Quantum Field Theory in Curved Space-time

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Bibliography

- Birrell&Davies “Quantum fields in curved space”
- Fabbri&Navarro-Salas “Modeling BH Evaporation”
- Frolov&Novikov “BH Physics”
- Fulling “Aspects of QFT in Curved S-t”
- Mukhanov&Winitzki “Quantum Effects in Gravity”
- Parker&Toms “QFT in Curved S-t”
- Wald “QFT in Curved S-t and BH Thermodynamics”

Classical Field Theory

Action Principle

- Line-element & metric of a curved s-t:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad\qquad g \equiv det(g_{\mu\nu})$$

- Action functional for a tensor field

$$S[\phi(x)] = \int_\Omega d^4x \, \sqrt{-g} \, \mathcal{L}(\phi(x), \phi_{,\mu}(x), x)$$

region of s-t

Lagrangian density

- Action principle: EOM for $\phi(x)$ are given by extremizing the action wrt $\phi(x)$:

$$\frac{\delta S}{\delta \phi(x)} = 0$$

Example: GR

- Einstein-Hilbert action (the field is $g_{\mu\nu}$):

$$S_{GR}[g_{\mu\nu}] = \frac{c^3}{16\pi G} \int_\Omega d^4x \, \sqrt{-g} \, (\underbrace{R}_{\text{Ricci scalar}} + \underbrace{2\Lambda}_{\text{Cosmological const.}})$$

- Einstein-Hilbert action with matter action:

$$S[\phi, g_{\mu\nu}] = S_{GR}[g_{\mu\nu}] + S_m[\phi, g_{\mu\nu}]$$

- Action principle: $\hbar = c = 1$

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \longrightarrow$$ Einstein field eqs.:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Stress-energy tensor: $$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}$$

T_{00} : energy density

T_{0j} : momentum density $(\mu, \nu = 0, 1, 2, 3; \qquad j = 1, 2, 3)$

T_{ij} : (normal & shear) stress

Stress-energy Tensor

- Bianchi identity $\Longrightarrow \nabla_\nu(\text{l.h.s. of EFE}) = 0 \Longrightarrow$

  it must be: $\nabla_\nu T^{\mu\nu} = 0$

- If S_m is constructed from a Lagrangian which is a scalar and the fields obey the matter EOM, then $T^{\mu\nu}$ is conserved

Example: Scalar Field

- For a real (minimally-coupled) scalar field of mass m:

$$S[\phi; g_{\mu\nu}] = \int_\Omega d^4x\sqrt{-g}\,\left[-\underbrace{\frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu}}_{\text{K.E.}} - \underbrace{\frac{1}{2}m^2\phi^2}_{\text{P.E.}}\right]$$

$$\xrightarrow{\delta S/\delta\phi = 0} \quad \text{Klein-Gordon eq.:} \quad \Box\phi - m^2\phi = 0$$

$$\Box \equiv \nabla^\mu\nabla_\mu$$

$$\xrightarrow{\delta S/\delta g^{\mu\nu}} \quad T_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi^{;\alpha}\phi_{;\alpha} - \frac{1}{2}m^2\phi^2 g_{\mu\nu}$$

Example: Scalar Field in Flat s-t

- Klein-Gordon eq. for a massive real scalar classical field in flat s-t:

$$\Box\phi - m^2\phi = 0 \qquad\qquad \Box = -\partial_t^2 + \vec{\nabla}^2$$

- Apply spatial Fourier transform:

The modes $$\phi_{\vec{k}}(t) \equiv \int_{\mathbb{R}^3} \frac{d^3\vec{x}}{(2\pi)^{3/2}}\; e^{-i\vec{k}\vec{x}}\phi(\vec{x},t)$$

satisfy the eq. for the simple harmonic oscillator:

$$\frac{d^2\phi_{\vec{k}}}{dt^2} + \omega_k^2\phi_{\vec{k}} = 0$$

oscillator frequency: $\omega_k \equiv \sqrt{k^2 + m^2}$

- The general solution may be expressed via the inverse Fourier transform as

$$\phi(x) = \int_{\mathbb{R}^3} d^3\vec{k} \left(a_{\vec{k}}\, u_{\vec{k}}(x) + \underbrace{a^*_{\vec{k}}\, u^*_{\vec{k}}(x)}_{} \right)$$

so that $\phi(x) \in \mathbb{R}$

$$u_{\vec{k}}(x) \equiv \frac{e^{-i\omega_k t + i\vec{k}\vec{x}}}{\sqrt{16\pi^3\omega_k}}$$

Fourier coefficients

- The field may be viewed as an infinite set of decoupled simple harm. oscs., one for each $\vec{k}$

- In particular, the Hamiltonian is that for a set of harm. oscs.:

$$H = \int_{\mathbb{R}^3} d^3\vec{x}\; T_{00} = \frac{1}{2}\int_{\mathbb{R}^3} d^3\vec{x}\; \left(\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

$$H = \int_{\mathbb{R}^3} d^3\vec{k}\; \frac{\omega_k}{2} \left[a^*_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} a^*_{\vec{k}}\right]$$

QFT in Flat Space-time

I. Formalism

Canonical Quantization

- A real scalar field may be viewed as an infinite collection of simple harm. oscs., one for each $\vec{k}$

- In order to quantize the field (in the Heisenberg picture) we just quantize the harm. oscs.:

$a_{\vec{k}} \;\rightarrow\; \hat{a}_{\vec{k}}$: annihilation ops. $\qquad a^*_{\vec{k}} \;\rightarrow\; \hat{a}^\dagger_{\vec{k}}$: creation ops.

$$\Longrightarrow\; \phi(x) \;\rightarrow\; \hat{\phi}(x) = \int_{\mathbb{R}^3} d^3\vec{k}\; \left(\hat{a}_{\vec{k}}\, u_{\vec{k}}(x) + \hat{a}^\dagger_{\vec{k}}\, u^*_{\vec{k}}(x)\right)$$

$$\Longrightarrow\; T_{\mu\nu}(x) \;\rightarrow\; \hat{T}_{\mu\nu}(x)$$

Commutation Relations

- Define the canonical field momentum in CFT: $\Pi(x) \equiv \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}}$

  and quantize: $\Pi(x) \rightarrow \hat{\Pi}(x)$

- Motivated by the commutation rlns. of QM: $[\hat{q}(t), \hat{p}(t)] = i$

  we impose the equal-time commutation rlns.:

$$\left[\hat{\phi}(\vec{x}, t), \hat{\Pi}(\vec{x}', t)\right] = i\delta^{(3)}(\vec{x} - \vec{x}')$$

$$\left[\hat{\phi}(\vec{x}, t), \hat{\phi}(\vec{x}', t)\right] = \left[\hat{\Pi}(\vec{x}, t), \hat{\Pi}(\vec{x}', t)\right] = 0$$

QFT Divergences

- The equal-time comm. rlns. are equivalent to the comm. rlns.:

$$\left[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right] = \delta^{(3)}\left(\vec{k} - \vec{k}'\right), \quad \left[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}\right] = 0 = \left[\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}\right]$$

$\hat{\phi}$ is an operator-valued distribution!

- So products like $\hat{\phi}^2(x)$ in $\hat{T}_{\mu\nu}(x)$ are just *formal* expressions and will plague the QFT with divergences

- Note: strictly, one should integrate it against a test-function $f(x)$ in order to get a well-defined operator:

$$\hat{\phi}(f) \equiv \int d^4x \; f(x)\hat{\phi}(x)$$

Hilbert Space

- The Minkowski vacuum state is defined via $\hat{a}_{\vec{k}} \, |M\,\rangle = 0, \quad \forall \vec{k}$

- Excited states $$|\, n_1, n_2, \ldots \rangle = \prod_s \frac{1}{\sqrt{n_s!}} \left(\hat{a}^{+}_{\vec{k}_s} \right)^{n_s} \, |\, M\rangle$$

span a Hilbert space (Fock representation). They correspond to states with n_s quantum particles with momentum $\vec{k}_s \;\; \forall s$

Scalar Product

- Define $$(\phi_1, \phi_2) \equiv i \int_{\mathbb{R}^3, t=t_0} d^3\vec{x}\, [\phi_1^* \partial_t \phi_2 - \phi_2 \partial_t \phi_1^*]$$

- it is a scalar product

- if ϕ_1 & ϕ_2 are slns. of K-G eq, then it is independent of t_0

- The Minkowski modes $u_{\vec{k}}(x)$ are orthonormal:

$$\left(u_{\vec{k}}, u_{\vec{k}'}\right) = \delta^{(3)}\left(\vec{k} - \vec{k}'\right) = -\left(u^*_{\vec{k}}, u^*_{\vec{k}'}\right)$$

$$\left(u_{\vec{k}}, u^*_{\vec{k}'}\right) = 0$$

Quantization Procedure

(1) Choose a set $\{u_i(x), u_i^*(x), \forall i\}$ of slns. of field eq. that is complete and orthonormal:

$$(u_i, u_j) = \delta_{ij} = -\left(u_i^*, u_j^*\right), \quad \left(u_i, u_j^*\right) = 0$$

(2) Then any op. sln. of the field eq. may be expanded as

$$\hat{\phi}(x) = \sum_i \hat{a}_i u_i(x) + \hat{a}_i^\dagger u_i^*(x)$$

where $\hat{a}_i = \left(u_i, \hat{\phi}\right), \quad \hat{a}_i^\dagger = -\left(u_i^*, \hat{\phi}\right)$

(3) Then imposing comm. rlns. for $\hat{a}_i, \ \hat{a}_i^\dagger$ is equivalent to imposing comm. rlns. for $\hat{\phi}, \ \hat{\Pi}$

(4) Define a vacuum by $\hat{a}_i \left|0\right\rangle = 0, \forall i$

excited states by $|\, n_i\rangle \equiv \left(\hat{a}_i^\dagger\right)^n |\, 0\rangle/\sqrt{n!}$ etc

Hamiltonian

- Hamiltonian: $$\hat{H} = \int_{\mathbb{R}^3} d^3\vec{k}\ \frac{\omega_k}{2} \left[\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}\hat{a}^\dagger_{\vec{k}}\right]$$

- Zero-point energy density:

$\langle M \mid \hat{H} \mid M \rangle$ per unit volume=

$$\frac{1}{2}\int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3}\omega_k = \frac{1}{4\pi^2}\int_0^\infty dk\ k^2\sqrt{k^2+m^2}$$

has ultra-violet divergence since there is an infinite amount of harm. oscs., each with zero-point energy $\dfrac{\omega_k}{2}$

- Origin: $\hat{\phi}(x)$ is an operator-valued distribution

Normal Ordering

- Renormalization: this infinite cannot be detected by measuring transitions between states -> it can be subtracted away by normal ordering: place all annihilation ops. to the right of the creation ops.

$$\hat{H} = \int_{\mathbb{R}^3} d^3\vec{k}\ \frac{\omega_k}{2}\left[\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}\hat{a}^\dagger_{\vec{k}}\right] \longrightarrow\ :\hat{H}: = \int_{\mathbb{R}^3} d^3\vec{k}\ \omega_k \hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}}$$

$\langle M \mid : \hat{H} : \mid M \rangle = 0$ in agreement with experiments

- Normal-ordering is equivalent to subtracting the exp.val. in Minkowski vacuum:

$$\langle \psi \mid : \hat{\phi}(x)^2 : \mid \psi \rangle = \langle \psi \mid \hat{\phi}(x)^2 \mid \psi \rangle - \langle M \mid \hat{\phi}(x)^2 \mid M \rangle$$

Bogolyubov Transformations

- Consider two complete and orthonormal sets of slns. of the field eqs.: $\{u_i(x), u_i^*(x), \forall i\}$ and $\{v_i(x), v_i^*(x), \forall i\}$

- Then

$$\hat{\phi}(x) = \sum_i \hat{a}_i u_i(x) + \hat{a}_i^\dagger u_i^*(x) = \sum_r \hat{b}_r v_r(x) + \hat{b}_r^\dagger v_r^*(x)$$

and, since $\{u_i(x), \forall i\}$ form a complete set,

$$v_r(x) = \sum_i \alpha_{ri}\ u_i(x) + \beta_{ri}\ u_i^*(x) \longleftarrow \text{Bogolyubov transf.}$$

Bogolyubov coeffs.: $\alpha_{ri},\ \beta_{ri} \in \mathbb{C}$

Vacuum States

- The relationship between the creation/annihilation ops. is then

$$\hat{b}_r = \left(v_r, \hat{\phi}\right) = \cdots = \sum_i \alpha^*_{ri}\ \hat{a}_i - \beta^*_{ri}\ \hat{a}^\dagger_i$$

- Different choices of complete sets yield different 'vacuum' quantum states of the matter field:

$$\hat{a}_i\,|0\,\rangle_u = 0, \quad \forall i \qquad \text{and} \qquad \hat{b}_r\,|0\,\rangle_v = 0, \quad \forall r$$

- The number of quantum v-particles of mode-type r in the u-vacuum is

$$_u\langle 0 \mid \hat{b}^\dagger_r \hat{b}_r \mid 0\rangle_u = \cdots = \sum_i |\beta_{ri}|^2$$

QFT in Flat Space-time

II. Unruh Effect

Rindler Observers

- Consider 2-D flat s-t: $ds^2 = -dt^2 + dy^2 = -du dv$

  null coords.: $u \equiv t - y, \quad v \equiv t + y$

  where t is the proper time of inertial observers

- Rindler obs.: obs. with uniform acceleration $a^2 \equiv a^\alpha a_\alpha = const$

  and proper time τ . They have

$$t(\tau) = \frac{\sinh(a\tau)}{a}, \quad y(\tau) = \frac{\cosh(a\tau)}{a} \quad \to y^2 - t^2 = a^{-2}$$

(hyperbola)

Rindler Frame

- Coordinate transf. to Rindler coords. $\{\eta, \zeta\}$:

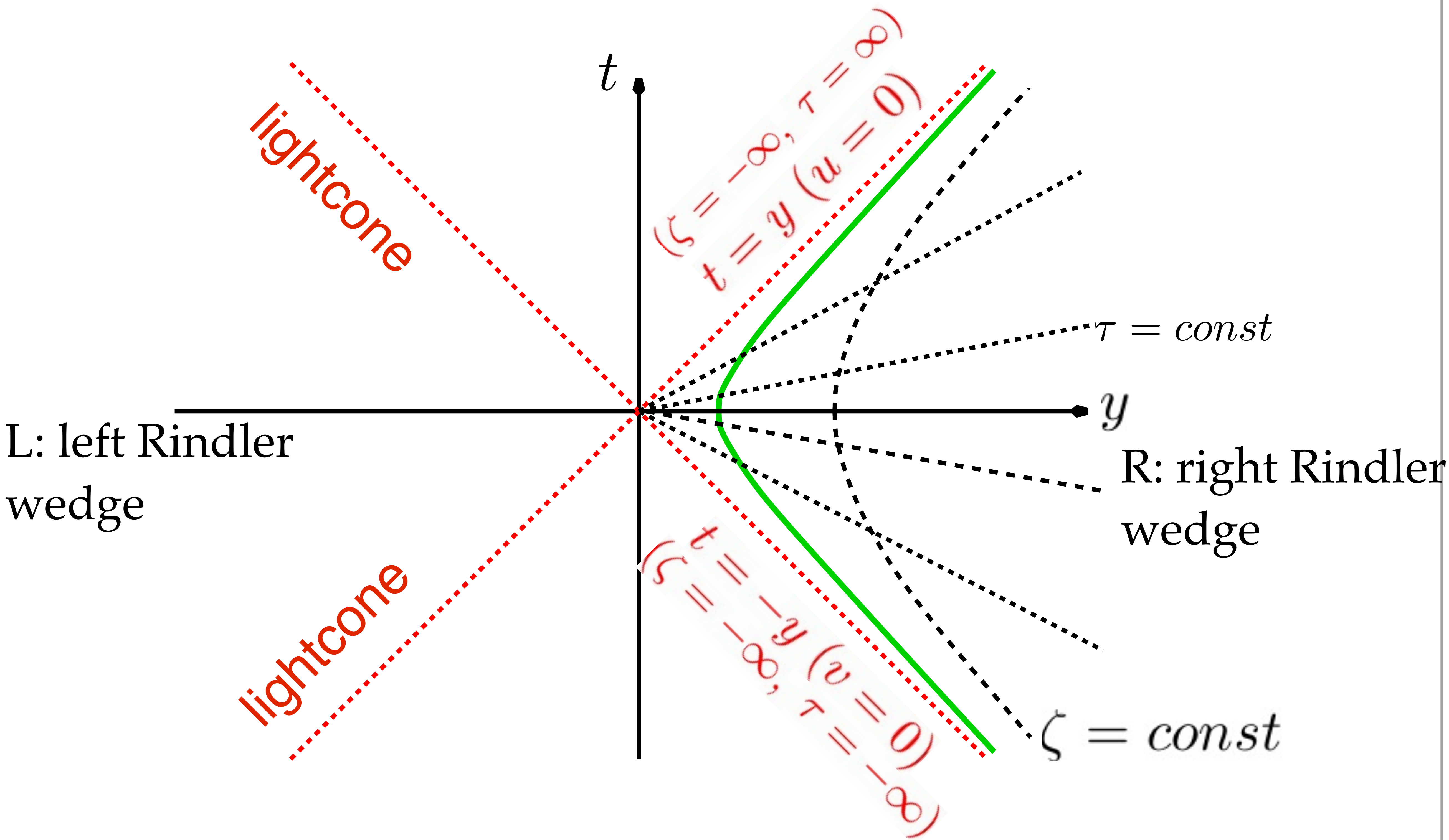
$$\alpha t = e^{\alpha\zeta} \sinh(\alpha\eta), \;\; \alpha y = e^{\alpha\zeta} \cosh(\alpha\eta) \qquad \alpha > 0$$

$$\rightarrow ds^2 = \underset{\uparrow}{e^{2\alpha\zeta}} \left(-d\eta^2 + d\zeta^2\right)$$

conformal factor

- Rindler obs. are at $\zeta = const$

- Range $\tau, \zeta \in (-\infty, \infty)$ only cover $y > |t|$: the right Rindler wedge. Similar transformations can be used to cover the whole s-t



- Rindler obs. 'perceive an event horizon' at the Killing horizons at $\zeta = -\infty$

Wave Eq.

- K-G eq. in the inertial frame: $\left(-\partial_t^2 + \partial_y^2\right)\phi = 0 \quad \longrightarrow$

$$\phi_k = \frac{e^{-i\omega t + iky}}{\sqrt{4\pi\omega}} = \frac{1}{\sqrt{4\pi\omega}} \begin{cases} e^{-i\omega u}, & k > 0 \\ e^{-i\omega v}, & k < 0 \end{cases}$$

$$\forall k \in \mathbb{R}, \qquad \omega \equiv |k|$$

These modes:

- are pos. freq. modes wrt ∂_t : $\quad \partial_t \phi_k = -i\omega\phi_k$

- form a complete set in the whole s-t: $\hat{\phi} = \int_{\mathbb{R}} dk \, \left(\hat{a}_k \phi_k + \hat{a}_k^\dagger \phi_k^*\right)$

- K-G eq. in the Rindler frame: $\left(-\partial_\eta^2 + \partial_\zeta^2\right)\phi = 0$

$$\bar{\phi}^R_{\bar{k}} = \frac{e^{-i\bar{\omega}\eta + i\bar{k}\zeta}}{\sqrt{4\pi\bar{\omega}}} = \frac{1}{\sqrt{4\pi\bar{\omega}}}\begin{cases}(-\alpha u)^{i\bar{k}/\alpha}, & \bar{k} > 0 \\ (\alpha v)^{i\bar{k}/\alpha}, & \bar{k} < 0\end{cases} \quad \text{in R}$$

$$\bar{\phi}^R_{\bar{k}} = 0 \quad \text{in L}$$

$$\forall \bar{k} \in \mathbb{R}, \quad \bar{\omega} \equiv |\bar{k}|$$

These modes:

- are pos. freq. modes wrt ∂_η : $\partial_\eta \bar{\phi}^R_{\bar{k}} = -i\bar{\omega}\bar{\phi}^R_{\bar{k}}$

- form a complete set in R only -> We can also find the mode slns.:

$$\bar{\phi}^L_{\bar{k}} = \frac{1}{\sqrt{4\pi\bar{\omega}}}\begin{cases}(-\alpha v)^{i\bar{k}/\alpha}, & \bar{k} > 0 \\ (\alpha u)^{i\bar{k}/\alpha}, & \bar{k} < 0\end{cases} \quad \text{in L}$$

$$\bar{\phi}^L_{\bar{k}} = 0 \quad \text{in R}$$

- Together they form a complete set in the whole s-t:

$$\hat{\phi} = \int_{\mathbb{R}} d\bar{k} \ \left(\hat{b}^R_{\bar{k}} \bar{\phi}^R_{\bar{k}} + \hat{b}^L_{\bar{k}} \bar{\phi}^L_{\bar{k}} + \text{h.c.} \right)$$

- Vacuum states:

- Minkowski vacuum: $\hat{a}_k \mid M\rangle = 0, \ \forall k \in \mathbb{R}$

- Rindler vacuum: $\hat{b}^R_{\bar{k}} \mid R\rangle = \hat{b}^L_{\bar{k}} \mid R\rangle = 0, \ \forall \bar{k} \in \mathbb{R}$

- How many Rindler particles does the Minkowski vacuum contain? Unruh'76

- The Minkowski modes $\phi_k = \frac{1}{\sqrt{4\pi\omega}} \begin{cases} e^{-i\omega u}, & k > 0 \\ e^{-i\omega v}, & k < 0 \end{cases}$

are analytic and bounded in the lower half of the complex u- and v-planes

- Any lin.comb. of pos. freq. (wrt ∂_t) modes ϕ_k (ie, trivial Bogolyubov transf. $\beta_{kk'} = 0$) will have this property but it won't if any negat. freq. mode ϕ_k^* is included -> characterization of Minkowski pos.freq. modes: they are analytic and bounded in the lower u- and v-planes

- Rindler modes for $\bar{k} > 0$: $\phi_{\bar{k}}^R \propto \begin{cases} (-\alpha u)^{i\bar{k}/\alpha}, & u < 0 \text{ (eg, in R)} \\ 0, & u > 0 \text{ (eg, in L)} \end{cases}$

are not analytic

- But it can be shown that the modes

$$u_{\bar{k}}^{R} \equiv \frac{e^{\pi\bar{k}/(2\alpha)}}{\sqrt{2\sinh\left(\pi\bar{k}/\alpha\right)}}\left[\bar{\phi}_{\bar{k}}^{R} + e^{-\pi\bar{k}/\alpha}\bar{\phi}_{-\bar{k}}^{L^*}\right]$$

are analytic and bounded in the lower half of the u- and v-planes

- For $\bar{k} > 0$ they are concentrated in R, so also construct

$$u_{\bar{k}}^{L} \equiv \frac{e^{-\pi\bar{k}/(2\alpha)}}{\sqrt{2\sinh\left(\pi\bar{k}/\alpha\right)}}\left[\bar{\phi}_{-\bar{k}}^{R^*} + e^{\pi\bar{k}/\alpha}\bar{\phi}_{\bar{k}}^{L}\right]$$

which are also analytic and bounded in the lower half of the u- and v-planes, and, for $\bar{k} > 0$, they are concentrated in L

- We can expand $\hat{\phi} = \int_{\mathbb{R}} d\bar{k} \;\left(\hat{a}^R_{\bar{k}} u^R_{\bar{k}} + \hat{a}^L_{\bar{k}} u^L_{\bar{k}} + \text{h.c.} \right)$

  then $\hat{a}^R_{\bar{k}} \mid M\rangle = \hat{a}^L_{\bar{k}} \mid M\rangle = 0, \;\forall \bar{k} \in \mathbb{R}$

- We have explicitly constructed the Bogolyubov transf.:

$$\hat{b}^R_{\bar{k}} = \left(\hat{\phi}, \bar{\phi}^R_{\bar{k}}\right) = \frac{1}{\sqrt{2\sinh\left(\pi\bar{k}/\alpha\right)}} \left[e^{\pi\bar{k}/(2\alpha)} \hat{a}^R_{\bar{k}} + e^{-\pi\bar{k}/(2\alpha)} \hat{a}^{L\dagger}_{-\bar{k}} \right]$$

- Therefore the number of Rindler particles contained in the Minkowski vacuum is

$$\langle M \mid \hat{b}^{R\dagger}_{\bar{k}} \hat{b}^R_{\bar{k}} \mid M\rangle = \frac{1}{e^{2\pi\bar{k}/\alpha} - 1}$$

  this is a thermal Planck spectrum of Rindler particles with temperature $T_0 = \dfrac{\alpha}{2\pi k_B}$

- It can be shown that the ‘renormalized’ $\langle R \mid : \hat{T}_{\mu\nu} : \mid R \rangle$ diverges at the Killing horizons (in the regular inertial frame)

- Therefore, the Rindler vacuum is an unphysical state

- A ‘quantum particle detector’ sees as a vacuum state that state which is defined using pos.freq. modes wrt its proper time [Unruh’76,Brown&Ottewill’83,Grove&Ottewill’83] (also true in curved s-t)

- So, if the field is in the Minkowski vacuum,

  - an inertial observer detects no particles

  - a Rindler observer detects particles as if he were in a thermal bath at the Unruh temperature

- Unruh effect not currently measurable, eg,

  for $T = 1K < T_{CMB} \approx 3K$ we need $a \sim 10^{20} m/s^2$

QFT in Curved Space-time

I. Formalism

Semiclassical Einstein Eqs.

- Semiclassical theory of Quantum Gravity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G \, \langle \psi \mid \hat{T}_{\mu\nu} \mid \psi \rangle$$

matter fields are quantized (in some appropriate state $\mid \psi \rangle$) but gravitational field is not. We will keep the grav. background classical and fixed

- This is valid in the limit that the length and time scales of the physical processes $\gg$ Planck length and time

$$\left(G\hbar/c^3\right)^{1/2} \sim 10^{-33} cm \quad \text{and} \quad \left(G\hbar/c^5\right)^{1/2} \sim 10^{-44} s$$

Renormalization

- However, $\langle\psi \mid \hat{T}_{\mu\nu} \mid \psi\rangle$ suffers from ultraviolet divergences since $\hat{\phi}(x)$ is an operator-valued distribution

- So we need to renormalize: $\langle\psi \mid \hat{T}_{\mu\nu} \mid \psi\rangle \to \langle\psi \mid \hat{T}_{\mu\nu} \mid \psi\rangle_{ren}$

- Energy now creates s-t curvature via Einstein eqs., so we cannot renormalize by simply subtracting constant infinities like we did in flat s-t

- Classically: $$\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = T_{\mu\nu}$$

$$S_m = \int_\Omega d^4x\sqrt{-g}\,\mathcal{L}_m$$

- Semiclassically: we define an effective action and Lagrangian

$$S_{eff} = \int_\Omega d^4x\sqrt{-g}\,\mathcal{L}_{eff}$$

s.t.

$$\frac{2}{\sqrt{-g}}\frac{\delta S_{eff}}{\delta g^{\mu\nu}} = \langle\psi \mid \hat{T}_{\mu\nu} \mid \psi\rangle$$

- The divergences in $\langle\psi \mid \hat{T}_{\mu\nu} \mid \psi\rangle$ can be traced back to divergences in $\mathcal{L}^{eff}$

- Before removing the divergences, one must use a regularization scheme to make sense of $\hat{\phi}^2(x)$

Eg, point-splitting lets temporarily $\hat{\phi}^2(x) \rightarrow \hat{\phi}(x)\hat{\phi}(x')$
then remove the divergences and afterwards take the limit $x' \rightarrow x$

$$\langle \psi \mid \hat{T}_{\mu\nu}(x) \mid \psi \rangle_{ren} = \lim_{x' \to x} \Big(\langle \psi \mid \hat{T}_{\mu\nu}(x, x') \mid \psi \rangle - T^{div}_{\mu\nu}(x, x') \Big)$$

How to calculate $T^{div}_{\mu\nu}(x, x')$?

- Using certain regularization schemes (like that in point-splitting) one can show that the divergences of $\mathcal{L}^{eff}$:

(a) only depend on the background $g_{\mu\nu}$ (they are independent of $\mid \psi \rangle$)

and...

(b) can be expressed as terms which are either:

(b1) proportional to R $\longrightarrow$ renormalize G

Recall: $\mathcal{L}^{grav} + \mathcal{L}^{eff} = -\dfrac{R}{16\pi G} - \dfrac{\Lambda}{8\pi G} + \mathcal{L}^{eff} =$

$$\Bigg(\underbrace{\frac{-R}{16\pi G} + \text{‘div. prop. to R’}}_{\dfrac{-R}{16\pi G_{ren}}}\Bigg) - \frac{\Lambda}{8\pi G} + \left(\mathcal{L}^{eff} - \text{‘div. prop. to R’}\right)$$

(b2) constant $\longrightarrow$ renormalize Λ

$$\mathcal{L}^{grav} + \mathcal{L}^{eff} = \Bigg(\underbrace{\frac{-R}{16\pi G} + \text{‘div. prop. to R’}}_{\dfrac{-R}{16\pi G_{ren}}}\Bigg) + \Bigg(\underbrace{\frac{-\Lambda}{8\pi G} + \text{‘const. div.’}}_{\dfrac{-\Lambda_{ren}}{8\pi G_{ren}}}\Bigg) +$$

$$\left(\mathcal{L}^{eff} - \text{‘div. prop. to R’} - \text{‘const. div.’}\right)$$

The finite, renormalized values G_{ren} and Λ_{ren} (which have 'absorbed' the divergences) are given by experiments.

(b3) 4th (adiabatic) order in derivatives of $g_{\mu\nu}$

$$R^2, R_{;\mu\nu}, \Box R, \Box R_{\mu\nu}, \ldots$$

$\longrightarrow$ A priori, Einstein eqs. might include terms of 4th order in derivatives of $g_{\mu\nu}$!

