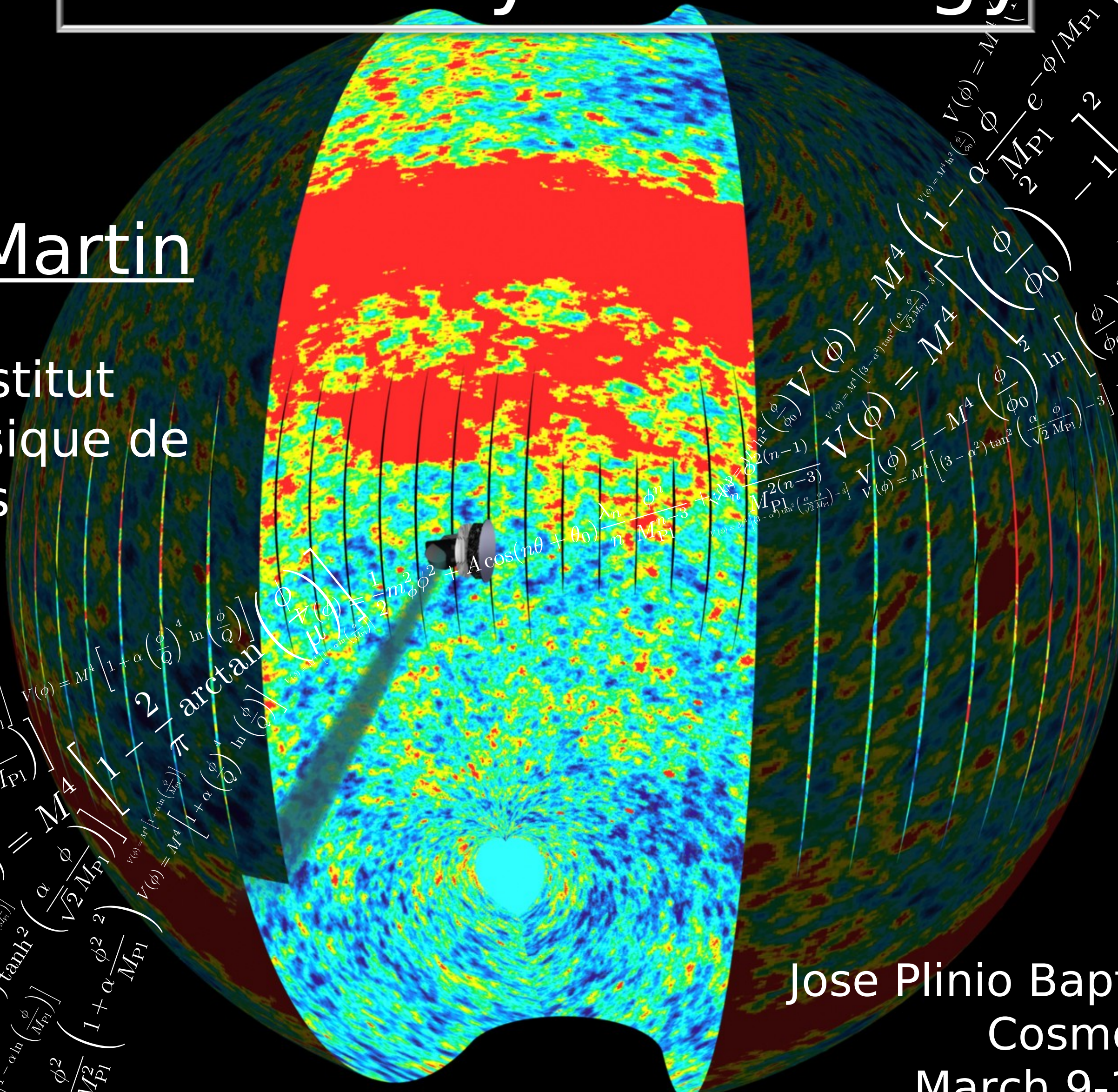
Inflationary Cosmology



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Four Lectures

- Lecture I: The Problems of the Hot Big Bang Model
- Lecture II: The Inflationary Solution
- Lecture III: Inflationary Perturbations of Quantum-Mechanical Origin
- Lecture IV: Inflation and Planck

Lecture I: problems of the standard hot Big Bang model

The standard model

The « hot Big Bang phase » is the standard cosmological model and provides a convincing description of the Universe on a wide range of energy scales.

