Slowly Rotating neutron stars in scalar-tensor theories of gravity

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October 4, 2021

Summary

- Gravitation
  - General Relativity
  - Modified Gravity: Escalar-Tensor Theories
  - Different Representations: Einstein X Jordan
  - Solar System Constraints
- Neutron Stars
  - Equilibrium in GR
  - Equation of State
  - Equilibrium in Scalar-Tensor Theory and Spontaneous Scalarization
- Rotating Neutron Stars
  - Hartle-Thorne Formalism
- Pulsars
  - Timing and Gravitational Tests
  - Scalar Charges
- Conclusions

Gravitation

The principal metric theory of gravity is GR. The action reads,

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + S^{(m)}[g_{\mu\nu}, \Psi], \tag{1}$$

and the field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2}$$

where $T_{\mu\nu} \equiv -(2/\sqrt{-g})\delta S^{(m)}/\delta g^{\mu\nu}$ and the matter follows

$$\nabla^{\mu} T_{\mu\nu} = 0. \tag{3}$$

Gravitation

GR is one of the most beautiful and successful theories of physics and has passed all the experimental tests until today. So, why modified it ?

- Quantum Gravity Effects
- Dark Components of the Universe

A good compass to navigate the alternative theories landscape is the Lovelock Theorem.

Theorem

In a four dimensional space-time, the only second order tensor, symmetric and divergence-free, build from the metric $g_{\mu\nu}$ and its derivatives to second order, is the Einstein tensor plus a cosmological constant.

Alternative theories must introduce new degrees of freedom !!!

Gravitation

Escalar-Tensor Theories is one of the most simple and well motivated alternative to GR. It was inspired by Dirac's ideia of a varing gravitational coupling. First, consider the action

$$S = \int d^4x \sqrt{-g}(\phi R + L_m), \tag{4}$$

where $\phi = 1/16\pi G$. Brans Dicke action (Prototype Theory)

$$S = \int d^4x \sqrt{-g} \left(\phi R - \frac{\omega}{\phi} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + L_m \right). \tag{5}$$

The general form of the action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(F(\Phi)R - \frac{Z(\Phi)}{2} \nabla_\mu \Phi \nabla^\mu \Phi - V(\Phi) \right) + S^m[e^{2a(\Phi)} g_{\mu\nu}, \Psi_m]. \tag{6}$$

Gravitation

We now use the freedom expressed by conformal transformations and the redefinition of the scalar field:

$$g_{\mu\nu} \to \Omega^2(\Phi) g_{\mu\nu}, \tag{7}$$

$$\Phi \to f(\Phi). \tag{8}$$

Jordan Representation:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(F(\Phi) R - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi - V(\Phi) \right) + S^m[g_{\mu\nu}, \Psi_m], \tag{9}$$

Einstein Representation:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi - V(\Phi) \right) + S^m[e^{2a(\Phi)} g_{\mu\nu}, \Psi_m]. \tag{10}$$

In the Einstein representation $(\varphi, g^*_{\mu\nu})$, the field equations are in a more mathematical friendly form

$$R^*_{\mu\nu} - \frac{1}{2}R^* g^*_{\mu\nu} - 2\partial_\mu\varphi\partial_\nu\varphi + g^*_{\mu\nu}g_*^{\alpha\beta}\partial_\beta\varphi\partial_\alpha\varphi = 8\pi G T^*_{\mu\nu} - 2U(\varphi)g^*_{\mu\nu}, \tag{11}$$

$$\Box^*\varphi = -4\pi G\alpha(\varphi)T^* + \frac{dU(\varphi)}{d\varphi}, \tag{12}$$

where

$$\alpha(\varphi) \equiv -\frac{1}{2}\frac{d\ln F(\Phi(\varphi))}{d\varphi}. \tag{13}$$

The relation between the two representations

$$g^*_{\mu\nu} \equiv F(\Phi)g_{\mu\nu}, \tag{14}$$

$$\left(\frac{d\varphi}{d\Phi}\right)^2 \equiv \frac{3}{4F(\Phi)^2}\left(\frac{dF(\Phi)}{d\Phi}\right)^2 + \frac{1}{4F(\Phi)}. \tag{15}$$