The resulting $\langle\psi \mid \hat{T}_{\mu\nu} \mid \psi\rangle_{ren}$, which is constructed from $\mathcal{L}^{eff}$ after the removal of the above divergences, satisfies Wald's axioms (Wald'77): the renormalized $\langle\psi \mid \hat{T}_{\mu\nu} \mid \psi\rangle_{ren}$ should satisfy:

(1) covariant conservation $\langle\psi \mid \hat{T}_{\mu\nu} \mid \psi\rangle_{ren}{}^{;\nu} = 0$ so it can be on

rhs of Einstein eqs. (true because derived from scalar lagrangian)

(2) causality: only depends on metric at pts. $\in J^{-}(x)$

(3) If $\langle\psi \mid \Phi\rangle = 0$, then $\langle\psi \mid \hat{T}_{\mu\nu} \mid \Phi\rangle_{ren} = \langle\psi \mid \hat{T}_{\mu\nu} \mid \Phi\rangle$

(this is because it's finite, so no need to do anything)

(4) In flat s-t with $\mathbb{R}^4$ topology: $\langle M \mid \hat{T}_{\mu\nu} \mid M\rangle_{ren} = 0$

'Uniqueness': If two different energy-momentum tensors satisfy (1)->(3), their difference can be 'absorbed' by the renormalization procedure mentioned.

Scalar product

- Consider a globally hyperbolic s-t -> Cauchy surfaces Σ_t labelled by a parameter t. Then

$$(\phi_1, \phi_2) \equiv i \int_{\Sigma_t} d\Sigma^\mu \left(\phi_1^* \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1^*\right)$$

- defines a scalar product

- is independent of the Cauchy surface if ϕ_i are slns. of the Klein-Gordon eq.

Quantization Procedure: like in flat s-t

(1) Choose a set $\{u_i(x), u_i^*(x), \forall i\}$ of slns. of field eq. that is complete and orthonormal:

$$\left(u_i, u_j\right) = \delta_{ij} = -\left(u_i^*, u_j^*\right), \quad \left(u_i, u_j^*\right) = 0$$

ie, u_i have positive norm

(2) Then any op. sln. of the field eq. may be expanded as

$$\hat{\phi}(x) = \sum_i \hat{a}_i u_i(x) + \hat{a}_i^\dagger u_i^*(x)$$

where $\quad \hat{a}_i = \left(u_i, \hat{\phi}\right), \quad \hat{a}_i^\dagger = -\left(u_i^*, \hat{\phi}\right)$

(3) Then

$$\left[\hat{\phi}(\vec{x}, t), \hat{\Pi}(\vec{x}', t)\right] = i\delta^{(3)}(\vec{x} - \vec{x}') \longleftrightarrow \left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij}, \quad \text{etc}$$

(4) Define a vacuum by $\hat{a}_i\,|0\rangle = 0, \forall i$

excited states by $\quad |\, n_i\rangle \equiv \left(\hat{a}_i^\dagger\right)^n |\, 0\rangle/\sqrt{n!} \qquad$ etc

Choice of modes

- Suppose χ is a timelike Killing vector, then

  - $\exists\, x_0$ s.t. it is the proper time of an observer with 4-velocity parallel to $\chi = \partial_{x_0}$

  - $\exists$ complete set $\{u_i(x), u_i^*(x), \forall i\}$ of slns. s.t.

  $\partial_{x_0} u_k = -i\omega_k u_k, \quad \omega_k > 0$ (pos. freq. modes wrt ∂_{x_0})

- Remember: an observer sees as empty a quantum state which is defined using pos.freq. modes wrt its proper time.
  So the obs. with 4-velocity $||\chi$ would see as empty the state defined using u_k

- In a non-stationary s-t, it is ‘natural’ to choose modes which are pos. freq. wrt the tlike Killing vector (supposing there’s one) in the asymptotic past & future

- The K-G scalar product is positive definite for pos. freq. modes -> it is an inner product for the vector space of pos. freq. slns. (one-particle Hilbert space)

QFT in Flat Space-time

II. Black Holes

Eternal Schwarzschild BH

- Line-element for a static, spherically-symmetric black hole of mass M (Schwarzschild):

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

Event horizon at $r = 2M$

- Killing vectors: slike ∂_φ (axisymmetry) and tlike ∂_t (stationarity)

- Null coords.: $u \equiv t - r_*, \; v \equiv t + r_*$

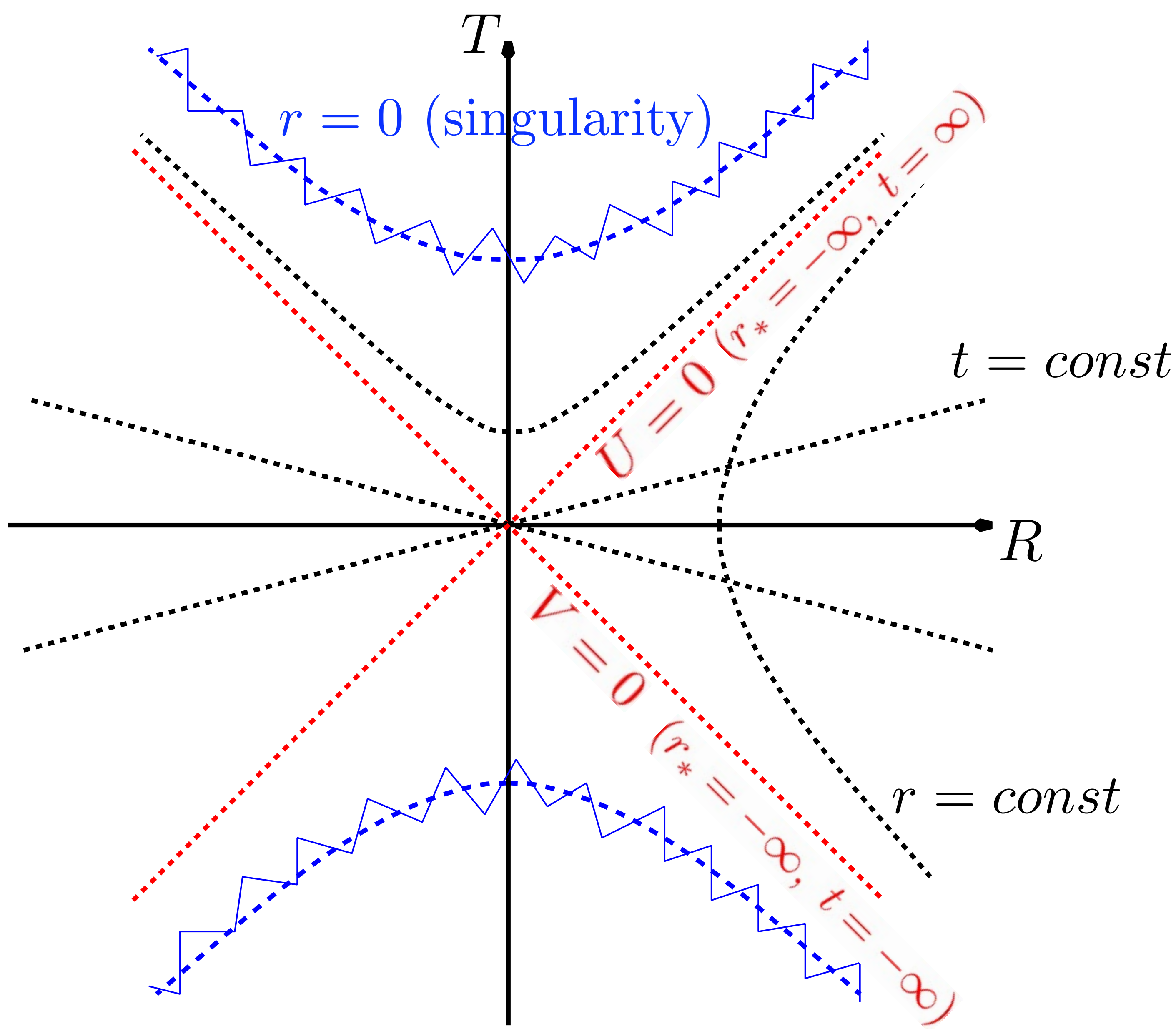
$$r_* \equiv r + 2M \ln\left(\frac{r}{2M} - 1\right) \in (-\infty, \infty) \quad \text{for} \quad r \in (2M, \infty)$$

Radially in/outgoing null geodesics are at u,v=const.

- Kruskal coords. to cover beyond EH: $U \equiv -e^{-\kappa u}, \; V \equiv e^{\kappa v}$

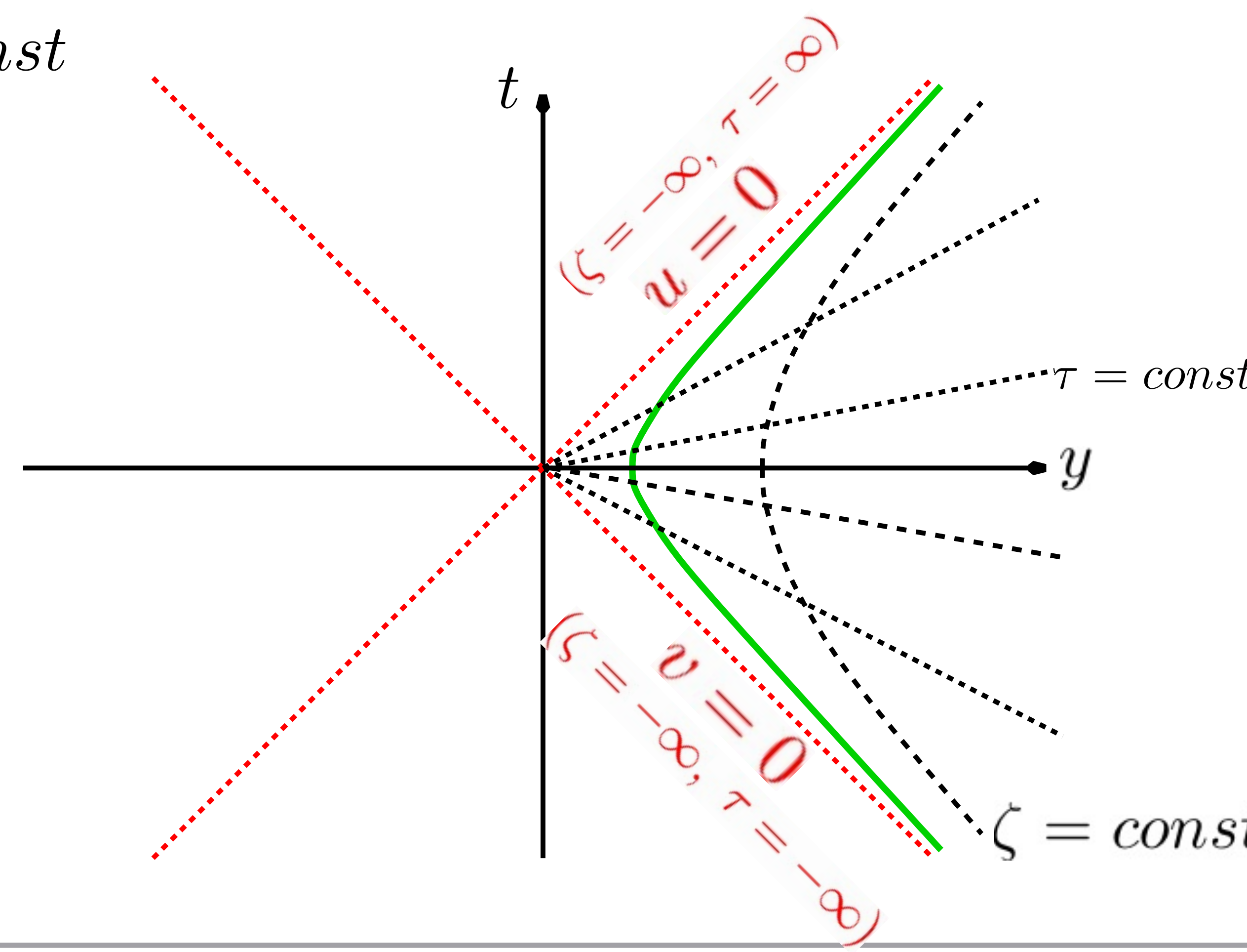
$\kappa \equiv \dfrac{1}{4M}$ surface gravity=force done at infinity to hold unit mass above horizon

Similar transformations to cover whole s-t $U, \; V \in (-\infty, \infty)$

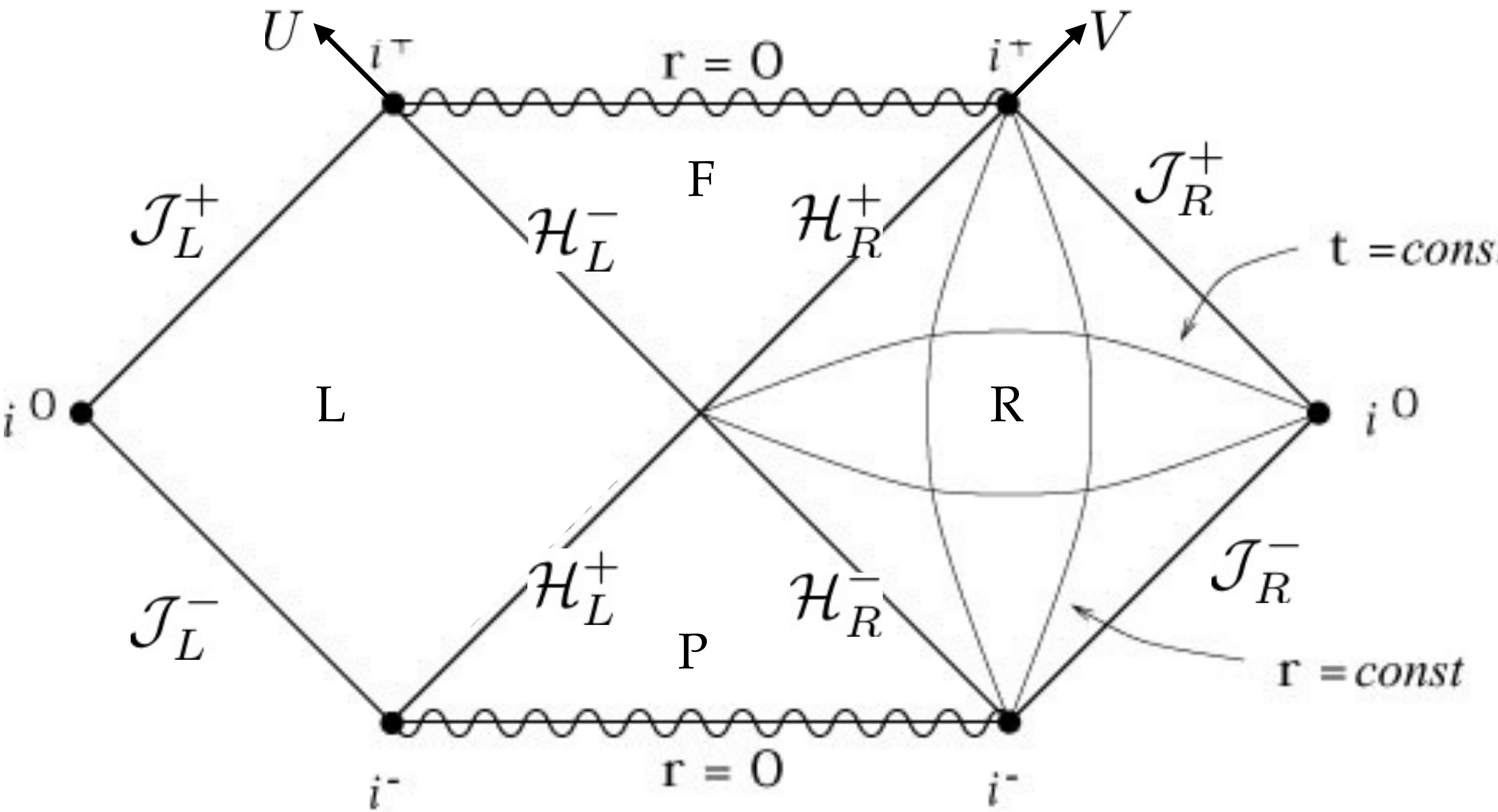


<- Schwarzschild

flat/Rindler ->



- Compactify s-t via $\bar{U} \equiv \arctan U, \;\; \bar{V} \equiv \arctan V \in (-\pi/2, \pi/2)$