The model is based on three assumptions:

1- Gravity is described by General Relativity

2- Cosmological principle: the Universe is homogeneous and isotropic (on large scales)

3- Matter/energy is given by different sources to be listed in the following

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1- Gravity is described by General Relaltivity

Geometry   Matter/energy

$$G_{\mu\nu}[g_{\mu\nu}] = \frac{8\pi G}{c^4} \sum_{i=1}^{N} T_{\mu\nu}^{(i)}$$

c, G: Relativistic theory of gravitation

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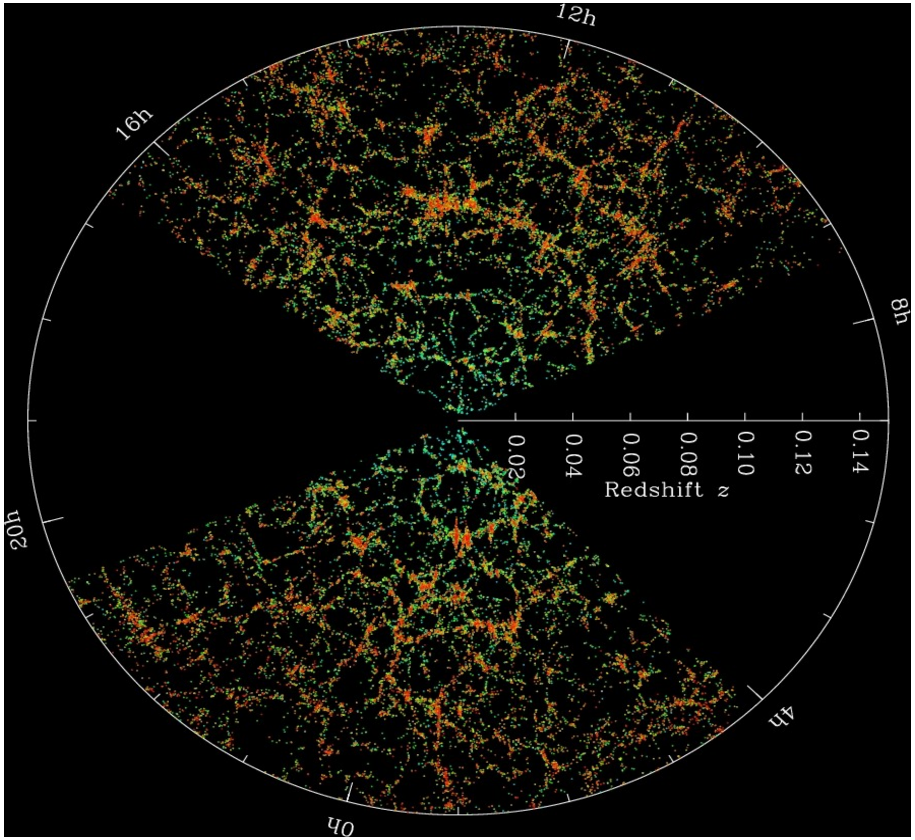

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2- Cosmological principle: the Universe is homogeneous and isotropic (on large scales)

$$G_{\mu\nu}[g_{\mu\nu}] = \frac{8\pi G}{c^4} \sum_{i=1}^{N} T_{\mu\nu}^{(i)}$$

This leads to the FLRW metric

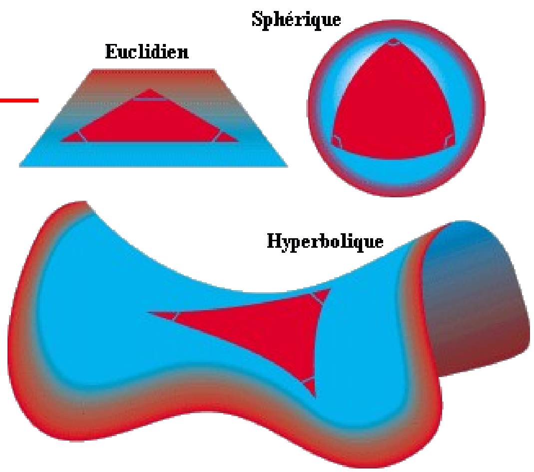
mological Principle : the Universe is (on large scales …) homogeneous a ropic.Technically, this implies that the metric is of the FLRW type