Gravitation

We focus in the coupling function and make the potential $U(\varphi) = 0$. A non-minimal couling in the Jordan Representation

$$F(\Phi) = 1 - 8\pi\xi\Phi^2. \tag{16}$$

The relation between the fields

$$\frac{d\varphi}{d\Phi} = 2\sqrt{\pi}\frac{\sqrt{1 - 8\pi\xi(1 - 6\xi)\Phi^2}}{1 - 8\pi\xi\Phi^2}. \tag{17}$$

We define a coupling function already in the Einstein representation that mimics the latter

$$F(\varphi) = \left[\cosh\left(2\sqrt{3}\xi\varphi\right)\right]^{-1/(3\xi)}, \tag{18}$$

$$\alpha(\varphi) = \frac{1}{\sqrt{3}}\tanh\left(2\sqrt{3}\xi\varphi\right). \tag{19}$$

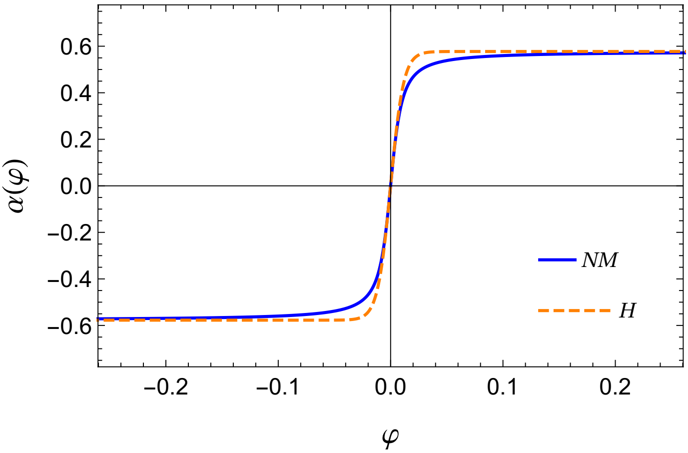

Figure 1: Effective Coupling $\alpha(\varphi)$, considering the Nonminimal coupling (NM) blue and the hyperbolic (H) in orange dashed, both with $\xi = 25$.

Gravitation

Solar system experiments constrains the coupling and the constant ξ. Parametrized Pos Newtonian Formalism:

$$g_{00} = -1 + (2U - 2\beta U^2 + (2\gamma + 2)\Phi_1 + (6\gamma - 4\beta + 2)\Phi_2 + 2\Phi_3 + 6\gamma\Phi_4) + \\ + O(\epsilon^3) \quad (20)$$

$$g_{0i} = -(2\gamma + 3/2)V_i - W_i/2 + O(\epsilon^{5/2}) \quad (21)$$

$$g_{ij} = (1 + 2\gamma U)\delta_{ij} + O(\epsilon^2). \quad (22)$$

$$F(\varphi) = F(\varphi_\infty) + (dF/d\varphi)_{\varphi_\infty}(\varphi - \varphi_\infty) + O[(\varphi - \varphi_\infty)^2]. \quad (23)$$

$$1 - \gamma = 2\frac{\alpha^2}{1 + \alpha^2} = \frac{(dF/d\Phi)^2}{F/2 + 2(dF/d\Phi)^2}, \quad (24)$$

where $1 - \gamma \lesssim 2.3 \times 10^{-5}$ implies $|\xi\varphi_\infty| < 0.0017$. Thus, from now on $\varphi_\infty = 0$.

Neutron Stars

Stellar core made of cold and catalysed matter.
Static and spherical symmetric space-time:

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{25}$$

Perfect Fluid:

$$T_{\mu\nu} = \epsilon U_\mu U_\nu + P(g_{\mu\nu} + U_\mu U_\nu), \tag{26}$$

Equations of hydrostatic equilibrium (TOV):

$$\frac{d\nu(r)}{dr} = 2\frac{M(r) + 4\pi r^3 P}{r(r - 2M(r))}, \tag{27}$$

$$\frac{dM(r)}{dr} = 4\pi\epsilon r^2, \tag{28}$$

$$\frac{dP}{dr} = -\frac{\epsilon M(r)}{r^2}\left(1 + \frac{P}{\epsilon}\right)\left(1 + \frac{4\pi P r^3}{M(r)}\right)\left(1 - \frac{2M(r)}{r}\right)^{-1}, \tag{29}$$

