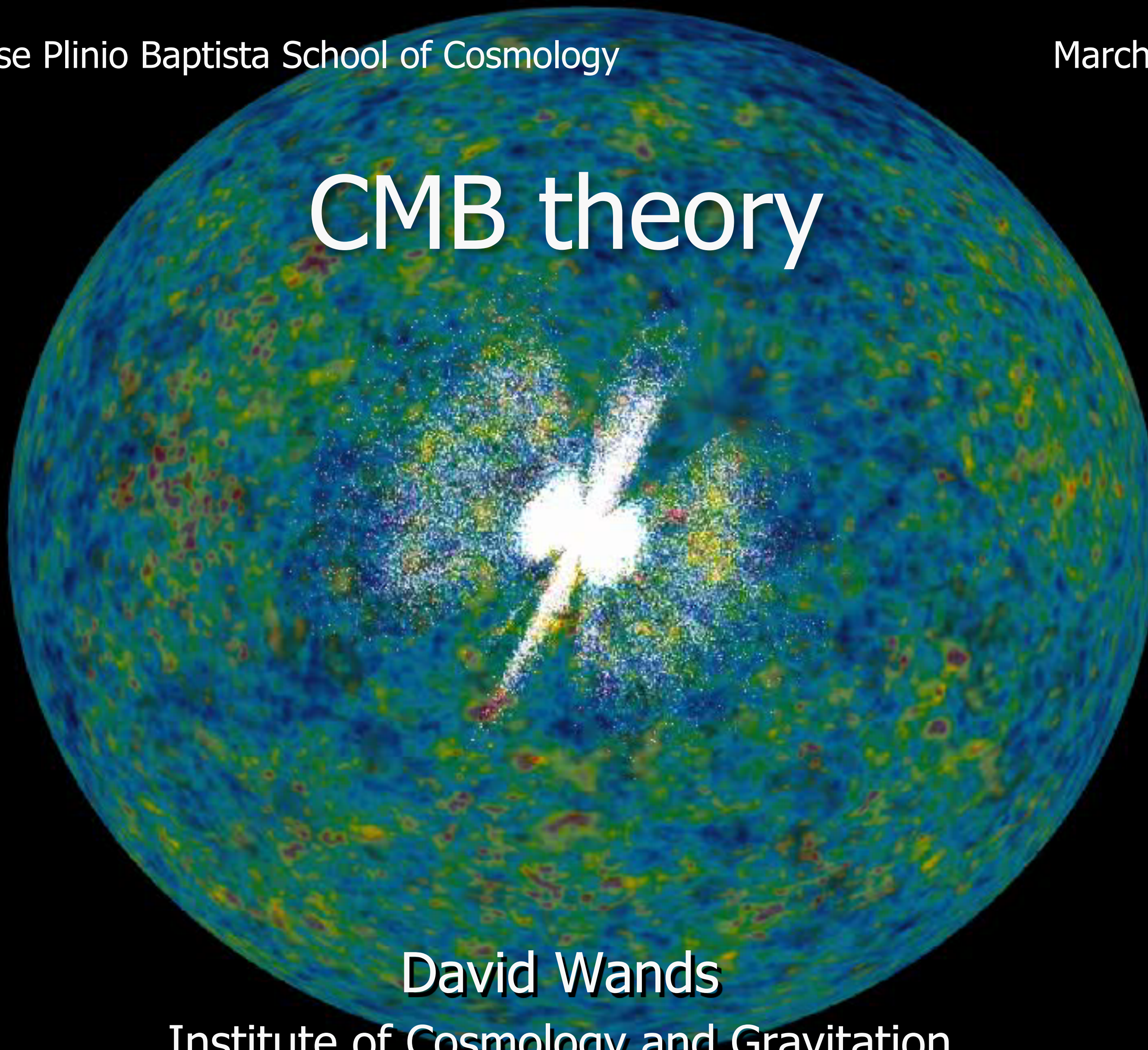
II Jose Plinio Baptista School of Cosmology March 2014

CMB theory



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Part 2: the anisotropic CMB

- Inhomogeneities
  - Fourier modes
  - Spherical harmonics
- CMB sphere
  - Sudden recombination
- Perturbed FLRW geometry
  - Sachs-Wolfe formula
  - perturbed dynamics
  - large scale anisotropies

homogeneous background dynamics

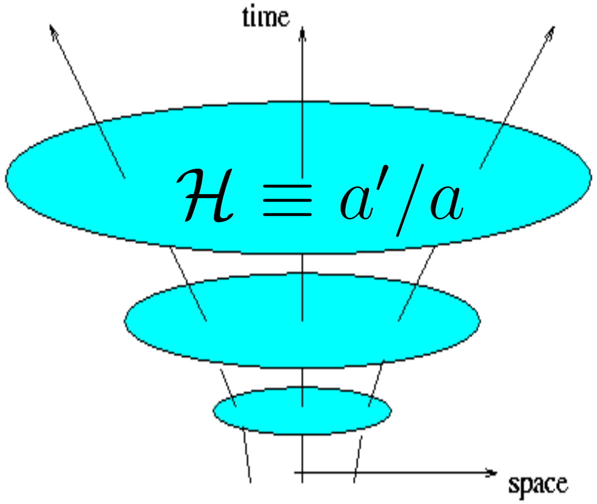
- Spatially flat metric: $ds^2 = -c^2 dt^2 + a^2 \delta_{ij} dx^i dx^j = a^2 \left[-d\eta^2 + \delta_{ij} dx^i dx^j \right]$