Maximally extended Schwarzschild s-t

Scalar Field

- Massless Klein-Gordon eq., $\Box\phi = 0$, separates by variables:

Gral. sln.: $$\phi(x) = \int_{-\infty}^{\infty} d\omega \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_\Lambda \phi_\Lambda(x)$$

$$\Lambda \equiv \{\ell, m, \omega\}$$

Field modes: $$\phi_\Lambda(x) = N_{\ell\omega} R_{\ell\omega}(r) Y_{\ell m}(\theta, \varphi) e^{-i\omega t}$$

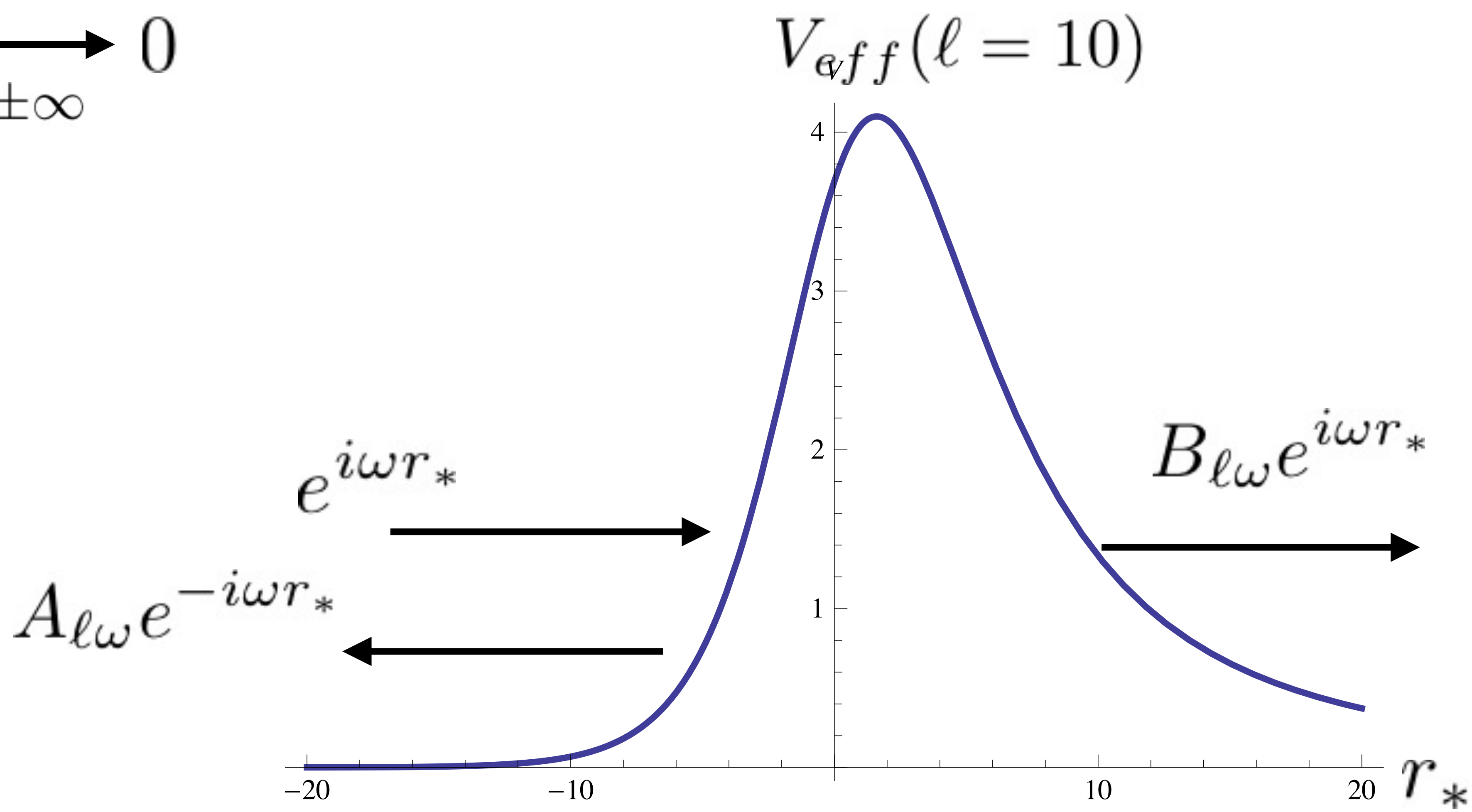
normalization const.

spherical harmonics

- Radial eq.: 2nd order linear ODE -> choose 2 lin. indep. slns.

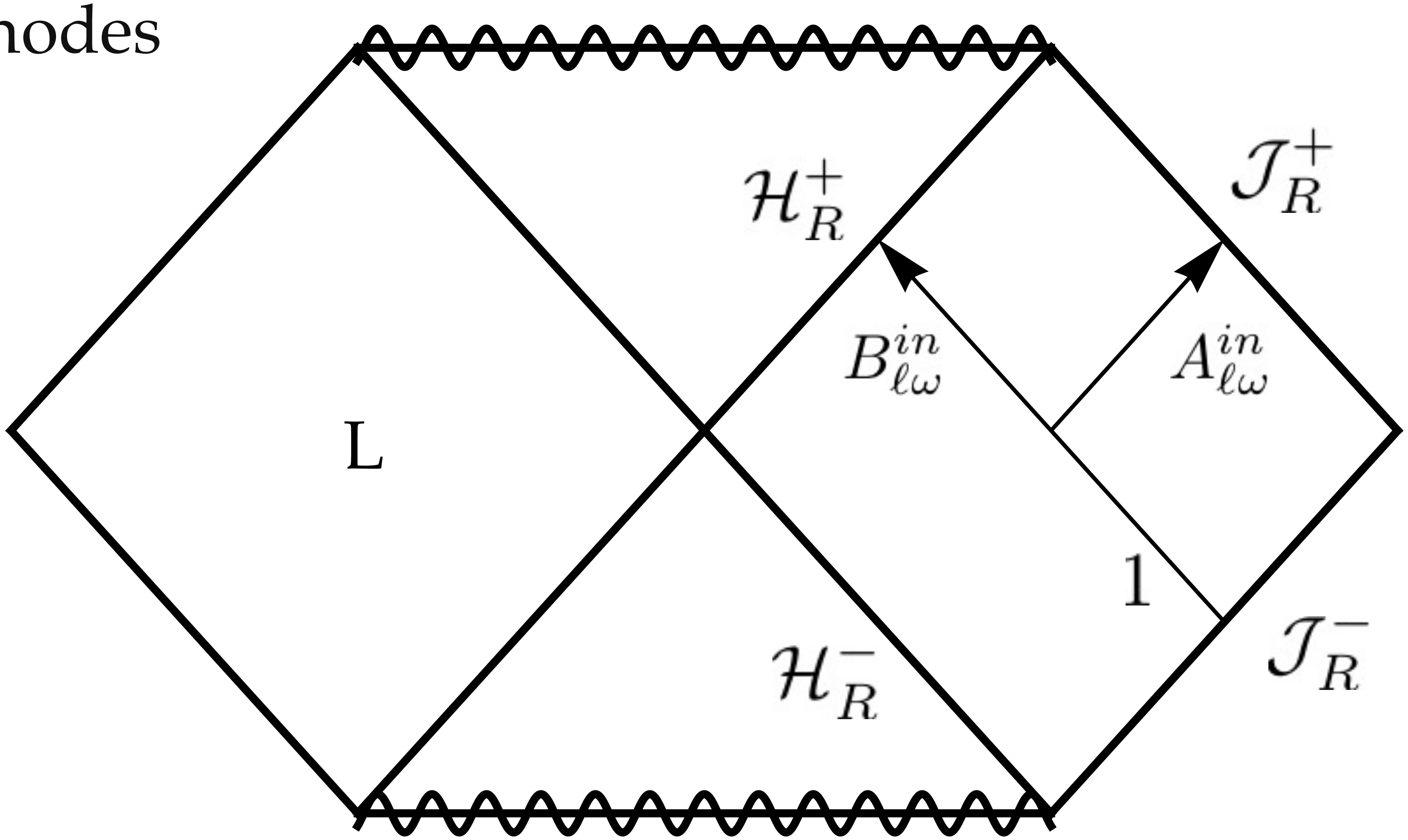
$$\left\{\frac{d^2}{dr_*^2}+\omega^2-\underbrace{\left(1-\frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2}+\frac{2M}{r^3}\right]}_{V_{eff}}\right\}R_{\ell\omega}=0$$

$$V_{eff}\xrightarrow[r_*\to\pm\infty]{}0$$



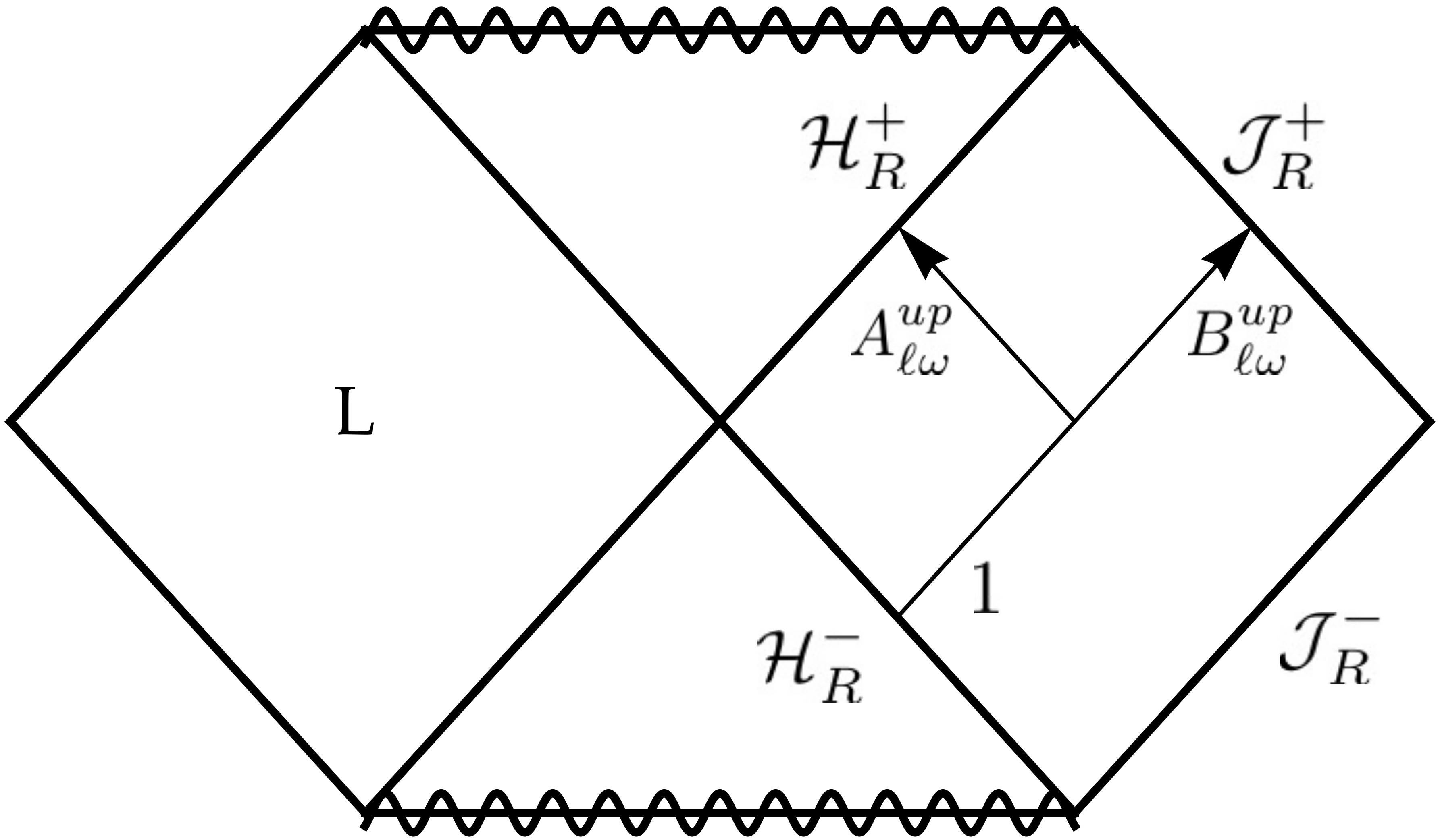
Modes

- 'in,R' modes



$$\phi_\Lambda^{in,R}(x) = 0, \ \forall x \in L \qquad \phi_\Lambda^{in,R}(x) \sim \frac{Y_{\ell m}(\Omega)}{r} \begin{cases} 0, & x \sim \mathcal{H}_R^-, \\ e^{-i\omega v}, & x \sim \mathcal{J}_R^-, \\ A_{\ell\omega}^{in} e^{-i\omega u}, & x \sim \mathcal{J}_R^+, \\ B_{\ell\omega}^{in} e^{-i\omega v}, & x \sim \mathcal{H}_R^+, \end{cases}$$