$$P = P(\epsilon, T) \tag{30}$$

Simple polytropic form

$$P = K\rho^{\Gamma}. \tag{31}$$

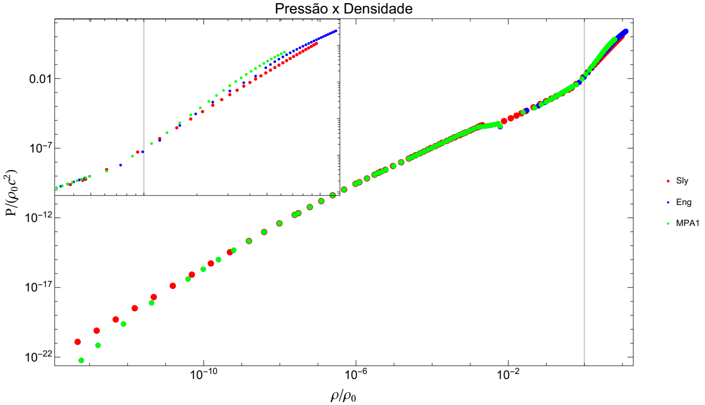

Figure 2: The different equations of state $\rho_0 = 2.7 \times 10^{14} \mathrm{g/cm^3}$

Neutron Stars

Piecewise polytropic

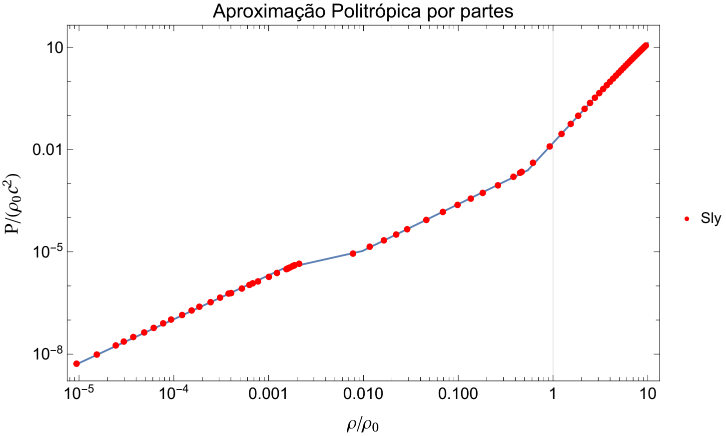

Figure 3: Piecewise approximation for the Sly equation of state (blue), in the vicinity of the density ρ_0, compared with the tabulated values (red).

Numerical results for the stellar structure

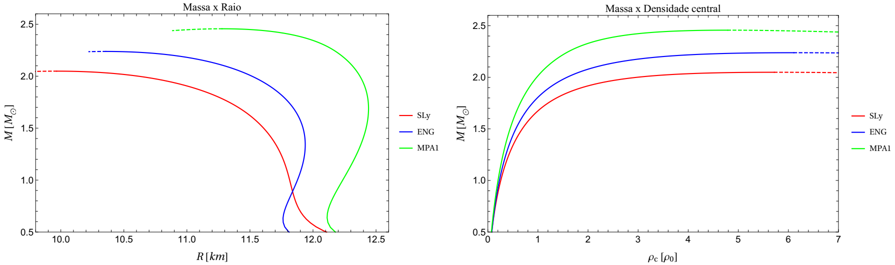

Figure 4: Neutron Star Structure. The unstable configurations are dashed.

Neutron Star

In the context of Scalar-Tensor theory, the field equations (Modified TOV)

$$M' = 4\pi r^2 F^{-2}\epsilon + \frac{1}{2} r(r-2M)(\varphi')^2, \tag{32}$$

$$\nu' = \frac{8\pi r^2 F^{-2} P}{r-2M} + r(\varphi')^2 + \frac{2M}{r(r-2M)}, \tag{33}$$

$$P' = -(\epsilon + P)\left[\frac{4\pi r^2 F^{-2} P}{r-2M} + \frac{r(\varphi')^2}{2} + \frac{M}{r(r-2M)} + \alpha\varphi'\right], \tag{34}$$

$$\varphi'' = \frac{4\pi r F^{-2}}{r-2M}\left[\alpha(\epsilon - 3P) + r\varphi'(\epsilon - P)\right] - \frac{2(r-M)}{r(r-2M)}\varphi'. \tag{35}$$