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]$$

cosmic time

space of constant curvature

only one (time-dependent) undetermined function: the scale factor a(t)



n conformal time, the same metric reads

flat (Euclidean) space k=0

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \gamma_{ij}^{(3)} dx^i dx^j\right] = a^2(\eta)\left[-d\eta^2 + \delta_{ij}^{(3)} dx^i dx^j\right]$$

conformal time $\eta = \int \frac{dt}{a(t)}$

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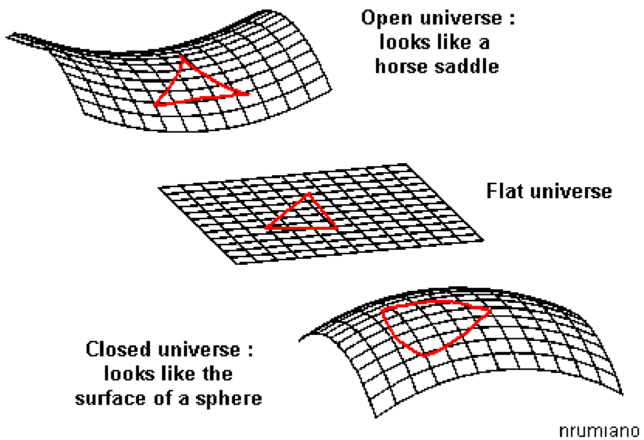
$$G_{\mu\nu}[g_{\mu\nu}] = \frac{8\pi G}{c^4}\sum_{i=1}^{N} T_{\mu\nu}^{(i)}$$

This leads to the FLRW metric

- The size of the Universe is described by a single function of time, the scale factor a(t)

- The spatial curvature of the Universe can be zero (flat) positive or negative



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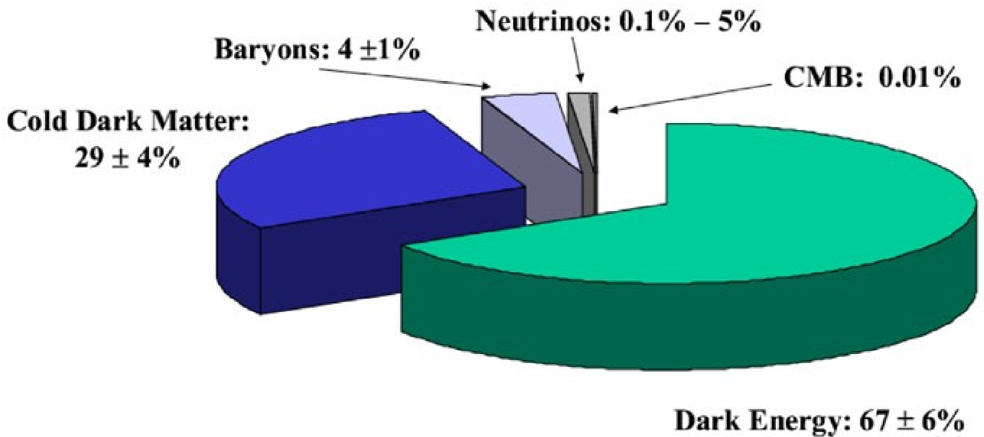

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The Einstein equations reduce to ordinary non-linear differential equations

Hubble parameter= Expansion rate of the Universe ⟵

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{c^4} \sum_{i=1}^{N} \rho^{(i)}$$

$H_0 \sim 70$ km/s/Mpc

$$\dot{\rho}^{(i)} + 3H\left(\rho^{(i)} + p^{(i)}\right) = 0$$

energy density of fluid “i”

Pressure of fluid “i”