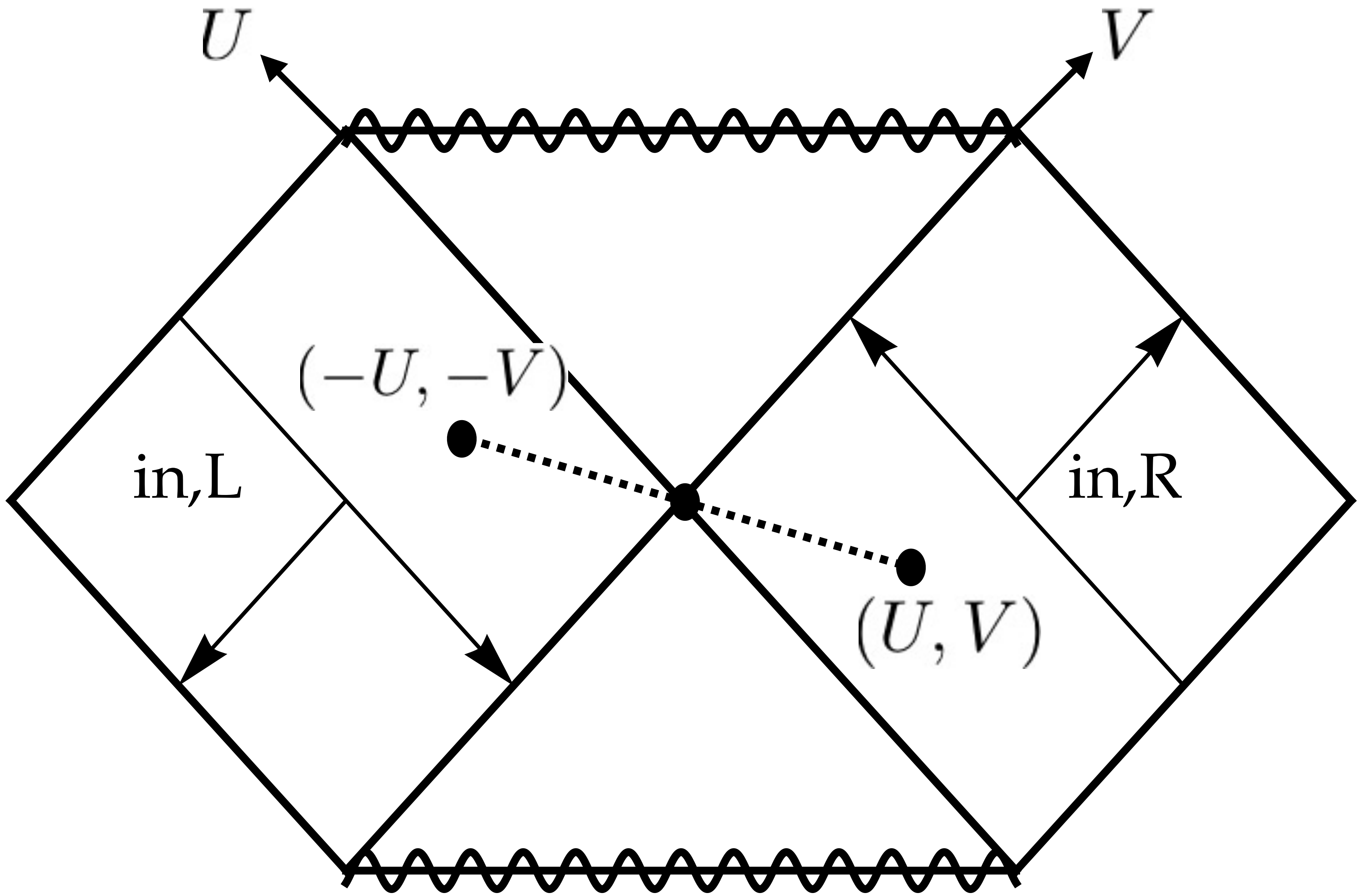
• 'up,R' modes



$$\phi_\Lambda^{up,R}(x) = 0,\ \forall x \in L \qquad \phi_\Lambda^{up,R}(x) \sim \frac{Y_{\ell m}(\Omega)}{r} \begin{cases} e^{-i\omega u}, & x \sim \mathcal{H}_R^-, \\ 0, & x \sim \mathcal{J}_R^-, \\ B_{\ell\omega}^{up} e^{-i\omega u}, & x \sim \mathcal{J}_R^+, \\ A_{\ell\omega}^{up} e^{-i\omega v}, & x \sim \mathcal{H}_R^+, \end{cases}$$

- ‘in&up,L’ modes

$$\phi_{\Lambda}^{in/up,L}(x) = 0,\ \forall x \in R$$



$$\phi_{\Lambda}^{in/up,L}(U,V,\Omega) = \phi_{\Lambda}^{in/up,R^*}(-U,-V,\Omega)$$

Boulware State

- On the maximally extended Schwarzschild s-t,

$\left\{\phi_\Lambda^{in/up,L/R} \ \& \text{ c.c.}, \ \forall\Lambda\right\}$ form a set that is:

- orthonormal: $\left(\phi_\Lambda^{in/up,L/R}, \phi_{\Lambda'}^{in/up,L/R}\right) = \delta_{\Lambda,\Lambda'}$

- complete:

$$\hat{\phi}(x) = \sum_{\ell,m} \int_0^\infty d\omega \left\{\hat{a}_\Lambda^{in,L} \phi_\Lambda^{in,L} + \hat{a}_\Lambda^{up,L} \phi_\Lambda^{up,L} + \hat{a}_\Lambda^{in,R} \phi_\Lambda^{in,R} + \hat{a}_\Lambda^{up,R} \phi_\Lambda^{up,R} + h.c.\right\}$$

- Boulware state is defined via

$$\hat{a}_\Lambda^{in/up,L} \mid B\rangle = 0 = \hat{a}_\Lambda^{in/up,R} \mid B\rangle, \quad \forall\Lambda$$

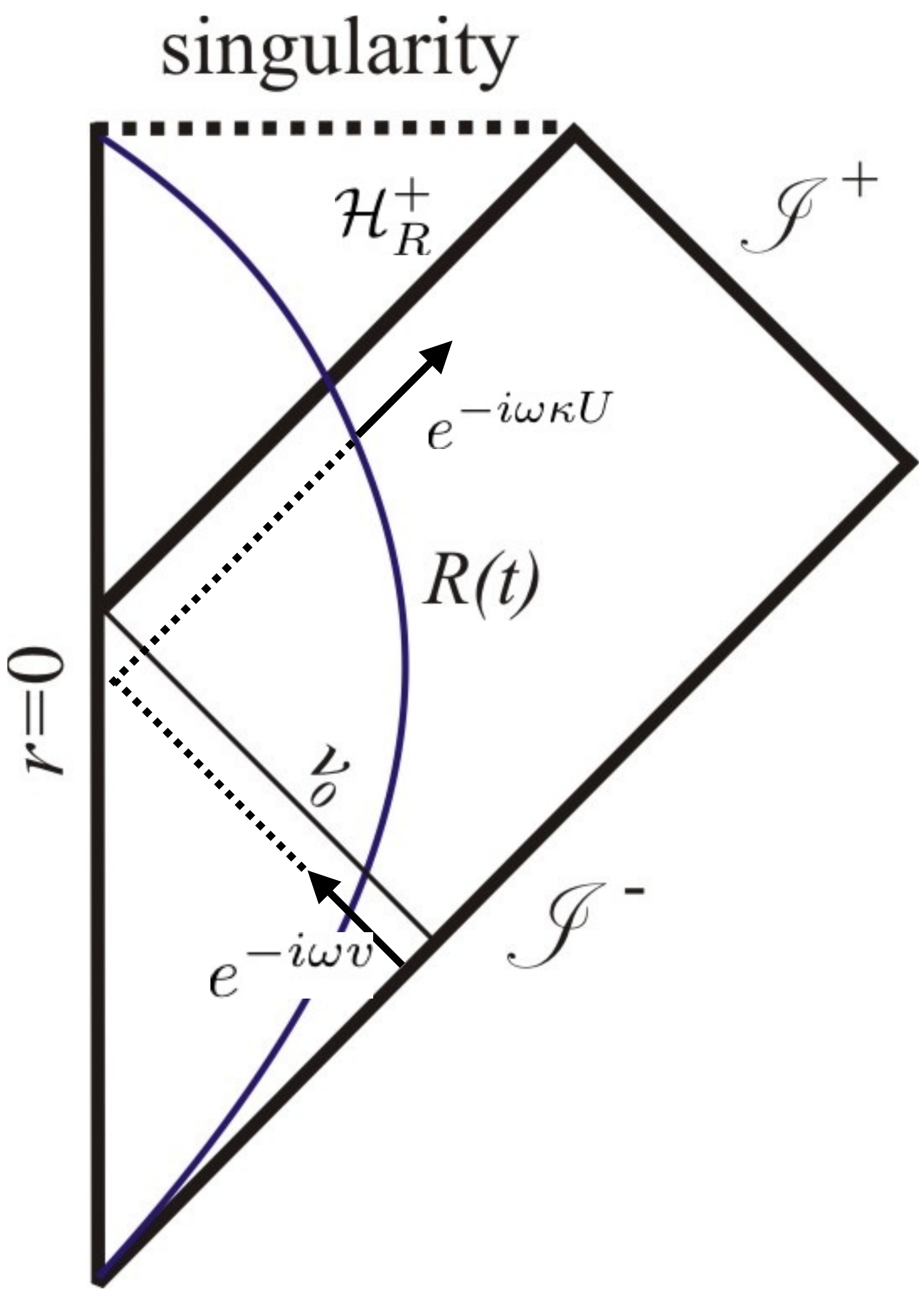
[Remember: an observer sees as empty a quantum state which is defined using pos.freq. modes wrt its proper time]

- Boulware state is defined using modes that are pos. freq. wrt ∂_t

$\Longrightarrow$ it is seen as empty by static obs. (ie, $r, \theta, \varphi = const$)

Gravitational Collapse

- Astrophysical BH's are not eternal, they are formed from gravitational collapse of a star



- Hawking'75 showed that modes $\phi_\Lambda^{in,R} \sim e^{-i\omega v}$ at $\mathcal{J}_R^-$

  behave like $e^{-i\omega\kappa U}$

  ie, they are pos. freq. modes wrt ∂_U, as they leave the star, leading to the 'evaporation' of the BH via emission of quantum Hawking radiation

Unruh State

- So instead of using $\phi_\Lambda^{up,R}$ in eternal BH s-t, we could model BH evaporation by using modes that are pos. freq. wrt ∂_U on $\mathcal{H}_R^-$
- Eg, we could directly use $e^{-i\omega\kappa U}$ but in order to obtain Bogolyubov coeffs. Unruh'76 instead used the modes:

$$\phi_\Lambda^{UP} \equiv \frac{e^{\pi\omega/(2\kappa)}}{\sqrt{2\sinh(\pi\omega/\kappa)}}\left[\phi_\Lambda^{up,R} + e^{-\pi\omega/\kappa}\phi_\Lambda^{up,L^*}\right]$$

Same rln. as between Rindler&Minkowski modes in flat s-t!
Remember:

$$u_{\bar{k}}^R \equiv \frac{e^{\pi\bar{k}/(2\alpha)}}{\sqrt{2\sinh(\pi\bar{k}/\alpha)}}\left[\bar{\phi}_{\bar{k}}^R + e^{-\pi\bar{k}/\alpha}\bar{\phi}_{-\bar{k}}^{L^*}\right]$$

they define $|M\rangle$

they define $|R\rangle$

- Similarly define $\phi_\Lambda^{\overline{UP}}$ modes with $L \leftrightarrow R$ in ϕ_Λ^{UP}

- On the maximally extended Schwarzschild s-t, $\left\{\phi_\Lambda^{in,L/R}, \phi_\Lambda^{UP/\overline{UP}} \text{ \& c.c.}, \ \forall\Lambda\right\}$ form a set that is:

  - orthonormal

  - complete:

$$\hat{\phi}(x) = \sum_{\ell,m} \int_0^\infty d\omega \left\{ \hat{a}_\Lambda^{in,L} \phi_\Lambda^{in,L} + \hat{a}_\Lambda^{UP} \phi_\Lambda^{UP} + \hat{a}_\Lambda^{in,R} \phi_\Lambda^{in,R} + \hat{a}_\Lambda^{\overline{UP}} \phi_\Lambda^{\overline{UP}} + h.c. \right\}$$

- Unruh state is defined via

$$\hat{a}_\Lambda^{in,R/L} \mid U\rangle = \hat{a}_\Lambda^{UP} \mid U\rangle = \hat{a}_\Lambda^{\overline{UP}} \mid U\rangle = 0, \quad \forall\Lambda$$

- It can be shown that ϕ_{Λ}^{UP} & $\phi_{\Lambda}^{\overline{UP}}$, $\forall\omega > 0$, are pos. freq. wrt ∂_U on $\mathcal{H}_R^-$

- It is defined using modes that are pos. freq. wrt affine parameter U on $\mathcal{H}_R^-$ and affine parameter v on $\mathcal{J}_R^-$ $\Longrightarrow$ Unruh state is seen as empty by free-falling obs. in those regions

Hartle-Hawking State

- We can also define the IN version of the UP modes as:

$$\phi_\Lambda^{IN} \,\&\, \phi_\Lambda^{\overline{IN}} \;\text{ defined like }\; \phi_\Lambda^{UP} \,\&\, \phi_\Lambda^{\overline{UP}} \;\text{ with }\; \phi_\Lambda^{up,R/L} \to \phi_\Lambda^{in,R/L}$$

- On the maximally extended Schwarzschild s-t,
- $\left\{ \phi_\Lambda^{UP/IN/\overline{IN}/\overline{UP}} \;\&\; \text{c.c.}\; \forall\Lambda \right\}$ form a set that is:

  - orthonormal

  - complete:

$$\hat{\phi}(x) = \sum_{\ell,m} \int_0^\infty d\omega \left\{ \hat{a}_\Lambda^{IN} \phi_\Lambda^{IN} + \hat{a}_\Lambda^{UP} \phi_\Lambda^{UP} + \hat{a}_\Lambda^{\overline{IN}} \phi_\Lambda^{\overline{IN}} + \hat{a}_\Lambda^{\overline{UP}} \phi_\Lambda^{\overline{UP}} + h.c. \right\}$$

- Hartle-Hawking state is defined via

$$\hat{a}_\Lambda^{IN} \mid H\rangle = \hat{a}_\Lambda^{UP} \mid H\rangle = \hat{a}_\Lambda^{\overline{IN}} \mid H\rangle = \hat{a}_\Lambda^{\overline{UP}} \mid H\rangle = 0, \quad \forall\Lambda$$

- It can be shown that ϕ_{Λ}^{IN} & $\phi_{\Lambda}^{\overline{IN}}$, $\forall \omega > 0$, are pos. freq. wrt ∂_V on $\mathcal{H}_R^+$

- It is defined using modes that are pos. freq. wrt affine parameter U on $\mathcal{H}_R^-$ and affine parameter V on $\mathcal{H}_R^+$ $\Longrightarrow$ it is seen as empty by free-falling obs. on $\mathcal{H}_R^{\pm}$

States' Properties (Candelas'80)

- RSET $\langle B \mid \hat{T}^{\mu}{}_{\nu} \mid B \rangle_{ren}$

- goes to zero like $O\left(\frac{M^2}{r^6}\right)$ as $r \to \infty$

- diverges on $\mathcal{H}_R^{\pm}$ in a free-falling frame

Expected, since it is defined using inertial frame $\{t, r_*\}$, which diverges on EH [similar to Rindler state defined using Rindler frame $\{\tau, \xi\}$, which diverges on Killing horizons]

- Remember: Boulware state is seen as empty by static obs., which are accelerated obs. in Schwarzschild s-t [so similar to Rindler state seen as empty by accelerated obs. in flat s-t]

- Boulware state:

  - is the equivalent of the Rindler state in flat s-t

  - models a cold star

- RSET $\langle U|\, \hat{T}_{\mu}{}^{\nu}\, |U\rangle_{ren}$

- diverges at $\mathcal{H}^-$

- is regular at $\mathcal{H}^+$

- has a flux of thermal Hawking radiation at $\mathcal{I}^+$:

$$\sim \frac{L}{4\pi r^2}\begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad r \to \infty$$

$\mu, \nu = \{t, r, \theta, \varphi\}$

Γ_ω : 'greybody' factor

Luminosity: $L = \dfrac{1}{2\pi}\displaystyle\int_0^\infty d\omega \frac{\omega \cdot \Gamma_\omega}{e^{\omega/T_H} - 1}$

Hawking temperature: $T_H = \dfrac{1}{8\pi M}$

Expected, since UP - up modes (though not for ingoing modes) relationship same as between Rindler&Minkowski modes in flat s-t