Assymptotic behaviour of the scalar field:

$$\varphi(r) \xrightarrow{r\to\infty} \varphi_\infty + \omega/r + O(1/r^2), \qquad \alpha = -\frac{\omega}{M} = \frac{\partial \log M}{\partial \varphi_\infty}. \tag{36}$$

Results:

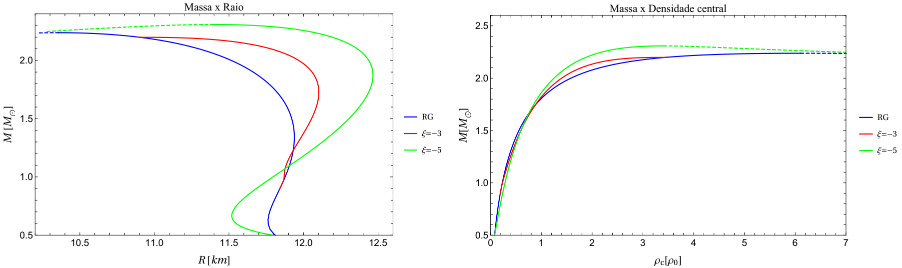

Figure 5: Mass radius diagram and Mass Central density for $\xi < 0$

A trivial solution $\varphi = 0$ always satisfied the field equations.

$$M' = 4\pi r^2 F^{-2}\epsilon + \frac{1}{2}r(r-2M)(\varphi')^2, \tag{37}$$

$$\nu' = \frac{8\pi r^2 F^{-2} P}{r-2M} + r(\varphi')^2 + \frac{2M}{r(r-2M)}, \tag{38}$$

$$P' = -(\epsilon + P)\left[\frac{4\pi r^2 F^{-2} P}{r-2M} + \frac{r(\varphi')^2}{2} + \frac{M}{r(r-2M)} + \alpha\varphi'\right], \tag{39}$$

$$\varphi'' = \frac{4\pi r F^{-2}}{r-2M}\left[\alpha(\epsilon - 3P) + r\varphi'(\epsilon - P)\right] - \frac{2(r-M)}{r(r-2M)}\varphi'. \tag{40}$$

The Spontaneous Scalarization depends upon

$$\Box^*\varphi = -4\pi G\alpha(\varphi)T^* \Longrightarrow \Box^{(0)}\delta\varphi = -8\pi\xi T^{(0)}\delta\varphi. \tag{41}$$

Results:

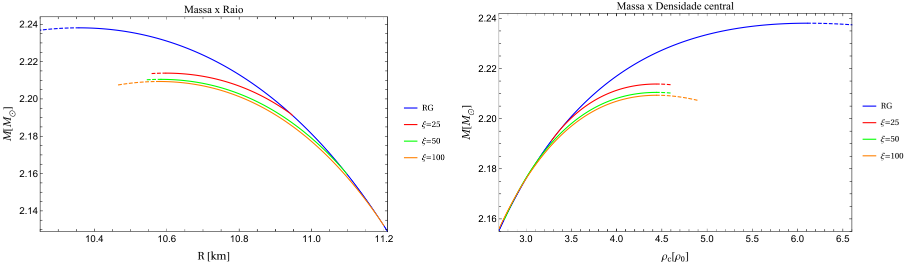

Figure 6: Mass radius relation for $\xi > 0$.

Consistent with the interpretation

$$G_{eff} = \frac{G}{F(\Phi(\varphi))} = \frac{G}{1 - 8\pi\xi\Phi^2}. \tag{42}$$

Neutron Stars

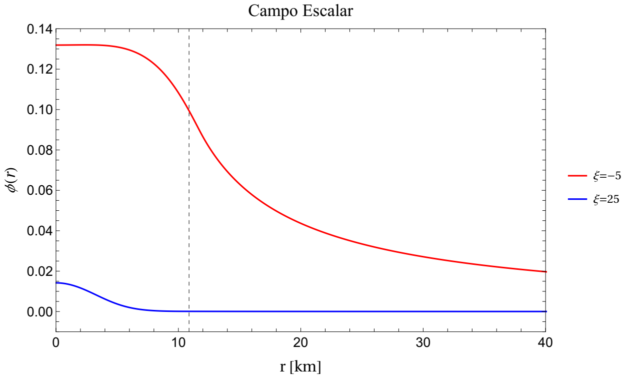

Figure 7: Two escalar field profiles for a star with $M = 2.2 M_\odot$. The approximate radius of the star is also shown.