Remarks on energy conservation

1- The last equation is in fact obtained from the conservation of the stress energy tensor of a perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$$

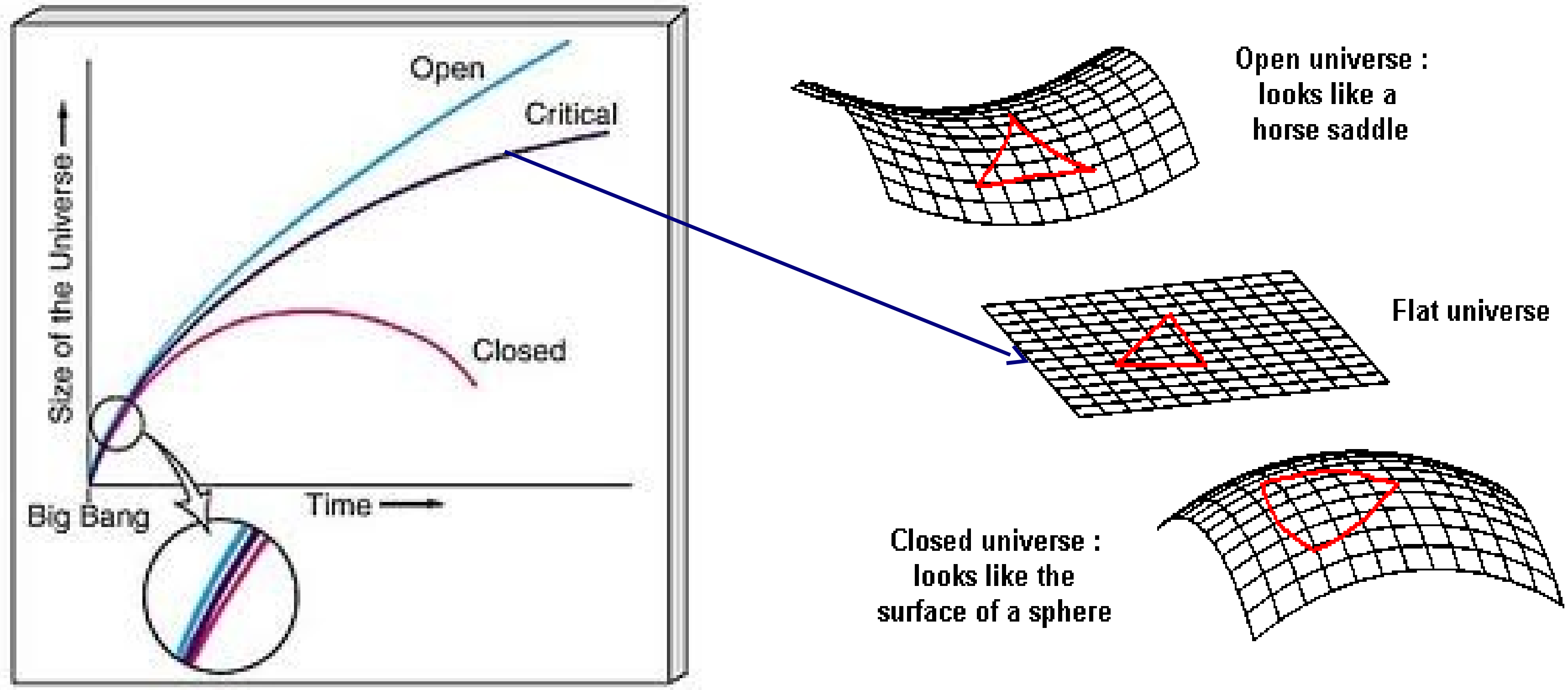
energy density (ρ)

pressure (p)