- Einstein energy constraint G_{00} + evolution G_{ij} equations:

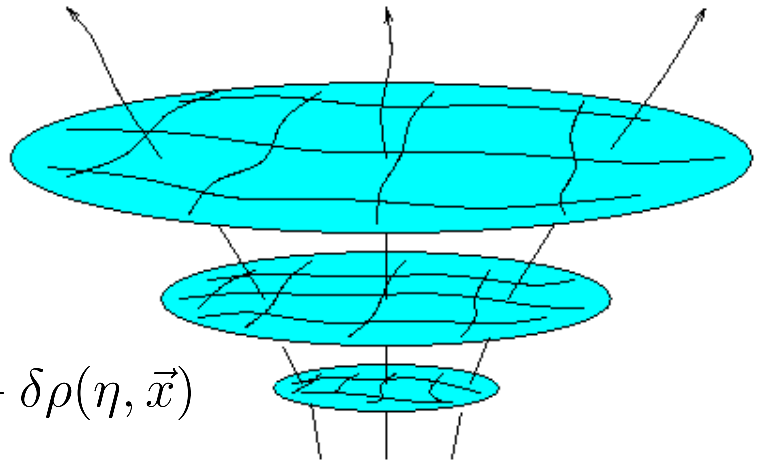
$$\begin{aligned} \mathcal{H}^2 &= \frac{8\pi G}{3} a^2 \rho \,, \\ \mathcal{H}' &= -\frac{4\pi G}{3} a^2 (\rho + 3P) \end{aligned}$$

- Energy conservation

$$\rho' = -3\mathcal{H} (\rho + P)$$



inhomogeneity



- Density field: $\rho(\eta) \to \rho(\eta, \vec{x}) = \bar{\rho}(\eta) + \delta\rho(\eta, \vec{x})$

- Fourier decomposition: $$\delta\rho(\eta, \vec{x}) = \frac{1}{(2\pi)^3} \int d^3\vec{k}\, \delta\rho_{\vec{k}}(\eta)\, e^{i\vec{k}.\vec{x}}$$
  - Different k-modes evolve independently at first-order

- Power spectrum: $$\langle \delta\rho_{\vec{k}_1} \delta\rho_{\vec{k}_2} \rangle = (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2) P(k_1)$$

brackets denote *ensemble average* (over many realisations)

Variance in real space $$\langle \delta\rho^2(\eta, \vec{x}) \rangle = \frac{1}{(2\pi)^3} \int d^3\vec{k}\, P(k) = \int d(\ln k) \mathcal{P}$$

dimensionless power (per log k) $$\mathcal{P}(k) \equiv \frac{k^3}{2\pi^2} P(k)$$

spectral tilt $$n - 1 \equiv \frac{d \ln \mathcal{P}}{d \ln k}$$

Anisotropies on sphere:

- Photon distribution (at first order): $f(p,\bar{T}) \to = f(p,\bar{T}(1+\Theta(\eta,\vec{x},\hat{p})))$

$$\Theta \equiv \frac{\delta T}{T} = \frac{1}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}$$

- Spherical harmonic decomposition:

$$\Theta(\eta,\vec{x},\hat{p}) = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{+\ell} a_{lm}(\eta,\vec{x})Y_{\ell m}(\hat{p})$$

  - » where $$\int d\Omega_n Y_{\ell m}(\hat{n})Y^*_{\ell' m'}(\hat{n}) = \delta_{\ell\ell'}\delta_{mm'}$$

- Angular power spectrum: $$\langle a^*_{\ell m}a_{\ell' m'}\rangle = \delta_{\ell\ell'}\delta_{mm'}C_\ell$$

Spherical harmonics: $\int d\Omega_n Y_{\ell m}(\hat{n}) Y^*_{\ell' m'}(\hat{n}) = \delta_{\ell\ell'}\delta_{mm'}$

