Derivative Expansion in a Two-Scalar Field Theory

Alícia G. Borges

Departamento de Física, ICE, Universidade Federal de Juiz de Fora

Quantum Summer 2025 - Iriri, ES, Brazil.

Contents

Based on : A.G. Borges and I. Shapiro, arXiv:2503.2369

Overview of the Problem

Certain cosmological models (see, e.g., [1]), such as assisted inflation, involve two scalar fields with a strong mass hierarchy. Given the high-energy scale of inflation, quantum effects must be considered, especially the impact of curvature on the effective potential and derivative terms.

A key aspect of such models is the potential decoupling of heavy fields at late times, effectively reducing the number of relevant degrees of freedom. This reinforces the importance of a quantum effective description, as only the lighter scalar's quantum effects may remain significant in later cosmological epochs.

In this work, we explore the one-loop quantum contributions to the effective action of two scalar fields with different masses in zero and second orders of derivative expansion, in curved spacetime. We follow an approach similar to extracting normal modes of the oscillator in classical mechanics.

The described reduction of the multi-scalar calculation to the single scalar case can be directly applied for a greater number of real scalars.

[1] D. Wands (Springer Berlin Heidelberg, 2008).

Single scalar field in flat space

For the classical theory for a one-component scalar field with self-interaction $\lambda\phi^4$, the Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\,\phi^4. \tag{1}$$

The Euclidean one-loop contribution to the effective action has the form [2],

$$\Gamma^{(1)}(\phi) = \frac{1}{2}\,\mathrm{Tr}\,\ln\,(\Box + M^2), \tag{2}$$

where $M^2 = m^2 + \frac{\lambda}{2}\,\phi^2(x)$.
The derivative expansion of the effective action is

$$\Gamma^{(1)}(\phi) = \int d^4x\sqrt{-g}\,\Big\{ - V^{(1)}(\phi) + \frac{1}{2}\,Z^{(1)}(\phi)\,g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + \ldots\Big\}. \tag{3}$$

[2] R. Jackiw (Phys. Rev. D, 1974).

By decomposing the field as $\phi = \phi_0 + \varphi$ in both equations, one can expand equation (3) up to second order in φ,

$$\Gamma^{(1)}(\phi_0 + \varphi) \; = \; \int d^4x \left\{ - \, V^{(1)}(\phi_0) \, - \, \frac{\partial V^{(1)}}{\partial \phi_0} \, \varphi - \frac{1}{2} \, \frac{\partial^2 V^{(1)}}{\partial \phi_0^2} \, \varphi^2 \right.$$
$$\left. + \, \frac{1}{2} \, Z^{(1)}(\phi_0)(\partial_\mu \varphi)(\partial^\mu \varphi) + \, ... \right\} . \quad (4)$$

Performing the same substitution in Eq. (2) we get

$$\Gamma^{(1)}(\phi_0 + \varphi(x)) - \Gamma^{(1)}(\phi_0) \; = \; \frac{1}{2} \, \mathrm{Tr} \, \ln \left[1 + \frac{\lambda/2}{p^2 + M^2} (2\phi_0 \varphi + \varphi^2) \right] . \quad (5)$$

Expanding the logarithm and commuting all momentum operators to the left, we compare the expansion of Eq. (4) and Eq. (5) order by order in powers of φ. Finally, the kinetic term [3, 4] is found in the form

$$Z^{(1)}(\phi_0) \; = \; \frac{\lambda^2 \phi_0^2}{12(4\pi)^2 M^2} . \quad (6)$$

[3] J. Iliopoulos, C. Itzykson e A. Martin (Rev. Mod. Phys. 1975).
[4] C. M. Fraser (Z. Phys. C, 1985).

Effective action of a single scalar field in curved space

In curved space, the action including the nonminimal term of the first order in curvature, has the form

$$S(\phi, g_{\mu\nu}) = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \phi^2 m^2 + \frac{1}{2} \xi R \phi^2 - \frac{\lambda}{4!} \phi^4 \right\}. \tag{7}$$

We use the local momentum representation and the modified propagator [5],

$$\bar{G}(y) = \int \frac{d^4k}{(2\pi)^4} e^{iky} \left[\frac{1}{p^2 + M_0^2} - \frac{\tilde{\xi} R}{(p^2 + M_0^2)^2} \right]. \tag{8}$$

where

$$M_0^2 = m^2 + \frac{\lambda}{2} \phi_0^2 \qquad \text{and} \qquad \tilde{\xi} = \xi - \frac{1}{6}. \tag{9}$$

Following the same expansion method as in flat space, but with curvature effects using the method of [5], we obtain the correction from the curvature in the form

$$Z(\phi) = \frac{\lambda^2 \phi_0^2}{12(4\pi)^2} \left(\frac{1}{M^2} + \frac{\tilde{\xi} R}{2M^4} \right). \tag{10}$$

[5] S. Bunch e L. Parker (Phys. Rev. D, 1979).