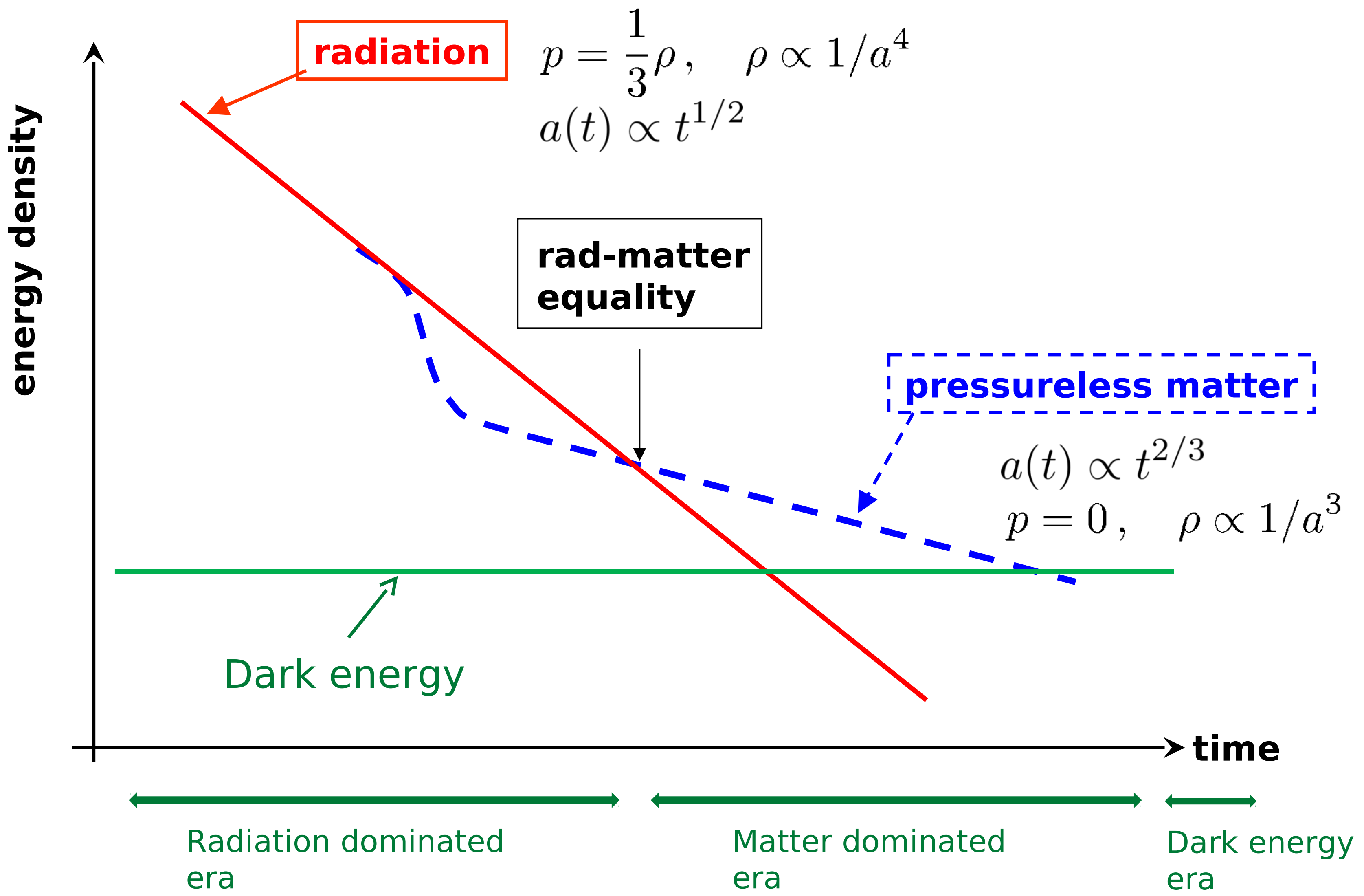
$$\nabla_\mu T^{\mu\nu} = 0 \longrightarrow \dot{\rho} + 3H(\rho + p) = 0$$

2- Conservation of the total stress energy tensor is equivalent to Einstein equations because of Bianchi identities.