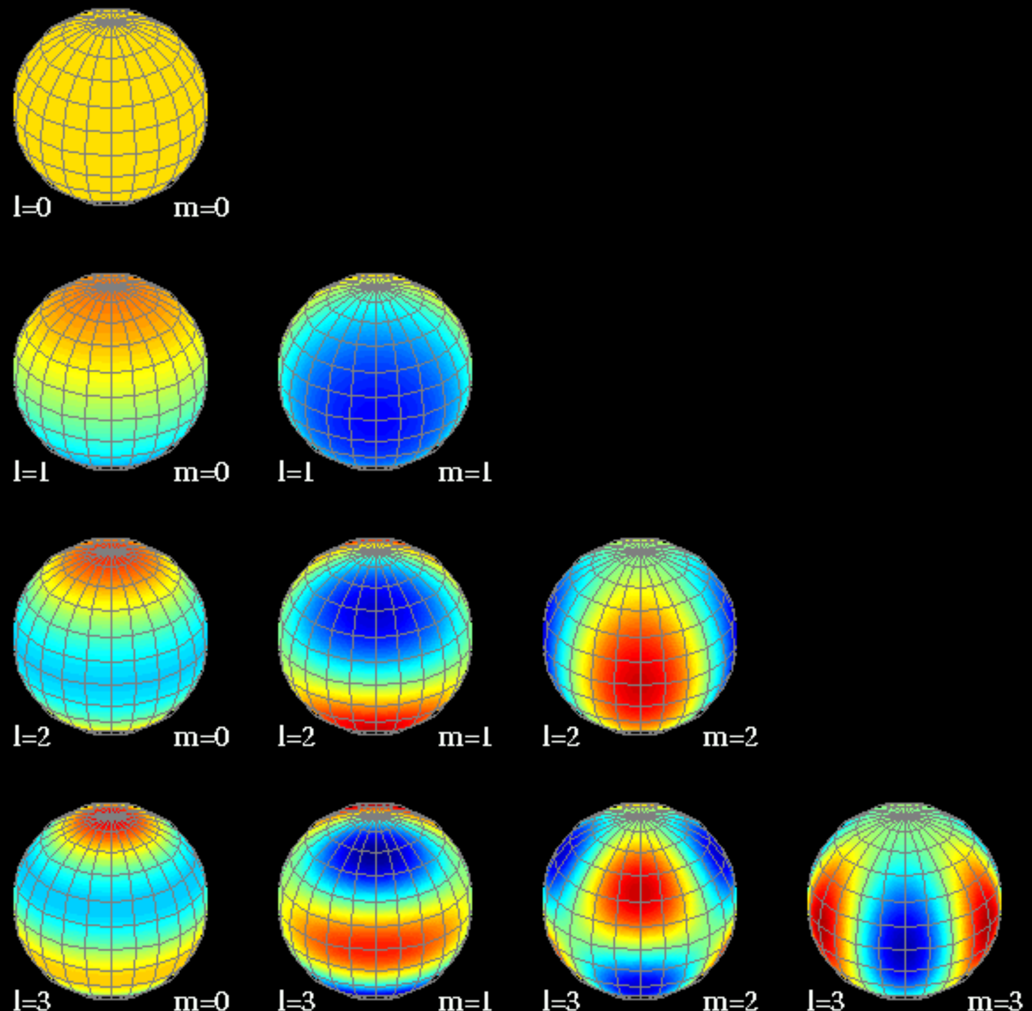
by Clem Pryke, Minnesota

monopole

dipole

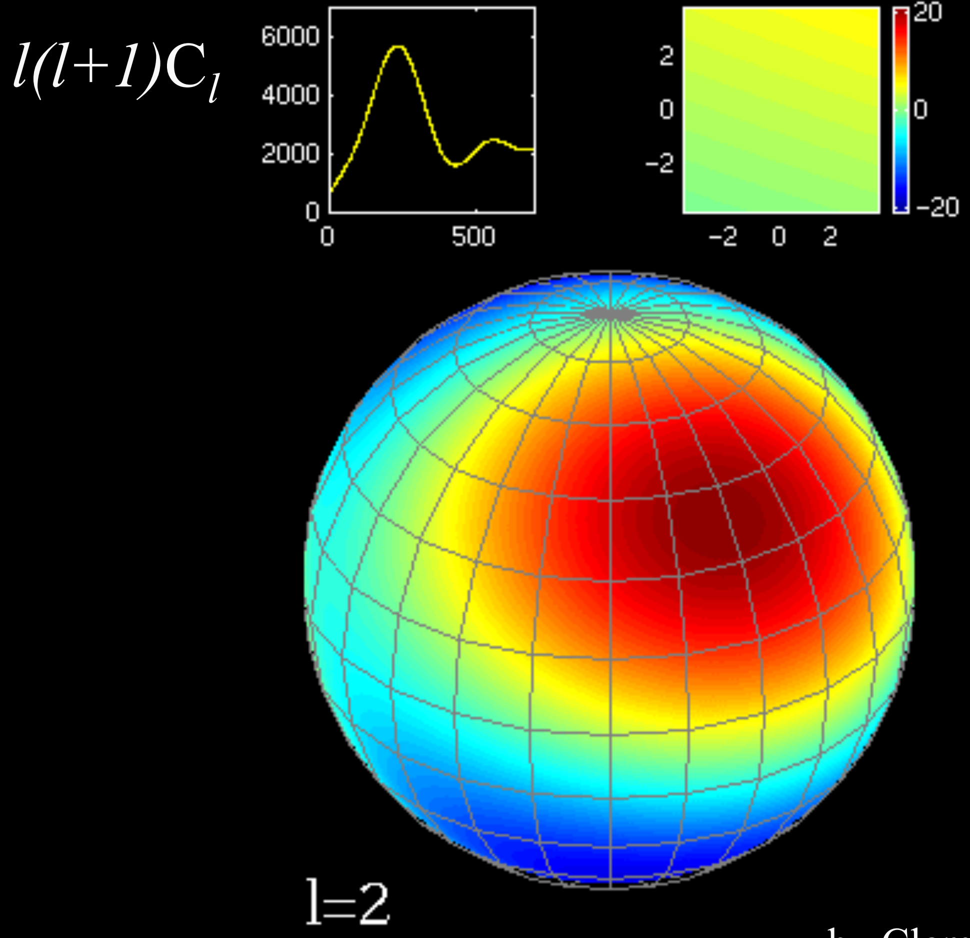
quadrupole

octopole



Angular power spectrum

$$\langle a^*_{\ell m} a_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell$$



by Clem Pryke, Minnesota

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Tight-coupling:

- Thomson scattering $\quad e^- + \gamma \quad \leftrightarrow \quad e^- + \gamma$

Thomson cross-section:

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65 \times 10^{-29}\,\mathrm{m}^2 \tag{44}$$

Mean free-path for photons

$$\lambda_T = \frac{1}{n_e\sigma_T} \tag{45}$$

Comoving mean free path near recombination ($x_e \approx 1$ when $z \approx 10^3$)

$$\lambda_T/a \approx 2.5\,\mathrm{Mpc} \tag{47}$$

On larger scales, $\lambda \gg \lambda_T$, photons are tightly coupled to electrons, and electrons are tightly coupled to baryons by Coulomb interaction.

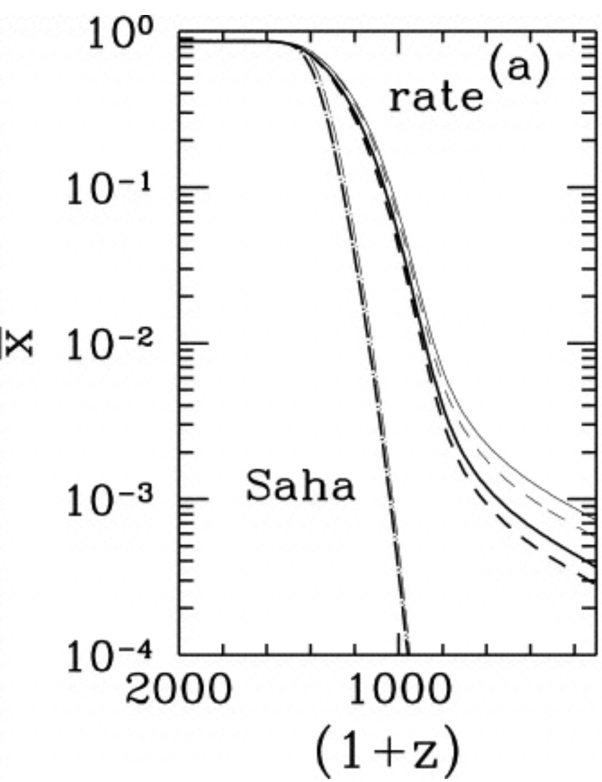
epoch of recombination

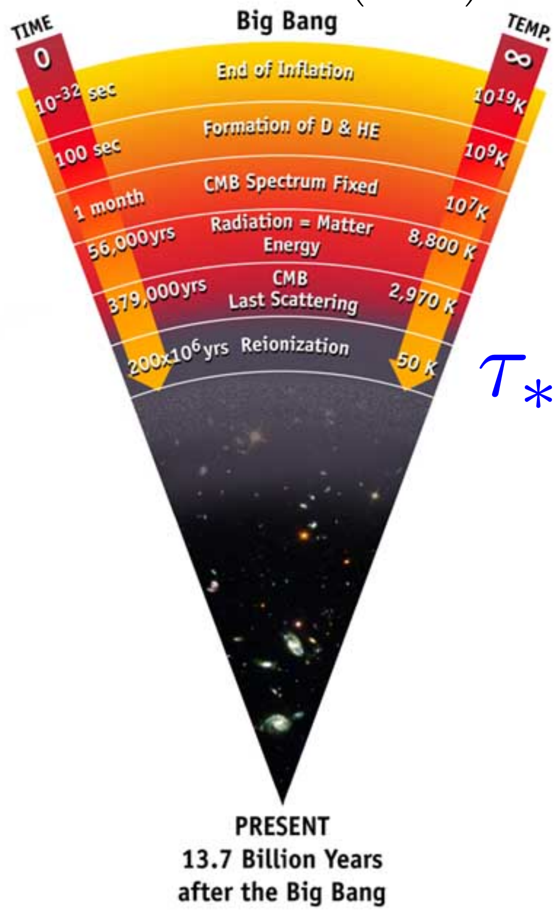
$$x_e = \frac{n_e}{n_e + n_H} \to 0$$

Recombination is epoch at which Thomson optical depth, $\tau = \int n_e \sigma_T dt$, reaches unity

$$\tau_* = 1 \quad \Rightarrow \quad 1 + z_* = 1089 \left(\frac{\Omega_m h^2}{0.14}\right)^{0.0105} \left(\frac{\Omega_b h^2}{0.024}\right)^{-0.028} . \quad (4$$