- Unruh state models an evaporating BH

- RSET $\langle H \mid \hat{T}_\mu{}^\nu \mid H \rangle_{ren}$

- has a bath of thermal radiation at $\mathcal{J}_R^\pm$:

$$\frac{1}{2\pi^2}\int_0^\infty d\omega \frac{\omega^3}{e^{2\pi\omega/\kappa}-1}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 \end{pmatrix}, \quad r \to \infty$$

$$\mu, \nu = \{t, r, \theta, \varphi\}$$

Expected, since UP - up and IN - in modes relationship same as between Rindler&Minkowski modes in flat s-t

- is regular at $\mathcal{H}_R^\pm$ (and everywhere else)

Expected, since it is defined using Kruskal frame $\{U, V\}$ which is regular in the whole s-t [similar to Minkowski state defined using inertial frame, regular in the whole flat s-t]

- is invariant under symmetries of BH s-t

- Remember: H-H state is seen as empty by free-falling obs. on $\mathcal{H}_R^{\pm}$, [similar to Minkowski state seen as empty by inertial obs. in flat s-t]

- Hartle-Hawking state:

  - is the equivalent of the Minkowski state in flat s-t

  - models a BH in thermal (unstable) equilibrium with its own radiation

Kerr Space-time

- Rotating black hole with angular momentum per unit mass a and mass M

$$ds^2 = -\frac{\Delta}{\Sigma}\left[dt^2 - a\sin^2\theta d\varphi\right]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[\left(r^2+a^2\right)d\varphi - a dt\right]^2$$

$$\Delta \equiv r^2 - 2Mr + a^2 \qquad \Sigma \equiv r^2 + a^2\cos^2\theta$$

- Event-horizon: $r_h = M + \sqrt{M^2 - a^2}$

- Angular velocity: $\Omega = \dfrac{a}{r_h^2 + a^2}$

Killing Vectors

- Axisymmetric s-t $\Longrightarrow$ ∂_φ is a spacelike killing vector

- Stationary s-t $\Longrightarrow$ ∂_t is a spacelike killing vector

- Any lin. comb. of ∂_t and ∂_φ is a Killing vector. In particular,

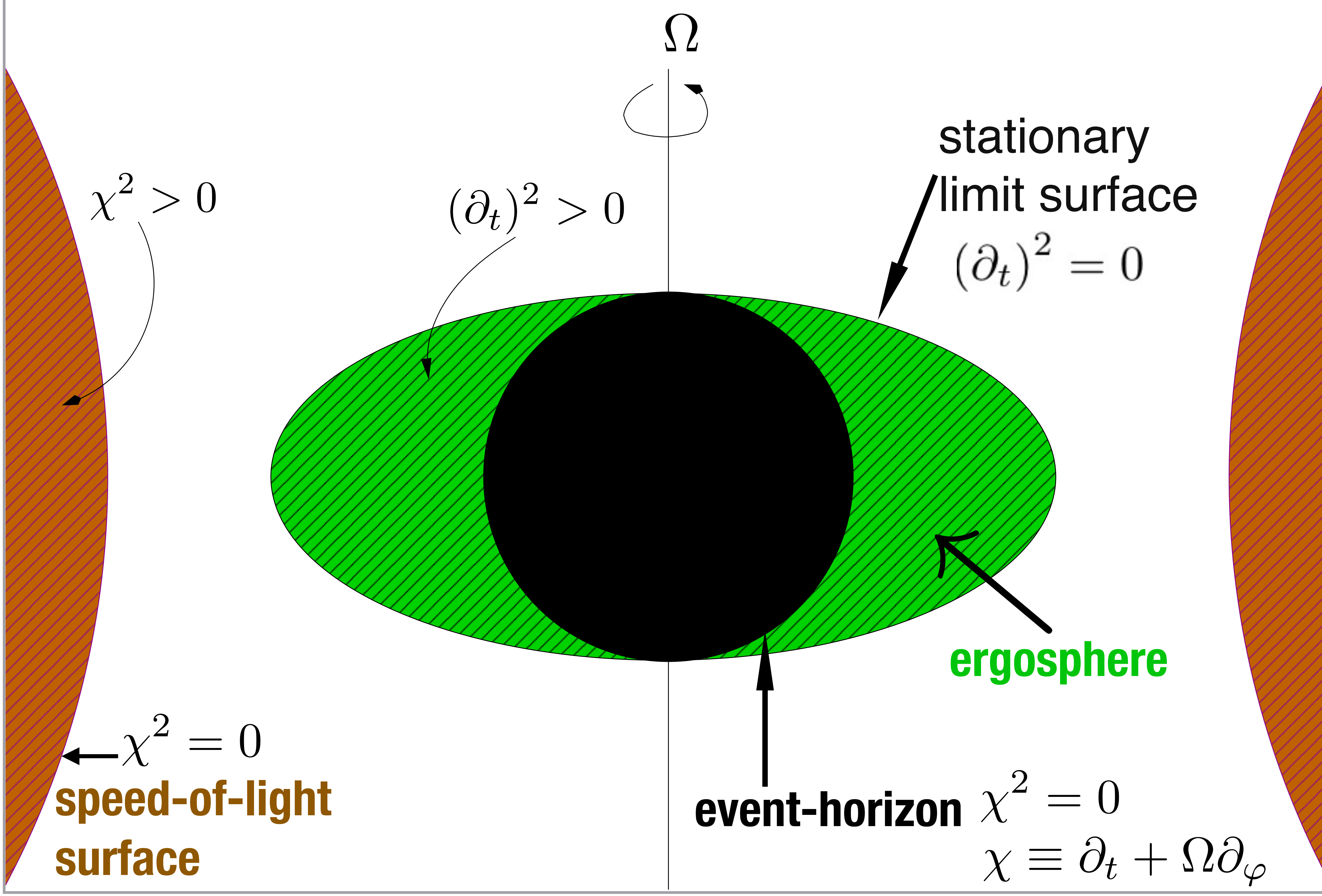
$\chi \equiv \partial_t + \Omega\partial_\varphi$ is a Killing vector that is the null generator of the horizon

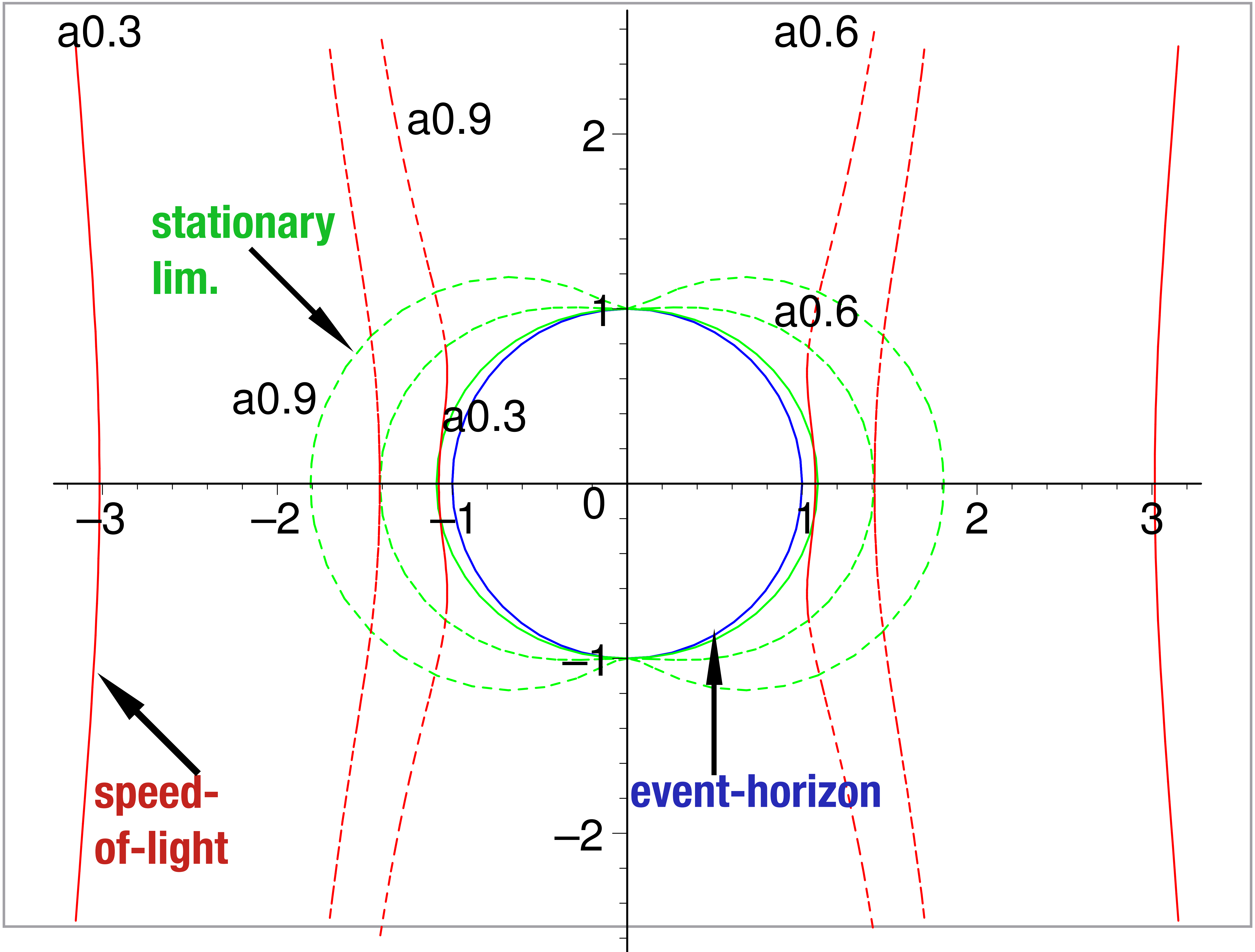
- However, there is no Killing vector which is timelike everywhere outside the horizon:

- ∂_t is timelike from infinity down to the stationary limit surface: where obs. cannot be static anymore (they must rotate)

- χ is timelike from just outside the horizon up to the speed-of-light surface: where obs. cannot rotate at Ω anymore (veloc.=c)

Hypersurfaces





a0.3
a0.6
a0.9
stationary
lim.
a0.6
a0.9
a0.3
speed-
of-light
event-horizon
2
1
0
−1
−2
−3
−2
−1
1
2
3

QFT in Kerr for Bosons

- Field eq.: $\text{``}\Box\text{''}\phi = 0$ separates by variables

Mode slns.: $\phi_\Lambda = R_\Lambda(r) S_\Lambda(\theta) e^{im\varphi - i\omega t}$

$$\Lambda \equiv \{\ell, m, \omega\}$$
$$-\Lambda \equiv \{\ell, -m, -\omega\}$$

- Suppose $\{\phi_\Lambda, \forall\Lambda\}$ is a complete set of slns., then you may expand the field operator as

$$\hat{\phi} = \sum_\Lambda \hat{a}_\Lambda \phi_\Lambda + \hat{a}^\dagger_\Lambda \phi_{-\Lambda}$$

- If $(\phi_{\pm\Lambda}, \phi_{\pm\Lambda'}) = \pm\delta_{\Lambda,\Lambda'}$ positive norm wrt a scalar product
negative

then commutation rlns. $\left[\hat{a}_\Lambda, \hat{a}^\dagger_{\Lambda'}\right] = \delta_{\Lambda,\Lambda'}$ follow from the equal

time commutation rlns. $\left[\hat{\phi}(\vec{x}, t), \hat{\Pi}(\vec{x}', t)\right] = i\delta^{(3)}(\vec{x} - \vec{x}')$

QFT in Kerr for Fermions

- Dirac field eq. $$\gamma^{\mu}\left(\partial_{\mu}-\Gamma_{\mu}\right)\phi=0$$

Dirac matrices

$$\{\gamma^{\mu},\gamma^{\nu}\}=2g^{\mu\nu}$$

Dirac 4-spinors

Spinor connection matrices

$$\partial_{\nu}\gamma^{\mu}+\Gamma^{\mu}_{\nu\kappa}\gamma^{\kappa}-\Gamma_{\nu}\gamma^{\mu}+\gamma^{\mu}\Gamma_{\nu}=0$$

The Dirac field eq. in Kerr separates by variables and decouples for the different spinor components

- Suppose $\{\phi_{\Lambda},\forall\Lambda\}$ is a complete set of slns., then you may expand the field operator as

$$\hat{\phi}=\sum_{\Lambda}\hat{a}_{\Lambda}\phi_{\Lambda}+\hat{b}^{\dagger}_{\Lambda}\phi_{-\Lambda}$$

- There is a conserved current $J^{\mu} = \left(\phi_1^{\dagger}\tilde{\gamma}^0\right)\gamma^{\mu}\phi_2$

$$\nabla_{\mu}J^{\mu} = 0$$

which we can use to constract an inner product as:

$$(\phi_1, \phi_2) = \int_{t=const} J^t dS$$

- If $(\phi_{\pm\Lambda}, \phi_{\pm\Lambda'}) = \delta_{\Lambda,\Lambda'}$ all positive norm wrt inner product

then anticommutation rlns. $\left\{\hat{a}_{\Lambda}, \hat{a}^{\dagger}_{\Lambda'}\right\} = \left\{\hat{b}_{\Lambda}, \hat{b}^{\dagger}_{\Lambda'}\right\} = \delta_{\Lambda,\Lambda'}$

follow from $\left\{\hat{\phi}(\vec{x},t), \hat{\Pi}(\vec{x}',t)\right\} = i\delta^{(3)}(\vec{x}-\vec{x}')$

Modes

- We will define in/up/UP/IN modes in Kerr similarly to the way we did in Schwarzschild, with two differences:

  - We will only consider the outer region of the BH

  - The radial eq. has the Schrodinger-like form:

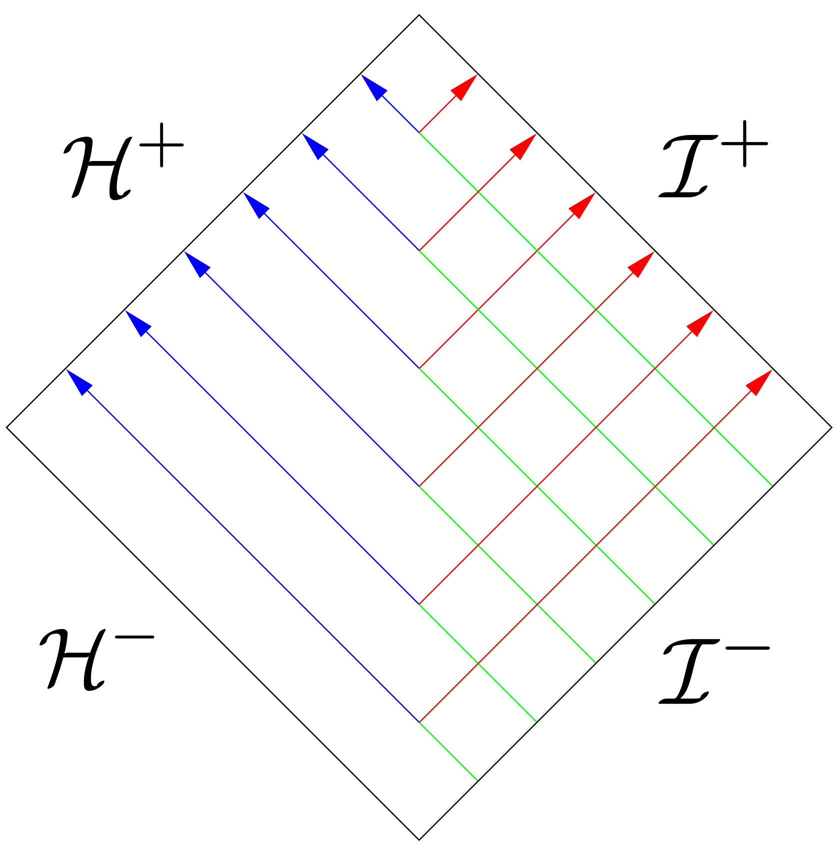
$$\frac{d^2R(r)}{dr_*^2} - U(r)R(r) = 0$$