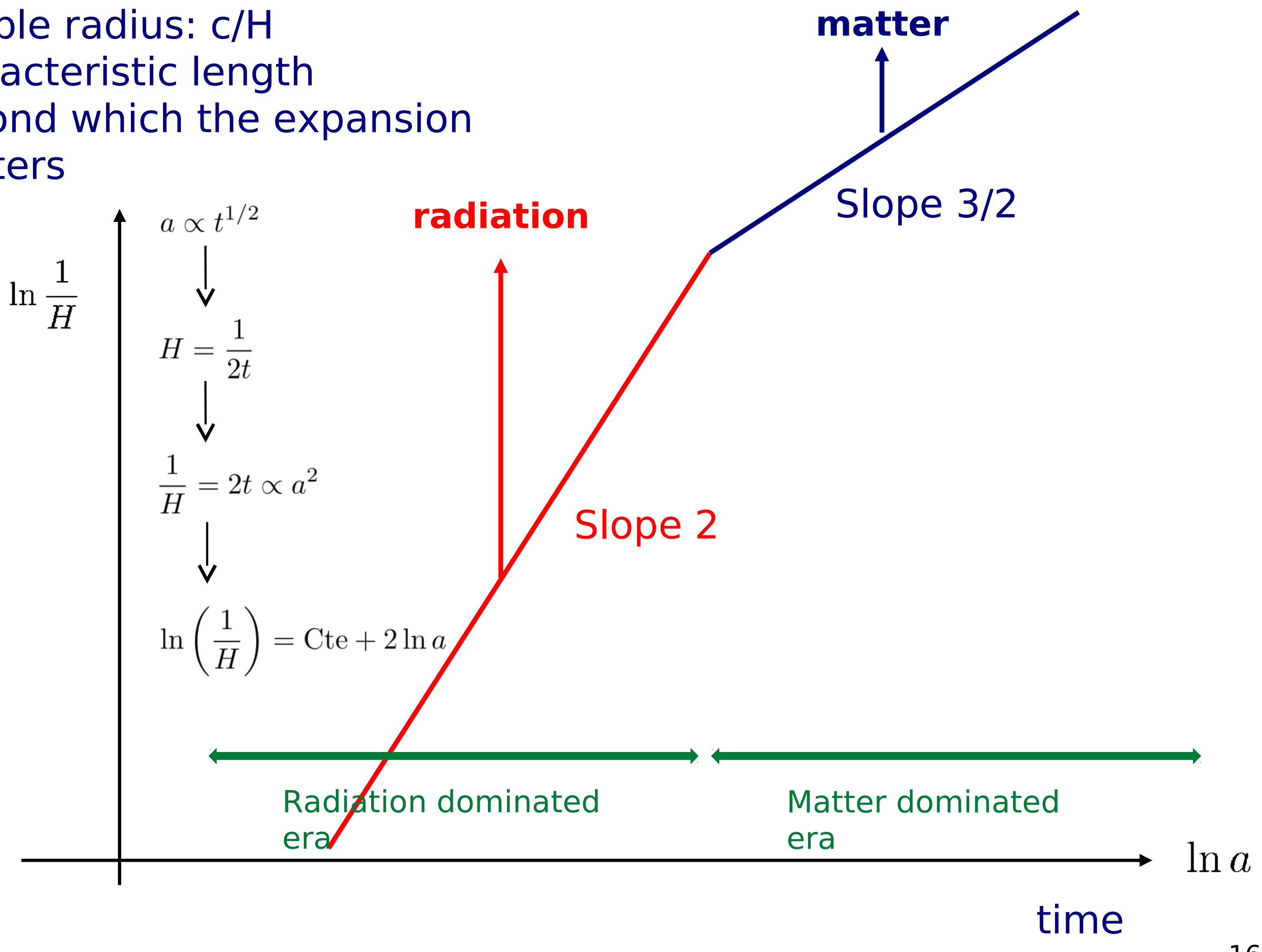
The fact that a=a(t) implies that the Universe is expanding. This expansion is now measured with very high accuracy



The Universe is measured to be spatially flat


radiation
$p = \frac{1}{3}\rho\,, \quad \rho \propto 1/a^4$
$a(t) \propto t^{1/2}$
energy density
rad-matter equality
pressureless matter
$a(t) \propto t^{2/3}$
$p = 0\,, \quad \rho \propto 1/a^3$
Dark energy
time
Radiation dominated era
Matter dominated era
Dark energy era

Hubble radius: c/H
characteristic length
beyond which the expansion
matters



A good model!