Effective Action for a two-scalar field theory

Consider a model with two real interacting scalar fields ϕ and χ, described by the covariant Lagrangian density

$$\mathcal{L} = \frac{1}{2}\, g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\, g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}\, m^2\phi^2 - \frac{1}{2}\, M^2\chi^2 + \frac{1}{2}\,\xi_1 R\phi^2 + \frac{1}{2}\,\xi_2 R\chi^2 - \frac{\lambda_1}{4!}\phi^4 - \frac{\lambda_2\chi^4}{4!} - \frac{\lambda_{12}}{8}\phi^2\chi^2\,. \tag{11}$$

Here m, M are distinct masses of the fields and λ_1, λ_2, λ_{12} are the coupling constants.

In what follows, we show how to reduce the quantum calculations in the theory with two masses to the single-mass case. Our approach is based on the background field method and the change of variables (diagonalization) which was previously used in [6]. The remainder of the procedure follows the same steps used in the single scalar case.

[6] I.L. Buchbinder, A.R. Rodrigues, E.A. dos Reis and I.L. Shapiro (Eur. Phys. J., 2019)

Diagonalization in the space of fields

We decompose scalar fields into background ϕ, χ and quantum σ_1, σ_2 counterparts as

$$\phi \longrightarrow \phi + \sigma_1 \,, \qquad \chi \longrightarrow \chi + \sigma_2 \,. \tag{12}$$

The one-loop contribution depends on the bilinear in quantum fields action

$$S^{(2)} = \frac{1}{2} \int d^4x \sqrt{-g} \begin{pmatrix} \sigma_1 & \sigma_2 \end{pmatrix} \hat{H} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} . \tag{13}$$

The Hessian matrix is non-diagonal and has a standard matrix form

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \hat{1}\Box + \begin{pmatrix} M_{11}^2 & M_{12}^2 \\ M_{21}^2 & M_{22}^2 \end{pmatrix} , \tag{14}$$

where

$$\hat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{cases} M_{11}^2 = m^2 - \xi_1 R + \frac{\lambda_1}{2}\phi^2 + \frac{\lambda_{12}}{4}\chi^2, \\ M_{12}^2 = M_{21}^2 = \lambda_{12}\,\phi\chi, \\ M_{22}^2 = M^2 - \xi_2 R + \frac{\lambda_2}{2}\chi^2 + \frac{\lambda_{12}}{4}\phi^2 \,. \end{cases} \tag{15}$$

To diagonalize matrix $\hat{H}$, we perform rotation in the space of fields,

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}. \tag{16}$$

After rotation, the bilinear action becomes

$$\begin{aligned} S^{(2)} = \frac{1}{2}\int d^4x\sqrt{-g}\Big\{&\rho_1\Box\rho_1 + \rho_2\Box\rho_2 \\ &+ \rho_1\big(M_{11}^2\cos^2\alpha + 2M_{12}^2\sin\alpha\cos\alpha + M_{22}^2\sin^2\alpha\big)\rho_1 \\ &+ \rho_2\big(M_{11}^2\sin^2\alpha - 2M_{12}^2\sin\alpha\cos\alpha + M_{22}^2\cos^2\alpha\big)\rho_2 \\ &+ \rho_1\big[(M_{22}^2 - M_{11}^2)\sin 2\alpha + 2M_{12}^2\cos 2\alpha\big]\rho_2\Big\}. \end{aligned} \tag{17}$$

Choosing

$$\theta = \cot 2\alpha = \frac{M_{22}^2 - M_{11}^2}{2M_{12}^2}, \tag{18}$$

we eliminate the non-diagonal components.