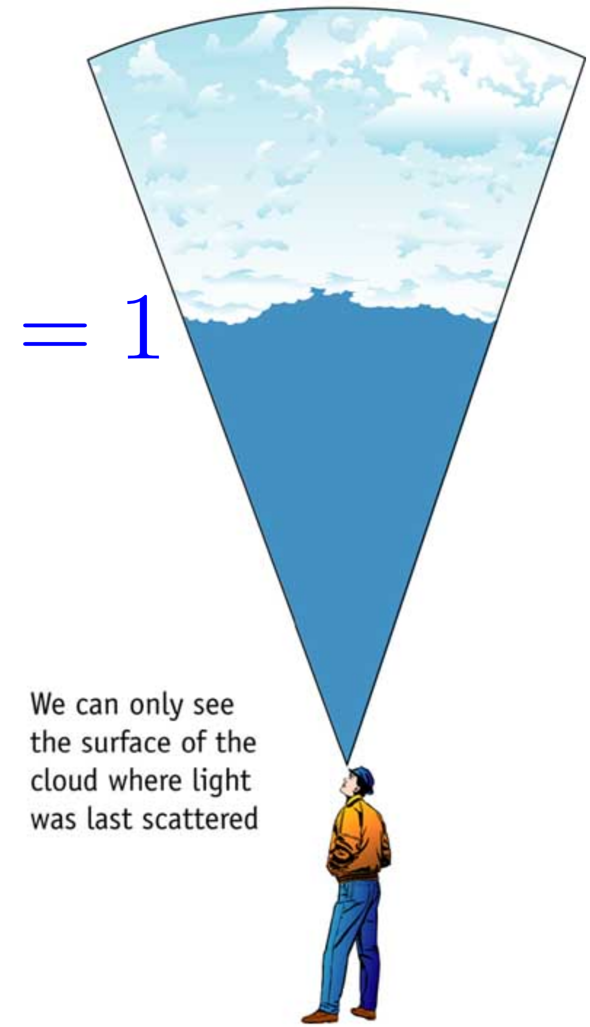
Free e- fraction

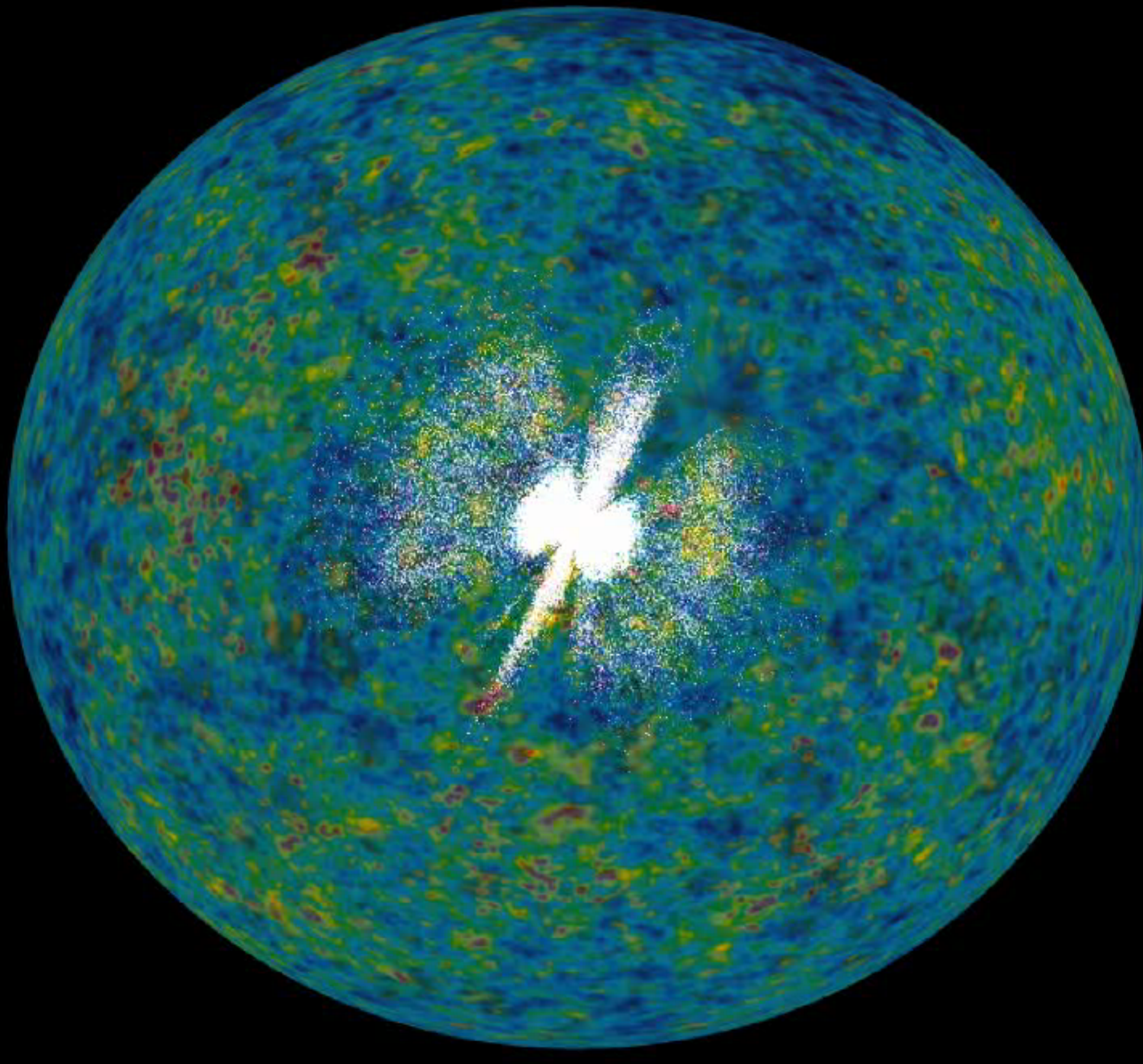




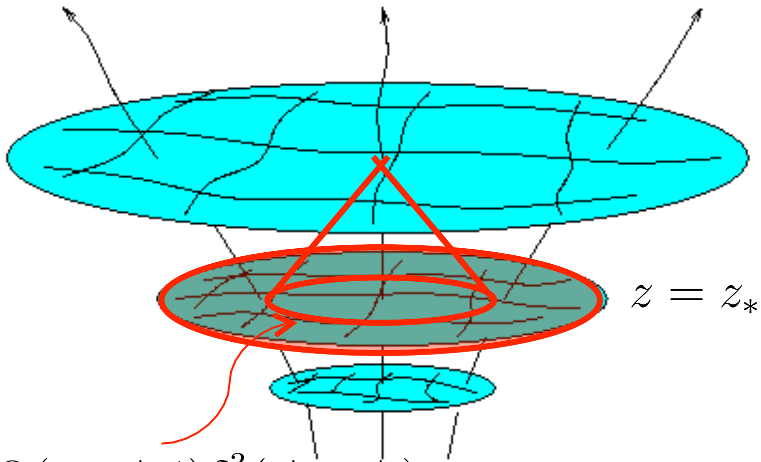
The cosmic microwave background Radiation's "surface of last scatter" is analogous to the light coming through the clouds to our eye on a cloudy day.

$$\tau_* = 1$$





Sudden recombination



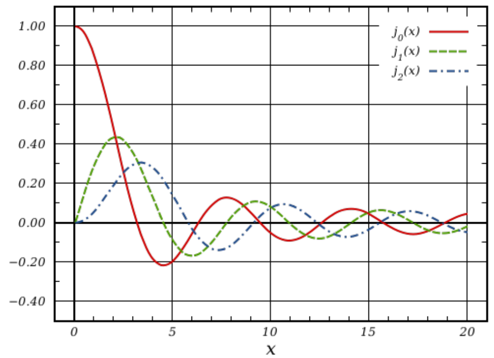
- Instant recombination at $\vec{x}_* = -r_* \hat{p}$

$$\Theta(\hat{p}) = \int d^3\vec{x}\,\Theta(\eta_*, \vec{x}, \hat{p})\delta^3(\vec{x} - \vec{x}_*)$$

- Fourier decomposition: $$\Theta(\eta, \vec{x}) = \frac{1}{(2\pi)^3} \int d^3\vec{k}\, \Theta_{\vec{k}}(\eta)\, e^{i\vec{k}.\vec{x}}$$
  - Plane-wave expansion in spherical harmonics