Main Scalar Charge

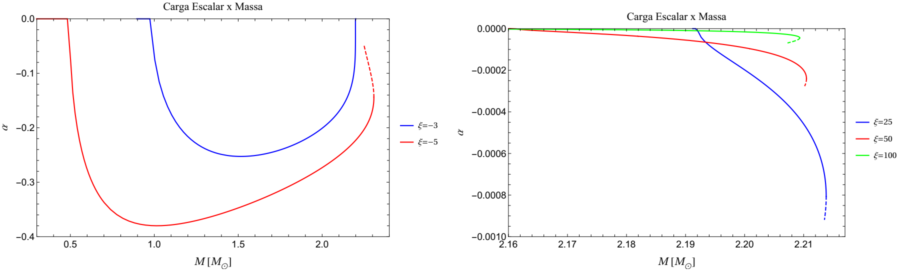

Figure 8: Left: $\xi < 0$ Right: $\xi > 0$.

We treat the rotation as a perturbation of the static solution

$$\Omega^2 \ll \left(\frac{c}{R}\right)^2 \frac{GM}{Rc^2} \quad \longrightarrow \quad \Omega R \ll c. \tag{43}$$

Space-Time with axial symmetry, stationary and assymptotic flat

$$ds^2 = -H^2 dt^2 + Q^2 dr^2 + r^2 K^2 [d\theta^2 + \sin^2\theta (d\phi - L dt)^2]. \tag{44}$$

$$L(r,\theta) = \omega(r,\theta) + O(\Omega^3). \tag{45}$$

Einstein's equations

$$R^t_\phi = 8\pi T^t_\phi. \tag{46}$$

$$T^t_\phi = (\varepsilon + P)e^{-\nu}(\Omega - \omega) r^2 \sin^2\theta, \qquad \bar{\omega} \equiv \Omega - \omega. \tag{47}$$

Rotating Neutron Stars

$$\frac{1}{r^4}\frac{d}{dr}\left(r^4 j\frac{d\bar{\omega}}{dr}\right)+\frac{4}{r}\frac{dj}{dr}=0, \qquad j(r)=e^{-(\nu+\lambda)/2}. \tag{48}$$

in the vaccum ($j=1$):

$$\bar{\omega}(r)=\Omega-\frac{2J}{r^3}, \qquad J=I\,\Omega. \tag{49}$$

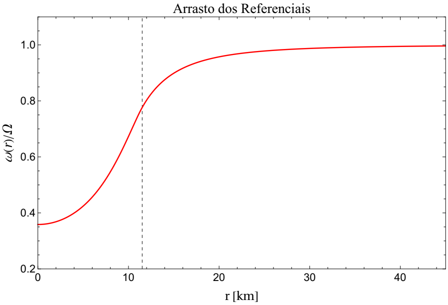

Figure 9: Function $\bar{\omega}/\Omega = 1 - 2J/\Omega r^3$

Moment of Inertia

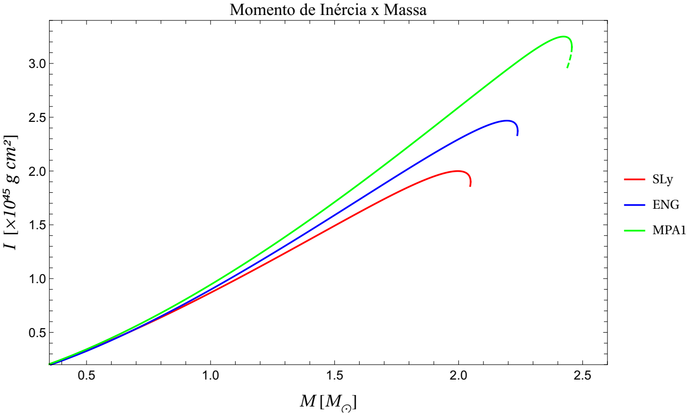

Figure 10: Moment of Inertia for the equations of state discussed. The function have a maximum before the maximal mass. Also, the unstable configurations are shown.