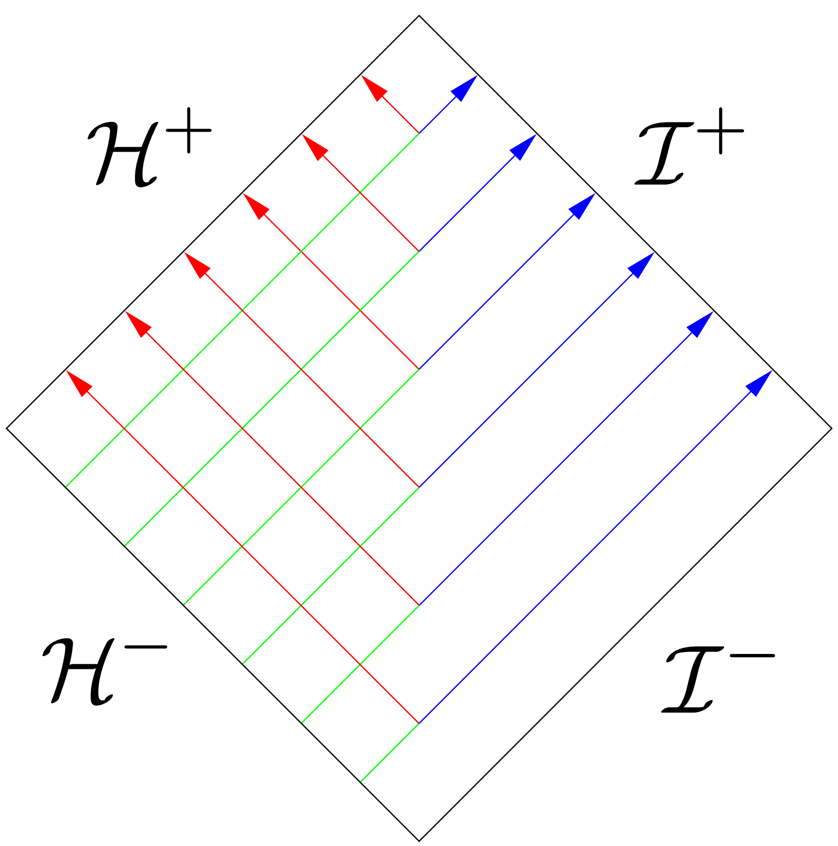
with

$$U(r) \to -\omega^2, \quad r \to \infty \qquad \text{but} \qquad U(r) \to -\tilde{\omega}^2, \quad r_* \to -\infty$$

$$\tilde{\omega} \equiv \omega - m\Omega$$

Pos. freq. modes: when Fourier-decomposed, the modes only have positive frequency





'in': p.f. at $\mathcal{I}^-$ wrt null coord. v $\quad \partial_v \phi_\Lambda^{in} \sim -i\omega\phi_\Lambda^{in} \quad \forall\omega > 0$

'up': p.f. at $\mathcal{H}^-$ wrt null coord. u $\quad \partial_u \phi_\Lambda^{up} \sim -i\tilde{\omega}\phi_\Lambda^{up} \quad \forall\tilde{\omega} > 0$

'IN': p.f. at $\mathcal{H}^+$ wrt Kruskal coord. V $\quad \forall\omega \in \mathbb{R}$

'UP': p.f. at $\mathcal{H}^-$ wrt Kruskal coord. U $\quad \forall\tilde{\omega} \in \mathbb{R}$

Classical: Boson Modes

- We are constrained to choosing the in/up modes as follows

- ‘in’ modes are pos/negat norm for pos/negat ω

  So Λ for ‘in’ modes involves only $\omega > 0$

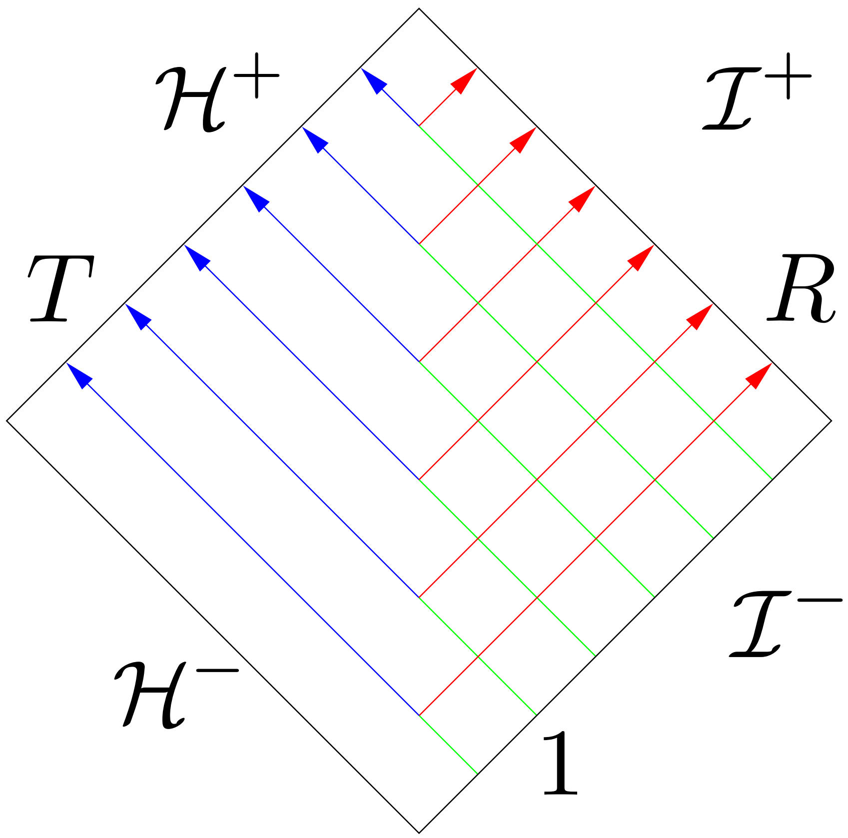
- ‘up’ modes are pos/negat norm for pos/negat $\tilde{\omega}$

  So Λ for ‘up’ modes involves only $\tilde{\omega} > 0$

Classical: Boson Modes

- Classical superradiance for $\omega\tilde{\omega} < 0$

$$1 - |R|^2 = \frac{\tilde{\omega}}{\omega} |T|^2$$



- 1st law of b-h thermodynamics:

$$dM = T_H d\left(\frac{A}{4}\right) + \Omega dJ$$

Send wavepacket in with $\dfrac{dM}{dJ} = \dfrac{\omega}{m} \quad \longrightarrow \quad \dfrac{\tilde{\omega}}{\omega} dM = T_H d\left(\dfrac{A}{4}\right)$

Area th.: $dA > 0$ if weak-energy cond. is satisfied (ie, energy density measured by an obs. is non-negat)

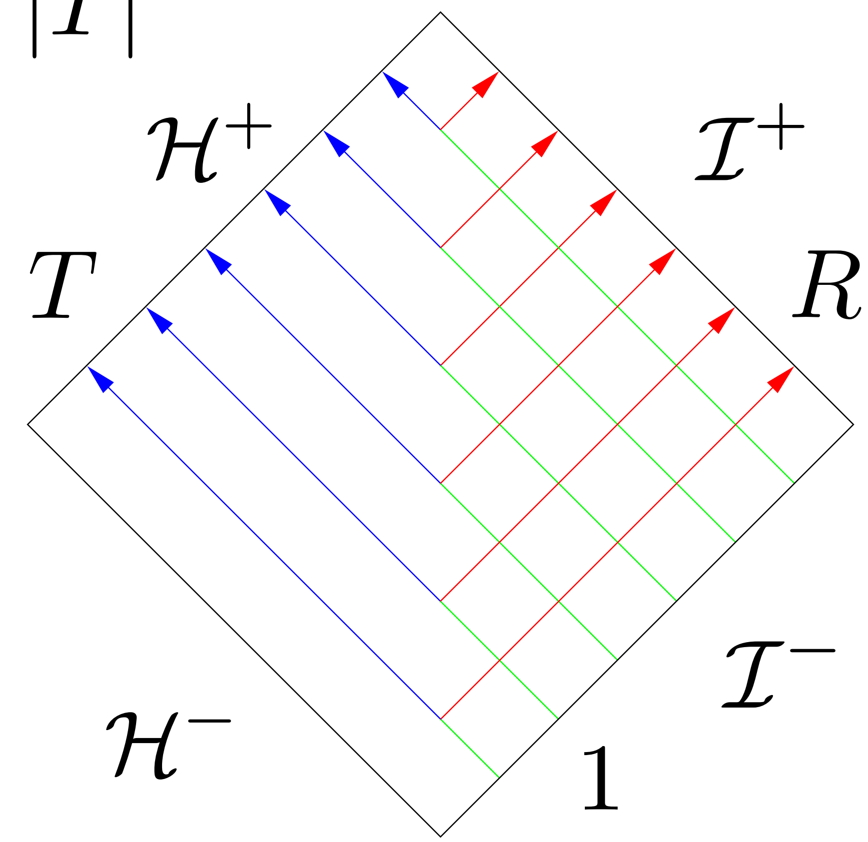
$\Rightarrow \quad dM < 0 \quad$ for $\quad \omega\tilde{\omega} < 0$

Classical: Fermion Modes

- All modes are positive norm $\forall \omega \in \mathbb{R}$

  -> freedom in choice of pos.freq. modes

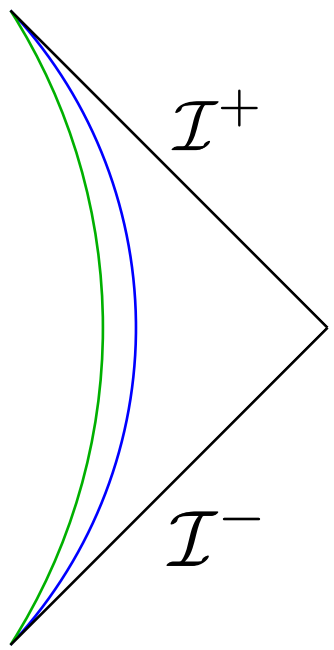
- No classical superradiance: $1 - |R|^2 = |T|^2$



- Fermions do not satisfy the weak-energy condition -> Area th. does not apply

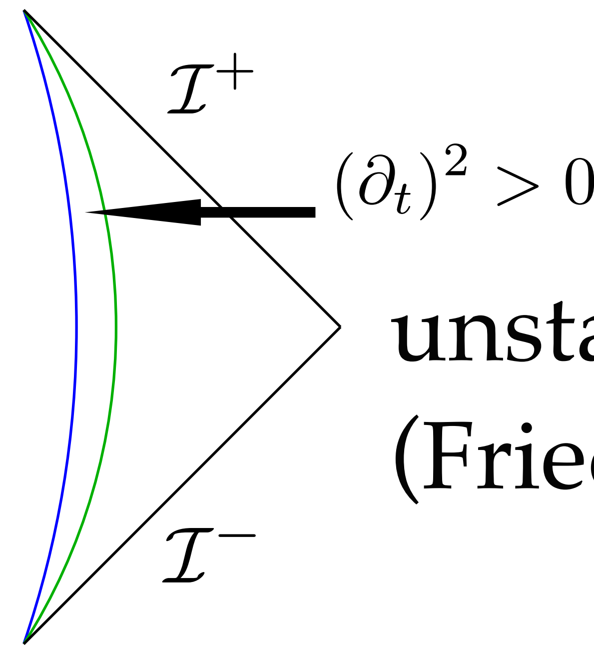
Classical stability for bosons

Mirror outside ergosphere



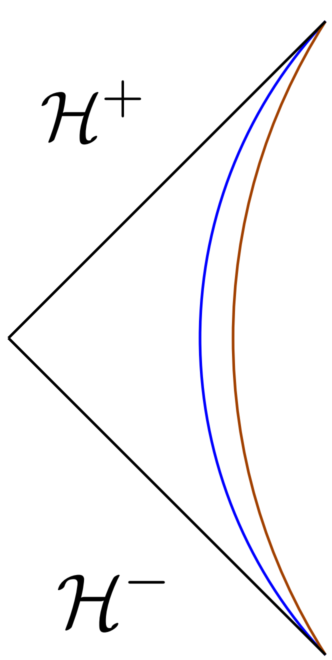
stable

Mirror inside ergosphere



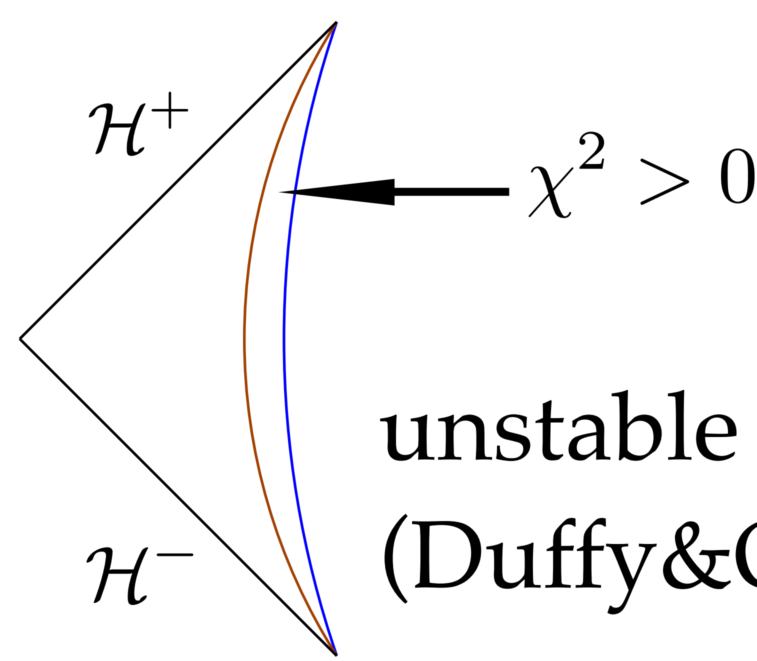
unstable
(Friedman'78)

Mirror inside SOL



stable

Mirror outside SOL



unstable
(Duffy&Ottewill'08)

Classical stability for fermions

stable in all cases!