$$e^{i\vec{k}.\vec{r}} = 4\pi \sum_{\ell,m} i^\ell j_\ell(kr) Y^*_{\ell m}(\hat{k}) Y_{\ell m}(\hat{r})$$

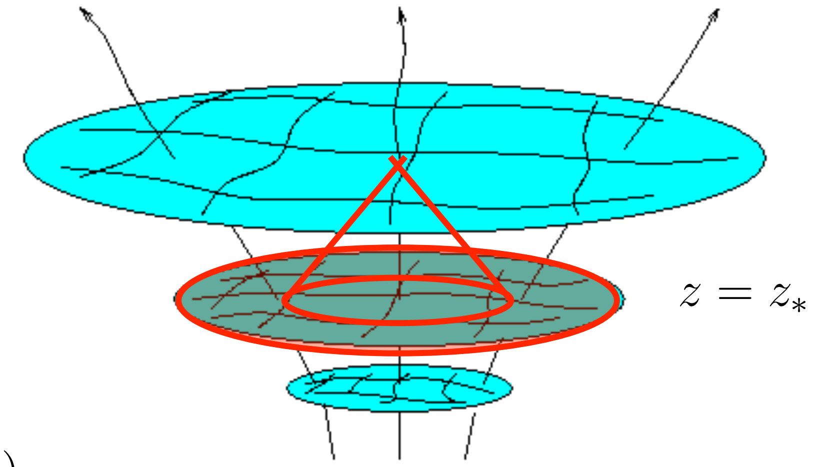
  - with spherical Bessel function $$j_\ell(x) = (\pi/2x)^{1/2} J_{\ell+(1/2)}(x)$$



- Spherical harmonic coefficient:

$$a_{\ell m} = \frac{1}{(2\pi)^3} \int d^3\vec{k}\, \Theta(\hat{k}) 4\pi i^\ell j_\ell(kr_*) Y^*_{\ell m}(\hat{k})$$

Angular power spectrum at sudden recombination



- Spherical harmonic coefficient:

$$a_{\ell m} = \frac{1}{(2\pi)^3} \int d^3\vec{k}\, \Theta(\hat{k}) 4\pi i^\ell j_\ell(kr_*) Y^*_{\ell m}(\hat{k})$$

- Angular power spectrum: $\langle a^*_{\ell m} a_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell$