Rotating Neutron Stars

In Scalar-Tensor theory, the differential equation

$$\bar{\omega}'' = \bar{\omega}' \left[-\frac{4}{r} + r(\varphi')^2 + \frac{4\pi r^2(\epsilon + P)}{F^2(r - 2M)} \right] + \frac{16\pi r(\epsilon + P)}{F^2(r - 2M)}(\Omega - \bar{\omega}). \quad (50)$$

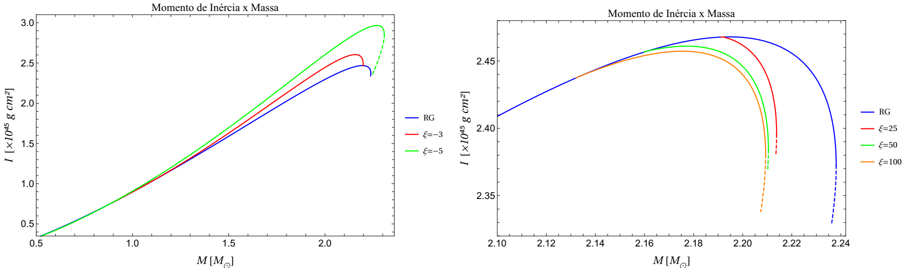

Figure 11: Moment of Inertia in Scalar Tensor Theories. Left: $\xi < 0$ Right: $\xi > 0$.

Timing ideia:

$$t^{BSS} = \tau - \frac{\Delta D}{f^2} + \Delta_{R_\odot} + \Delta_{S_\odot} + \Delta_{E_\odot} + \Delta_R + \Delta_S + \Delta_E, \qquad (51)$$

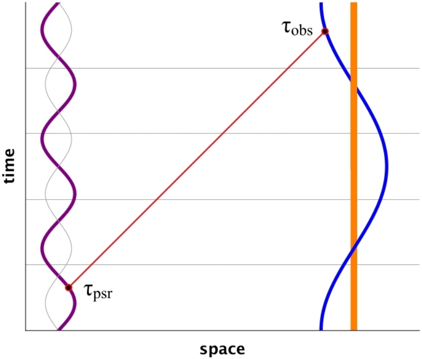

Figure 12: We have to relate the two time intervals. From Wex, 2014.

Pulsars

The orbit can be described by the pos-keplerian parameters

$$\dot{\omega}^{RG} = \frac{3n_b V_b^2}{1-e^2}, \tag{52}$$

$$\gamma^{RG} = \frac{e}{n_b}\left(1+\frac{m_c}{M}\right)\frac{m_c V_b^2}{M}, \tag{53}$$

$$r^{RG} = Gm_c, \tag{54}$$

$$s^{RG} = xn_b\frac{M}{m_c V_b}, \tag{55}$$

$$\dot{P}_b^{RG} = -\frac{192\pi}{5}\frac{m_p m_c}{M^2}\frac{1+73e^2/24+37e^4/96}{(1-e^2)^{7/2}}V_b^5, \tag{56}$$

where $n_b = 2\pi/P_b$, $V_b = (GMn_b)^{1/3}$ and $x = a\sin i$ is the projected of the major semi axis.

The PK parameters allows one of the most sucessful consistent tests of GR. For the double pulsar

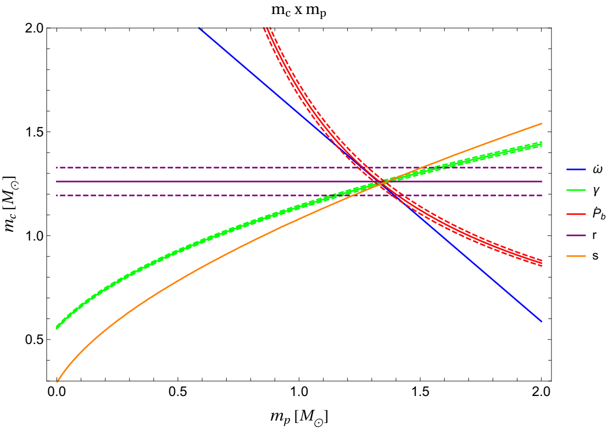

Figure 13: Best consistence test of General Relativity until now

Pulsars

In the context of Scalar-Tensor Theory, the PK parameters have a sensitivity in the scalar charges:

$$\dot{\omega}^{TE} = \frac{n_b}{1-e^2}\left(\frac{3-\alpha_p\alpha_c}{1+\alpha_p\alpha_c} - \frac{m_p\alpha_p^2\beta_c + m_c\alpha_c^2\beta_p}{2M(1+\alpha_p\alpha_c)^2}\right)\mathcal{V}_b^2, \tag{57}$$

$$\gamma^{TE} = \frac{e}{n_b}\left(\frac{1+\kappa_p\alpha_c}{1+\alpha_p\alpha_c} + \frac{m_c}{M}\right)\frac{m_c\mathcal{V}_b^2}{M}, \tag{58}$$

$$r^{TE} = \mathcal{G}m_c, \tag{59}$$

$$s^{TE} = xn_b\frac{M}{m_c\mathcal{V}_b}, \tag{60}$$

$$\dot{P}_b^{TE} = -2\pi\frac{m_pm_c}{M^2}\frac{1+e^2/2}{(1-e^2)^{5/2}}\frac{(\alpha_p-\alpha_c)^2\mathcal{V}_b^3}{1+\alpha_p\alpha_c}. \tag{61}$$

where $\mathcal{V}_b = [\mathcal{G}(1+\alpha_p\alpha_c)Mn_b]^{1/3}$.

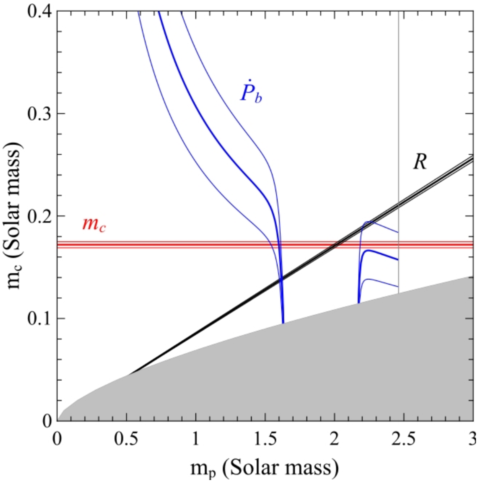

Figure 14: Conistence test of a theory with $\xi = -2.25$. The theory clearly fails the test. From Wex, 2014.

To better understand the scalar charge, consider the action

$$S_m = -\sum_{A=1}^{N} \int m_A(\varphi_A) d\tau_A^*, \tag{62}$$

where we define

$$\alpha_A \equiv \left.\frac{d \log m_A}{d\varphi_A}\right|_{\varphi_\infty}, \tag{63}$$

$$\beta_A \equiv \left.\frac{d\alpha_A}{d\varphi_A}\right|_{\varphi_\infty}, \tag{64}$$

$$k_A \equiv -\left.\frac{\partial \log I_A}{\partial\varphi_A}\right|_{\varphi_\infty}. \tag{65}$$

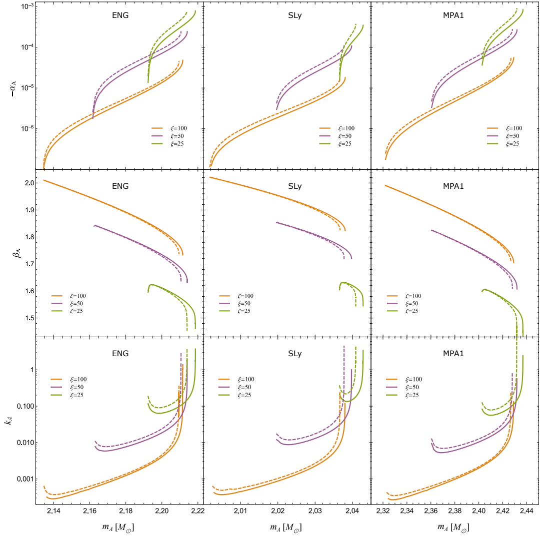

Figure 15: Scalar Charges for $\xi > 0$.

Conclusion

- Scalar-Tensor theories are the most famous and well motivated alternative to GR.
- The theory presented here, with a nonminimal coupling and constant $\xi > 0$, are capable to pass the pulsar timing tests.
- Another test of the theory can be made with the study of quasinormal modes of oscillation.