For simplicity, we introduce the new notations

$$a = \frac{1}{2} + \frac{\theta}{2\sqrt{1+\theta^2}}\,, \qquad b = \frac{1}{2} - \frac{\theta}{2\sqrt{1+\theta^2}}\,, \qquad c = \frac{1}{\sqrt{1+\theta^2}}. \tag{19}$$

In the new variables, the bilinear form of the action becomes

$$\hat{\mathcal{H}} = \begin{pmatrix} \Box + \Pi_1 & 0 \\ 0 & \Box + \Pi_2 \end{pmatrix} = \begin{pmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_2 \end{pmatrix}, \tag{20}$$

where,

$$\Pi_1 = aM_{11}^2 + bM_{22}^2 - cM_{12}^2\,, \qquad \Pi_2 = bM_{11}^2 + aM_{22}^2 + cM_{12}^2\,. \tag{21}$$

Using the relation

$$\text{ln Det}\,\hat{H} \;=\; \text{ln Det}\,\hat{\mathcal{H}} \;=\; \text{Tr ln}\,\hat{\mathcal{H}} \;=\; \text{Tr ln}\,\mathcal{H}_1 \;+\; \text{Tr ln}\,\mathcal{H}_2\,. \tag{22}$$

Thus, the one-loop contribution to the effective action is

$$\Gamma^{(1)}(\phi,\chi) \;=\; \frac{1}{2}\,\text{Tr ln}\,(\Box + \Pi_1) + \frac{1}{2}\,\text{Tr ln}\,(\Box + \Pi_2)\,. \tag{23}$$

Using normal modes in the space of background fields.

One can set $\phi \to \phi_0 + \varphi$ and $\chi \to \chi_0 + \varepsilon$ and the derivative expansion up to the second order in the scalar fields φ and χ is

$$\Gamma^{(1)}(\phi,\chi) \,=\, \int d^4x\sqrt{-g}\Big\{-V^{(1)}(\phi,\chi) + \frac{1}{2}\,Z^{(1)}_{\phi}(\partial\varphi)^2 + \frac{1}{2}\,Z^{(1)}_{\chi}(\partial\varepsilon)^2 + Z^{(1)}_{\phi\chi}(\partial\varphi)(\partial\epsilon) \,+\, ...\Big\}, \tag{24}$$

On the other hand, we have to expand Π_j $(j = 1, 2)$ as follows:

$$\Pi_j(\phi_0 + \varphi; \chi_0 + \varepsilon) \,=\, \Pi_{j0}(\phi_0,\chi_0) + \frac{\partial\Pi_j}{\partial\phi_0}\,\varphi + \frac{\partial\Pi_j}{\partial\chi_0}\,\varepsilon + \frac{1}{2}\,\frac{\partial^2\Pi_j}{\partial\phi_0^2}\,\varphi^2 + \frac{1}{2}\,\frac{\partial^2\Pi_j}{\partial\chi_0^2}\,\varepsilon^2 + \frac{\partial^2\Pi_j}{\partial\phi_0\partial\chi_0}\,\varphi\varepsilon + ... \tag{25}$$

The terms involving second derivatives do not contribute to the kinetic term. Thus, we need to keep only the terms of the first order in derivatives.

Furthermore, in the two first-derivative terms, the coefficients $\partial\Pi_j/\partial\phi_0$ and $\partial\Pi_j/\partial\chi_0$ depend solely on the constants ϕ_0 and χ_0. That being said, we can consider that

$$\frac{\partial\Pi_j}{\partial\chi_0} = \Pi_\chi(\chi_0) = \Pi_\chi \qquad \text{and} \qquad \frac{\partial\Pi_j}{\partial\phi_0} = \Pi_\phi(\phi_0) = \Pi_\phi \tag{26}$$

are constant coefficients of the linear expansion.

To derive the coefficients $Z^{(1)}$, we now introduce an auxiliary field ρ which is a linear combination of φ and ε, and expanding Π_j in a single field ρ,

$$\Pi_j(\phi_0+\varphi,\,\chi_0+\varepsilon) \;=\; \Pi_{0j}+\Pi_{\phi j}\varphi+\Pi_{\chi j}\varepsilon \;=\; \Pi_{0j}+\Pi_{\rho j}\rho, \tag{27}$$

where (for both $j=1,2$)

$$\Pi_\rho = \sqrt{\Pi_\phi^2+\Pi_\chi^2} \qquad \text{and} \qquad \rho = \frac{\Pi_\phi}{\Pi_\rho}\,\varphi + \frac{\Pi_\chi}{\Pi_\rho}\,\varepsilon\,. \tag{28}$$

Introducing ρ is analogous to what is called derivative along the given direction in the analysis of many variables. Together with the rotation, the procedure can be called double diagonalization.