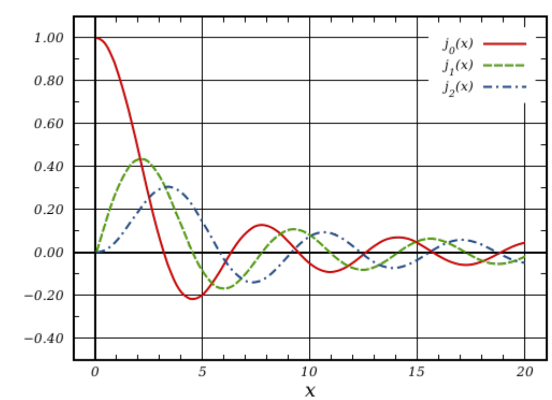
  – where $$C_\ell = \int d(\ln k) \mathcal{P}_\Theta(k) 4\pi j_\ell^2(kr_*)$$

  – and window function peaks at $l = kr_*$

$$W_\ell(k) \equiv 4\pi j_\ell^2(kr_*)$$

- For scale-invariant P(k)

$$\frac{\ell(\ell+1)}{2\pi} C_\ell = \mathcal{P}_\Theta(\ell/r_*)$$



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Perturbed FLRW geometry

Malik & Wands, Phys Rep (2009)

- Perturbed metric:

$$ds^2 = a^2 \left\{ -(1+2\mathcal{A})d\eta^2 + 2\nabla_i \mathcal{B}\, dx^i d\eta + \left[(1+2\mathcal{C})\delta_{ij} + 2\nabla_i\nabla_j\mathcal{E}\right] dx^i dx^j \right\}$$

  - 4 scalar metric perturbations $\mathcal{A} \quad , \quad \mathcal{B} \quad , \quad \mathcal{C} \quad , \quad \mathcal{E}$
  - 3 combinations independent of spatial threading: $\mathcal{A} \quad , \quad \mathcal{C} \quad , \quad \sigma \equiv \mathcal{E}' - \mathcal{B}$
  - 2 combination independent of time slicing: $\boxed{\Psi \equiv \mathcal{A} - \mathcal{H}\sigma - \sigma' \quad , \quad \Phi \equiv \mathcal{C} - \mathcal{H}\sigma}$
  - Vanishing anisotropic stress (e.g., tight-coupling)) $\boxed{\Psi + \Phi = 0}$

- Other gauge-invariant variables in conformal Newtonian (*E=B=0*) gauge

$$\boxed{\delta \equiv \frac{\delta\rho - \rho'\sigma}{\rho} \quad , \quad V \equiv v + \mathcal{E}'}$$

- but not unique gauge-independent variables
  - e.g., curvature perturbation in uniform-density gauge:

$$\boxed{\zeta = \mathcal{C} - \frac{\mathcal{H}}{\rho'}\delta\rho} = \Phi - \frac{\mathcal{H}\rho}{\rho'}\delta$$

Perturbed geodesics

- Perturbed metric in conformal Newtonian gauge:

$$ds^2 = a^2\left\{-(1+2\Psi)d\eta^2 + (1+2\Phi)\delta_{ij}dx^i dx^j\right\}$$

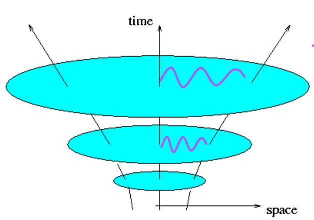
- Null trajectory: $$\frac{dx^i}{d\eta} = (1+\Psi-\Phi)\hat{p}^i$$

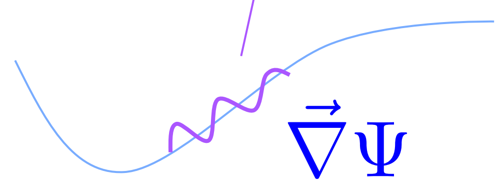
- 4-momentum: $$P^\mu = \frac{dx^\mu}{d\lambda}$$

- 3-momentum: $$p^2 = g_{ij}P^iP^j$$

- Geodesic equation: $$\frac{1}{p}\frac{dp}{d\eta} = -\left(\frac{1}{a}\frac{da}{d\eta} + \frac{\partial\Phi}{\partial\eta}\right) - \hat{p}^i\frac{\partial\Psi}{\partial x^i}$$

  – Perturbed Hubble redshift + gravitational potential redshift:





Sachs-Wolfe effect:

- Geodesic equation: $$\frac{1}{p}\frac{dp}{d\eta} = -\left(\frac{1}{a}\frac{da}{d\eta} + \frac{\partial\Phi}{\partial\eta}\right) - \hat{p}^i\frac{\partial\Psi}{\partial x^i}$$

- Total derivative along photon path: $$\frac{d\Psi}{d\eta} = \frac{\partial\Psi}{d\eta} + \hat{p}^i\frac{\partial\Psi}{\partial x^i}$$

$$\Rightarrow \frac{1}{p}\frac{dp}{d\eta} = -\frac{1}{a}\frac{da}{d\eta} - \frac{d\Psi}{d\eta} + \frac{\partial}{\partial\eta}(\Psi - \Phi)$$

- Integrate along path from recombination to observer

$$[\ln p]_*^0 = -[\ln a]_*^0 - [\Psi]_0^* + \int_*^0 (\Psi' - \Phi')d\eta$$

- First-order momentum perturbation

$$\left(\frac{\delta p}{p}\right)_0 = \left(\frac{\delta p}{p}\right)_* + \Psi_* - \Psi_0 + \int_*^0 (\Psi' - \Phi')d\eta$$

