = \frac{\bar{r}(\bar{r}-1)}{t^2} > 0$	$\frac{\ddot{b}}{b} = \frac{\tilde{r}(\tilde{r}-1)}{t^2} > 0$

For $\alpha = 0$

	Jordan Frame	Einstein Frame
Scale factor	$a \propto t^{\frac{2+2\omega}{4+3\omega}}$	$b \propto \tau^{\frac{\omega}{2}}$
Brans-Dicke field	$\Phi \propto t^{\frac{2}{4+3\omega}}$	-
Potential	$V(\phi) = V_0\phi^{-6(1+\omega)}$	$U(\psi) = \frac{(3+2\omega)(4+3\omega)}{(5+3\omega)^2} \exp\left\{-\frac{5+3\omega}{\sqrt{2(3+2\omega)(4+3\omega)}}\psi\right\}$

For $\alpha = 0$

	Jordan Frame	Einstein Frame
For attraction	$\bar{r} < 0;\ t \leq 0$ $\downarrow$ $-\frac{4}{3} < \omega < -1$	$\tilde{r} < 0;\ \tau \leq 0$ $\downarrow$ $-1/6 < \omega < 1/2$
	$\bar{r} > 1;\ t \geq 0$ $\downarrow$ $-1/2 < \omega < 0$	$\tilde{r} > 1;\ \tau \geq 0$ $\downarrow$ $\omega < 0$ or $\omega > 49/18$
For accelerated expansion	$\frac{\rho_0}{\Phi_0} > 0$ $\omega < -3/2$ and $-4/3 < \omega$	$\tilde{\rho}_0 > 0$ for all interval

For $\alpha = -1$

	Jordan Frame	Einstein Frame
Scale factor	$a \propto t^{\omega+\frac{1}{2}}$	$b \propto \tau^{\frac{\omega}{2}}$
Brans-Dicke field	$\Phi \propto t^2$	-
Potential	$\bar{V}_0$	U_0

For $\alpha = -1$

	Jordan Frame	Einstein Frame
For attraction	$\bar{r} < 0;\ t \leq 0$ $\downarrow$ $\omega < -1/2$	$\tilde{r} < 0;\ t \leq 0$ $\downarrow$ $\omega > 2$
	$\bar{r} > 1;\ \tau \geq 0$ $\downarrow$ $\omega < -1/3$ or $\omega > 4/9$	$\tilde{r} > 1;\ \tau \geq 0$ $\downarrow$ $\omega < 0$
For accelerated expansion	$\frac{\rho_0}{\Phi_0} > 0$ $\omega > 1/2; \omega < -3/2$ and $-5/6 < \omega < -1/2$	$\tilde{\rho}_0 > 0$ except for $-5/6 < \omega < 0$

For $\alpha = -2$

	Jordan Frame	Einstein Frame
Scale factor	$a \propto t^{\frac{2+6\omega}{4-9\omega}}$	$b \propto \tau^{\frac{12\omega}{49-18\omega}}$
Brans-Dicke field	$\Phi \propto t^{\frac{14}{4-9\omega}}$	-
Potential	$V(\phi) = V_0 \phi^{6\frac{(1+3\omega)}{7}}$	$U(\psi) = \frac{3(3+2\omega)(16+27\omega)}{(11-9\omega)^2} . \exp\left\{ -\frac{11-9\omega}{\sqrt{6(3+2\omega)(16+27\omega)}}\psi \right\}$

For $\alpha = -2$

	Jordan Frame	Einstein Frame
For attraction	$\bar{r} < 0;\ t \leq 0$ $\downarrow$ $\omega < -1/3$ or $\omega > 4/9$	$\tilde{r} < 0;\ \tau \leq 0$ $\downarrow$ $\omega < 0$ or $\omega > 49/18$
	$\bar{r} > 1;\ t \geq 0$ $\downarrow$ $\omega < -1/3$ or $\omega > 4/9$	$\tilde{r} > 1;\ \tau \geq 0$ $\downarrow$ $49/18 < \omega < 49/30$
For accelerated expansion	$\frac{\rho_0}{\Phi_0} > 0$ $\omega < -3/2$ and $-4/3 < \omega$	$\tilde{\rho}_0 > 0 \Rightarrow\ -27/16 < \omega < -3/2$

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