As a result, we obtain the kinetic contribution for the two-field theory in the $\mathcal{O}(R)$ approximation. In our case, or two scalars, we get, simply using the one-scalar result,

$$Z_\phi = \frac{1}{12(4\pi)^2}\left[\left(1+\frac{\tilde{\xi}_1 R}{2\Pi_{01}}\right)\frac{\Pi^2_{\phi 1}}{\Pi_{01}} + \left(1+\frac{\tilde{\xi}_2 R}{2\Pi_{02}}\right)\frac{\Pi^2_{\phi 2}}{\Pi_{02}}\right], \tag{29}$$

$$Z_\chi = \frac{1}{12(4\pi)^2}\left[\left(1+\frac{\tilde{\xi}_1 R}{2\Pi_{01}}\right)\frac{\Pi^2_{\chi_1}}{\Pi_{01}} + \left(1+\frac{\tilde{\xi}_2 R}{2\Pi_{02}}\right)\frac{\Pi^2_{\chi_2}}{\Pi_{02}}\right], \tag{30}$$

$$Z_{\chi\phi} = \frac{1}{12(4\pi)^2}\left[\left(1+\frac{\tilde{\xi}_1 R}{2\Pi^2_{01}}\right)\frac{\Pi_{\phi_1}\Pi_{\chi_1}}{\Pi_{01}} + \left(1+\frac{\tilde{\xi}_2 R}{2\Pi_{02}}\right)\frac{\Pi_{\phi_2}\Pi_{\chi_2}}{\Pi_{02}}\right]. \tag{31}$$

The coefficients Z_ϕ, Z_χ and $Z_{\chi\phi}$ are finite and independent of renormalization conditions.

Low-energy limit in the theory with hierarchy of the masses

After renormalization, we obtain for the effective potential

$$V_{ren}^{(1)} = \frac{\Pi_1^2(\phi,\chi)}{4(4\pi)^2} \ln\left[\frac{\Pi_1(\phi,\chi)}{\mu^2}\right] + \frac{\Pi_2^2(\phi,\chi)}{4(4\pi)^2} \ln\left[\frac{\Pi_2(\phi,\chi)}{\mu^2}\right]. \tag{32}$$

In the IR limit, the scalar performs small oscillations near the minimum, hence we assume $M^2 \gg \phi$. In the theory with mass hierarchy, $M^2 \gg m^2$, the mixing between the two quantum scalars vanishes, and the dominant contribution takes the form

$$V^{(1)} = \frac{1}{4(4\pi)^2}\left(m^2 + \frac{\lambda_1}{2}\phi^2 + \frac{\lambda_{12}}{4}\chi^2\right)^2 \ln\left(\frac{m^2 + \frac{\lambda_1}{2}\phi^2 + \frac{\lambda_{12}}{4}\chi^2}{\mu^2}\right). \tag{33}$$

Taking the limit $m \to 0$, the expression (33) becomes (up to the constant terms)

$$V^{(1)} = \frac{\lambda^2\rho^4}{16\,(4\pi)^2} \ln\left(\frac{\lambda\rho^2}{2\,\mu^2}\right), \quad \text{where} \quad \frac{\lambda}{2}\,\rho^2 = \frac{\lambda_1}{2}\phi^2 + \frac{\lambda_{12}}{4}\chi^2\,. \tag{34}$$

Thus, the effective potential boils down to the leading logarithmic contribution of a single scalar field ρ which is a mixing of the original scalars. Taking the corresponding limit for the kinetic term we get

$$Z_{IR}^{(1)} \; = \; \frac{\lambda^2}{12\,(4\pi)^2}\left(1 + \frac{\tilde{\xi} R}{2M^2}\right)\frac{\phi^2}{M^2}. \tag{35}$$

This formula shows that there is a quadratic decoupling and, for $|R| \ll M^2$, the R dependent term disappears with the double (quartic) rate.

Conclusion

This work presents a method for calculating the effective potential and first-order derivative expansion terms in a multi-scalar field theory with a mass hierarchy. The double diagonalization method simplifies the functional approach by diagonalizing the Hessian matrix and redefining the expansion basis in terms of directional derivatives of the background fields. This framework can be extended to multiple scalar fields.

In a two-scalar system with a strong mass hierarchy, the heavier field decouples in the infrared (IR) limit, diminishing its quantum contribution to the cosmological constant.

From a practical perspective, our results indicate that in assisted inflation models, the quantum effects of two scalar fields effectively reduce to those of the lightest scalar during the rapid expansion of the Universe. As the characteristic energy scale decreases, the heavy field gradually decouples.