- Observed temperature includes Doppler shift

$$\Theta = \frac{\delta p}{p} + \hat{n}.\vec{V}$$

Sachs-Wolfe formula:

$$\boxed{\Theta_{obs} = \frac{1}{4}\delta_{\gamma *} + \Psi_* - \hat{n}.\vec{V}_* + \int_*^0 (\Psi' - \Phi')d\eta - \Psi_0 + \hat{n}.\vec{V}_{obs}}$$

- Intrinsic Sachs-Wolfe effect (on last-scattering surface):

$$\frac{1}{4}\delta_{\gamma *} + \Psi_* - \hat{n}.\vec{V}_{\gamma *}$$

- Integrated Sachs-Wolfe effect (decay of potential along line of sight)

$$\int_*^0 (\Psi' - \Phi')d\eta$$

- Monopole: $-\Psi_0$

- Dipole: $\hat{n}.\vec{V}_{obs}$

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Linear perturbation dynamics:

scalar perturbations in arbitrary gauge

- Einstein energy+momentum constraints:

$$\begin{aligned} 3\mathcal{H}\left(-\mathcal{C}' + \mathcal{H}\mathcal{A}\right) + \nabla^2\left[\mathcal{C} - \mathcal{H}\sigma\right] &= -4\pi G a^2 \delta\rho \\ -\mathcal{C}' + \mathcal{H}\mathcal{A} &= -4\pi G a^2 (\rho + P)(v + \mathcal{B}) \end{aligned}$$

- Einstein evolution equations

$$\begin{aligned} \mathcal{C}'' + 2\mathcal{H}\mathcal{C}' - \mathcal{H}\mathcal{A}' - \left(2\mathcal{H}' + \mathcal{H}^2\right)\mathcal{A} &= -4\pi G a^2 \left(\delta P + \frac{2}{3}\nabla^2 \Pi\right) \\ \sigma' + 2\mathcal{H}\sigma - \mathcal{C} - \mathcal{A} &= 8\pi G a^2 \Pi \end{aligned}$$

- Energy+momentum conservation (continuity+Euler) equations

$$\delta\rho' + 3\mathcal{H}\left(\delta\rho + \delta P\right) + 3\left(\rho + P\right)\mathcal{C}' + (\rho + P)\nabla^2\left(v + \mathcal{E}'\right) = 0$$

$$(v + \mathcal{B})' + (1 - 3c_{\rm s}^2)\mathcal{H}(v + \mathcal{B}) + \phi + \frac{1}{\rho + P}\left(\delta P + \frac{2}{3}\nabla^2\Pi\right) = 0$$

Linear perturbation dynamics:

scalar perturbations in arbitrary gauge

-

-

- Energy conservation (continuity) equation

$$\delta\rho' + 3\mathcal{H}\left(\delta\rho + \delta P\right) + 3\left(\rho + P\right)\mathcal{C}' + \left(\rho + P\right)\nabla^2\left(v + \mathcal{E}'\right) = 0$$

Linear perturbation dynamics:

scalar perturbations in ~~arbitrary~~ gauge uniform density, $\delta\rho = 0$

- Energy conservation (continuity) equation

$$\cancel{\delta\rho'} + 3\mathcal{H}\left(\cancel{\delta\rho} + \delta P\right) + 3\left(\rho + P\right)\mathcal{C}' + \left(\rho + P\right)\nabla^2\left(v + \mathcal{E}'\right) = 0$$

Energy conservation on large scales:

- uniform-density gauge: $(\delta\rho = 0,\ \mathcal{C} = \zeta)$

$$3\mathcal{H}\delta P_{nad} + 3\left(\rho + P\right)\zeta' + \left(\rho + P\right)\nabla^2 V = 0$$

$$\boxed{\Rightarrow \quad \zeta' = -\mathcal{H}\frac{\delta P_{nad}}{\rho + P} - \frac{1}{3}\nabla^2 V}$$

  – Conserved quantities for barotropic fluids ($\delta P_{nad}=0$) on large scales ($V=0$)
    - radiation density perturbation $\zeta_\gamma = \frac{1}{4}\delta_\gamma + \Phi$
    - matter density perturbation $\zeta_m = \frac{1}{3}\delta_m + \Phi$

Initial conditions on large scales:

- Curvature and entropy perturbation:

$$\zeta = \frac{4\rho_\gamma\zeta_\gamma + 4\rho_m\zeta_m}{4\rho_\gamma + 3\rho_m} \quad , \quad S_m = 3(\zeta_m - \zeta_\gamma)$$

  - Adiabatic initial conditions:

$$\zeta = \zeta_\gamma = \zeta_m = \text{constant} \quad , \quad S_m = 0$$

  - Isocurvature initial conditions:

$$\zeta_\gamma = 0 \quad , \quad S_m = 3\zeta_m = \text{constant}$$

Intrinsic Sachs-Wolfe effect (on large scales)

$$\frac{\delta T}{T} = \frac{1}{4}\delta_{\gamma *} + \Psi_* = \zeta_\gamma + 2\Psi_*$$

end of part two

"Mr. Osborne, may I be excused? My brain is full."