(In preparation)

States in Kerr - bosons

Defined as in Schwarzschild but $\omega \to \tilde{\omega}$ for 'up' modes (because of positive norm)

- Boulware $\hat{a}^{in}_{\Lambda} \mid B\rangle = \hat{a}^{up}_{\Lambda} \mid B\rangle = 0, \quad \forall \Lambda$

[s=0: Unruh'74; Ottewill&Winstanley'00; s=1: Casals&Ottewill'05]

Properties:

- irregular at $\mathcal{H}^{\pm}$, regular elsewhere

- empty at $\mathcal{I}^{-}$ and $\mathcal{H}^{-}$

- has Unruh-Starobinsky radiation at $\mathcal{I}^{+}$ from superradiant modes

- there is no state empty at $\mathcal{I}^{\pm}$

- Unruh $\quad \hat{a}^{in}_{\Lambda} \mid U\rangle = \hat{a}^{UP}_{\Lambda} \mid U\rangle = 0, \quad \forall\Lambda$

Properties as in Schwarzschild:

- irregular at $\mathcal{H}^-$ but regular elsewhere

- empty at $\mathcal{I}^-$

- Hawking radiation at $\mathcal{I}^+$

at Hawking temperature: $T_H = \dfrac{r_h^2 - a^2}{4\pi r_h (r_h^2 + a^2)}$

States in Kerr for Bosons

- Hartle-Hawking

- Remember: in Schwarzschild, this state was regular everywhere, satisfied the symmetries of the s-t, contained a thermal bath at infinity, modeled a BH in thermal eq. with its own radiation

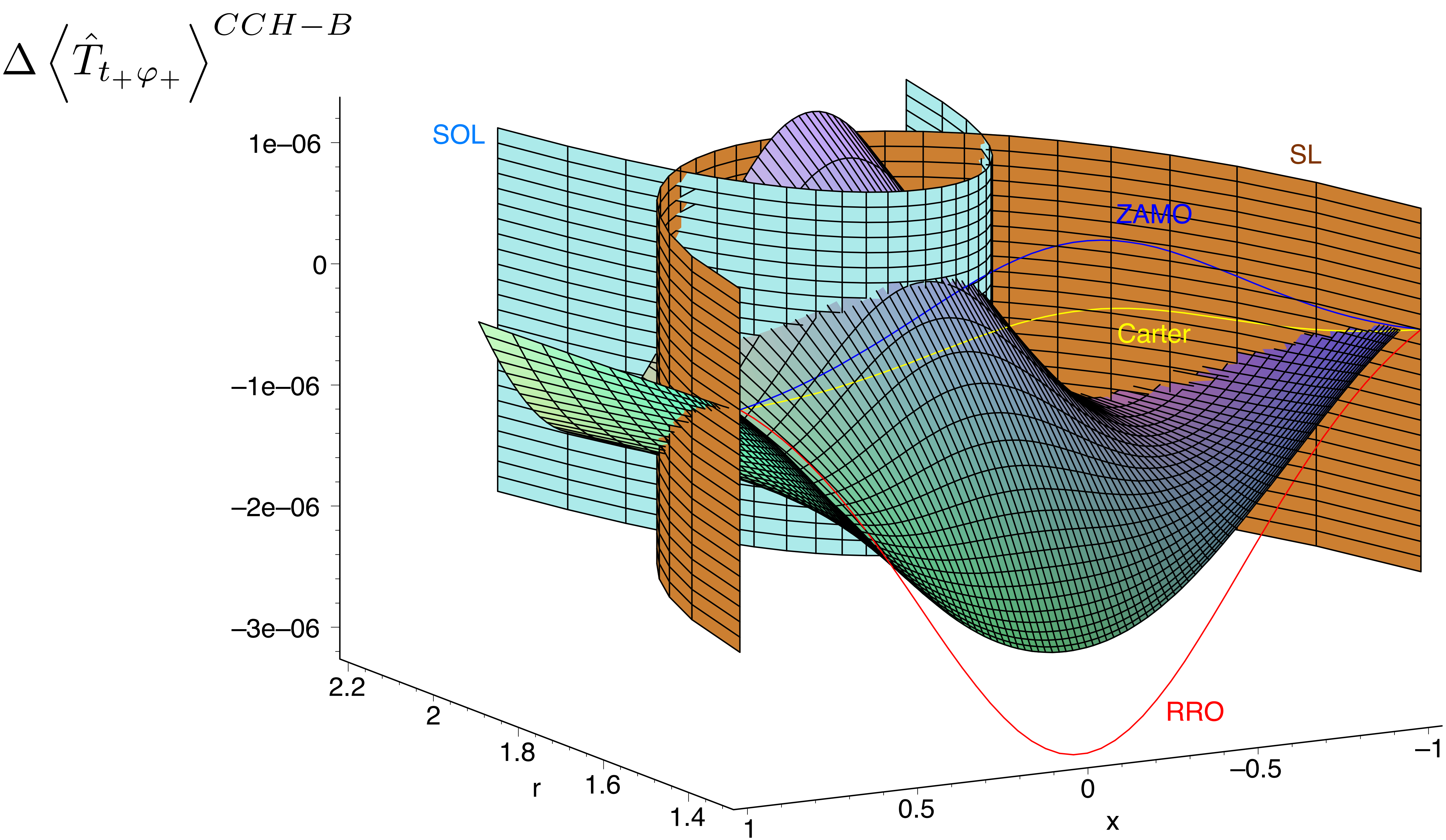
- This state, therefore, is the relevant one for, eg, the laws of BH thermodynamics; AdS/CFT correspondence, etc

- The attempt by Frolov&Thorne'89 is ill-defined everywhere due to a pole at $\tilde{\omega} = 0$ [Ottewill&Winstanley'00]

$$\langle U^- | \, \hat{\phi}^2 \, | U^- \rangle - \langle FT | \, \hat{\phi}^2 \, | FT \rangle \sim \sum_{\ell,m} \int_0^\infty d\omega \frac{|T|^2}{\omega \left(e^{\tilde{\omega}/T_H} - 1 \right)}, \quad r \to r_h$$

- Candelas, Chrzanowski & Howard'81 constructed a H-H-like state where modes are thermalized wrt their 'natural energy' (ie, 'in' modes wrt ω and 'up' modes wrt $\tilde{\omega}$)

Casals&Ottewill'05: CCH is regular everywhere but does not satisfy symmetries of space-time



- Kay&Wald'91: in a globally-hyperbolic s-t with a bifurcate Killing horizon, if there exists a state which is regular everywhere and has symmetries of spacetime then it is thermal (ie, Hartle-Hawking).

They prove that such a state does not exist in Kerr for bosons using superradiance....but what about fermions?

States in Kerr - fermions

[Casals,Dolan,Nolan,Ottewill,Winstanley'12]

Remember: 'in' & 'up' have positive norm $\forall\omega \in \mathbb{R}$

- Boulware: similar as for bosons. Eg, it has Unruh-Starobinsky radiation (even if fermions have no classical superradiance) at $\mathcal{I}^+$

  However, one may define:

$$\hat{a}^{in}_{\Lambda} |B\rangle = 0 \quad \forall\omega > 0$$

$$\hat{a}^{up}_{\Lambda} |B\rangle = 0 \quad \forall\omega > 0 \;\; \text{(for bosons: } \forall\tilde{\omega} > 0\text{)}$$

Expected to be empty at $\mathcal{I}^{\pm}$

- Unruh: Similar as for bosons

- ‘Hartle-Hawking’ is now well-defined!

  but where is it regular?

Boulware for Fermions

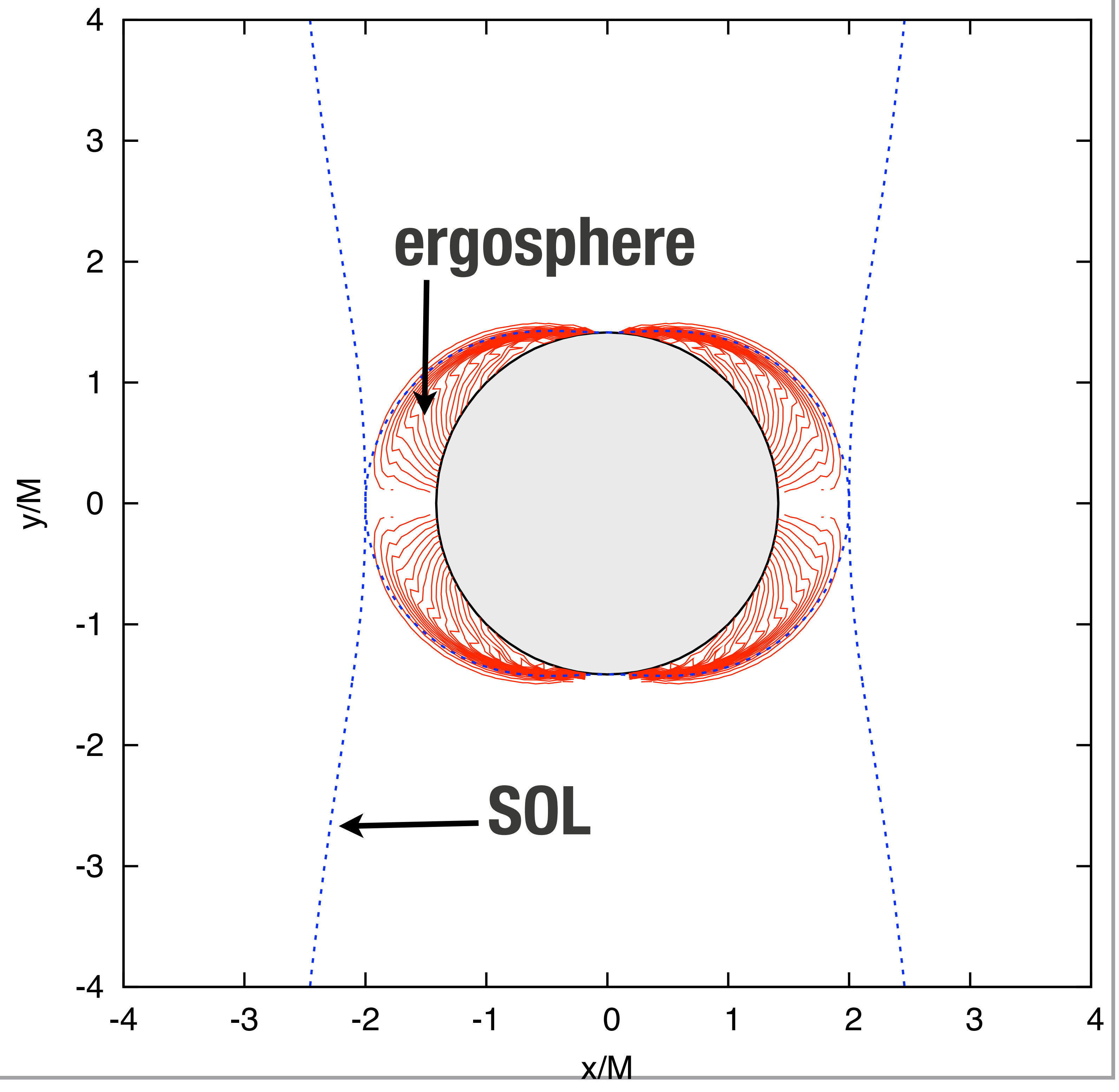
Particle number current $\left\langle \hat{J}^r \right\rangle^{U^- - B}$

Properties:

- Empty at radial infinity

- Regular on and outside SOL

- Divergence in ergosphere

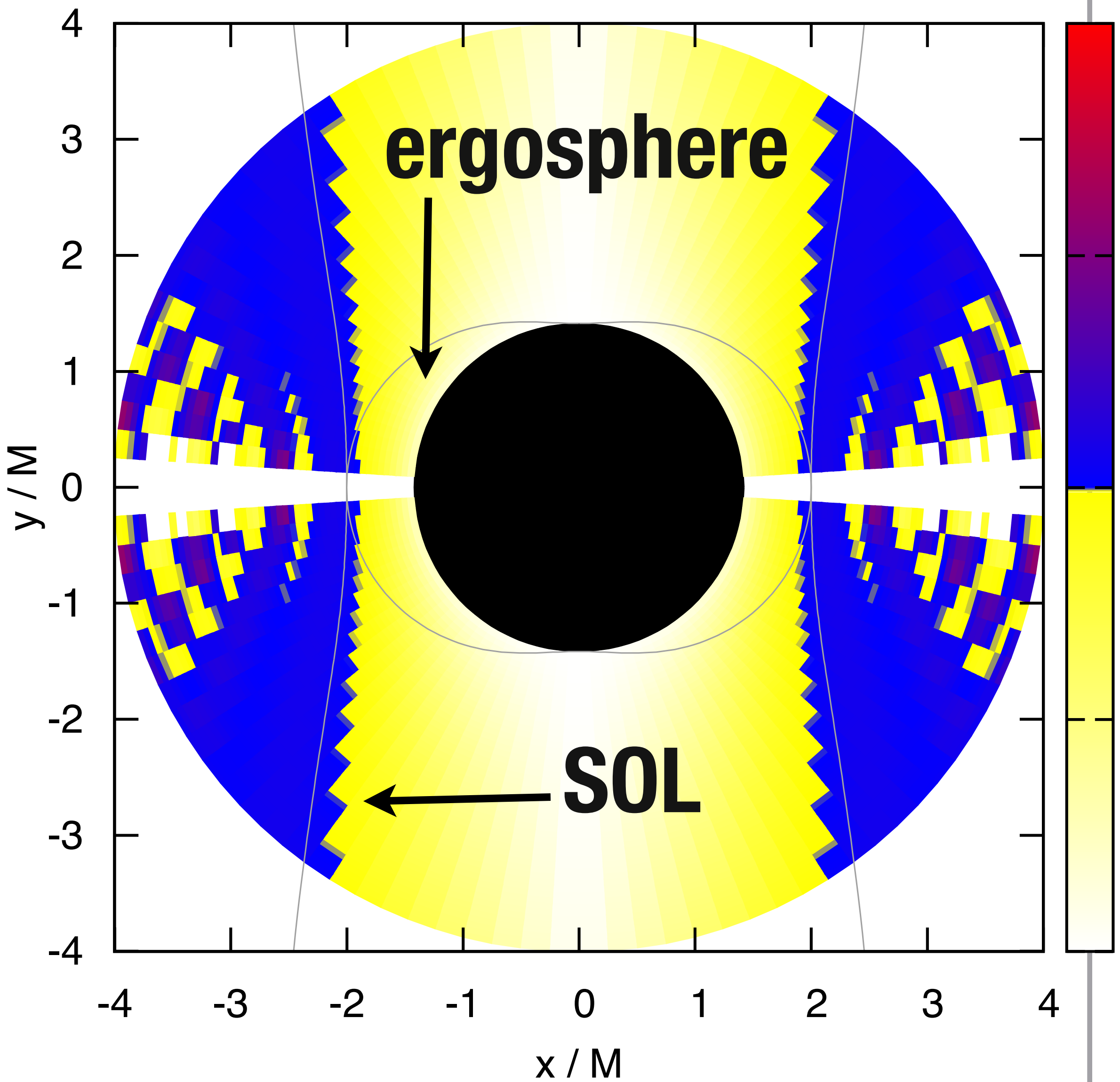


H-H for Fermions

Properties:

- It is well-defined

- Divergence on SOL due to large-modes: ℓ -> thermal bath rotating with horizon?



Conclusions in Kerr

- For bosons:

- No ‘Boulware empty at infinity

- Hartle-Hawking is ill-defined

- For fermions we can construct:

- A state (‘Boulware') empty at infinity, though div. in ergosphere

- A state (‘Hartle-Hawking’) that is in thermal though divergent on SOL

Obrigado