The standard model, though it is a simple construction, can account for a
large number of observations and/or experimental tests

- Expansion (Hubble diagram)

- CMB (this school!)

- Nucleosynthesis

- etc …

Puzzles

The standard model, despite its impressive achievements, suffers from a number of troubling puzzles

- Horizon problem

- Flatness problem

- Origin of the inhomogeneities in our Universe

- etc …

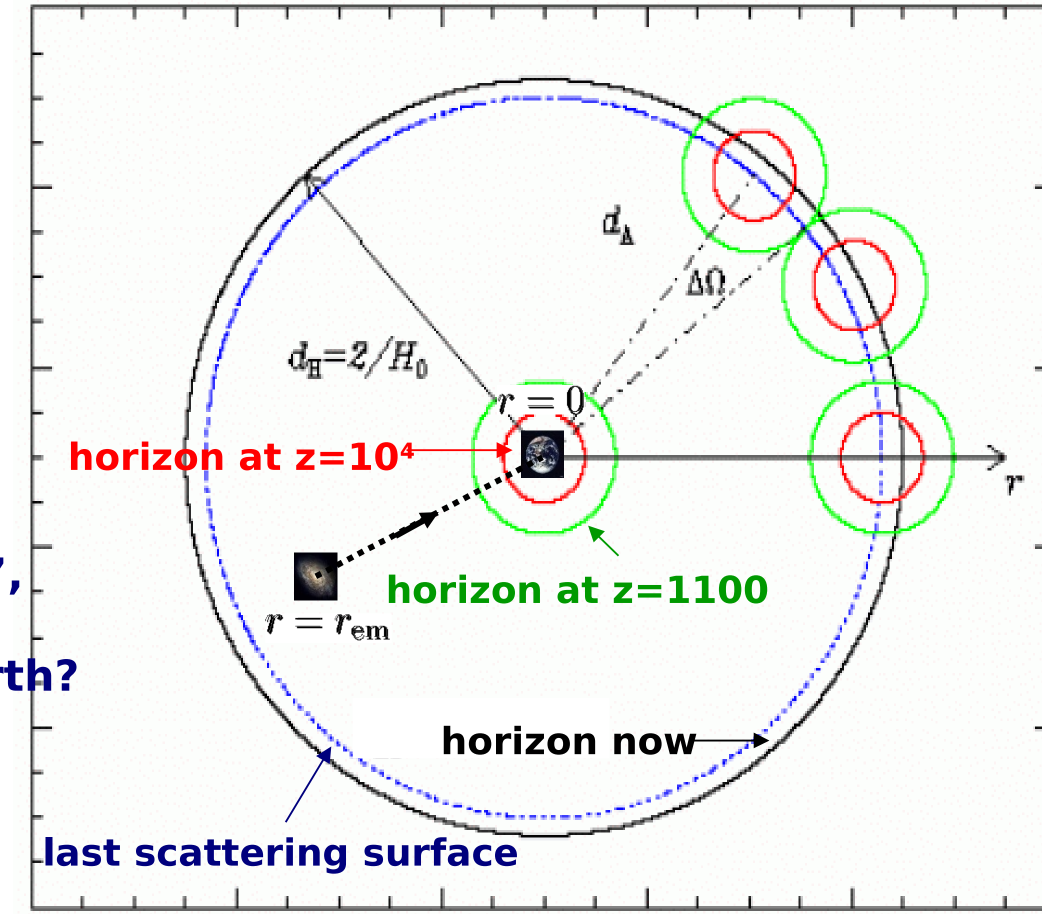
All this issues are related to the initial conditions

Horizon problem

Q1: what is the (proper) distance between the photon and Earth at time t?

$$d_{\rm proper}(t) = a(t)\left[r_{\rm em} - \int_{t_{\rm em}}^{t} \frac{{\rm d}\tau}{a(\tau)}\right]$$

Q2: <u>definition of the horizon</u>. At a given reception time, what is the coordinate distance to the furthest point where a "message", sent to us from there (ie "causal contact"), could have reached Earth?

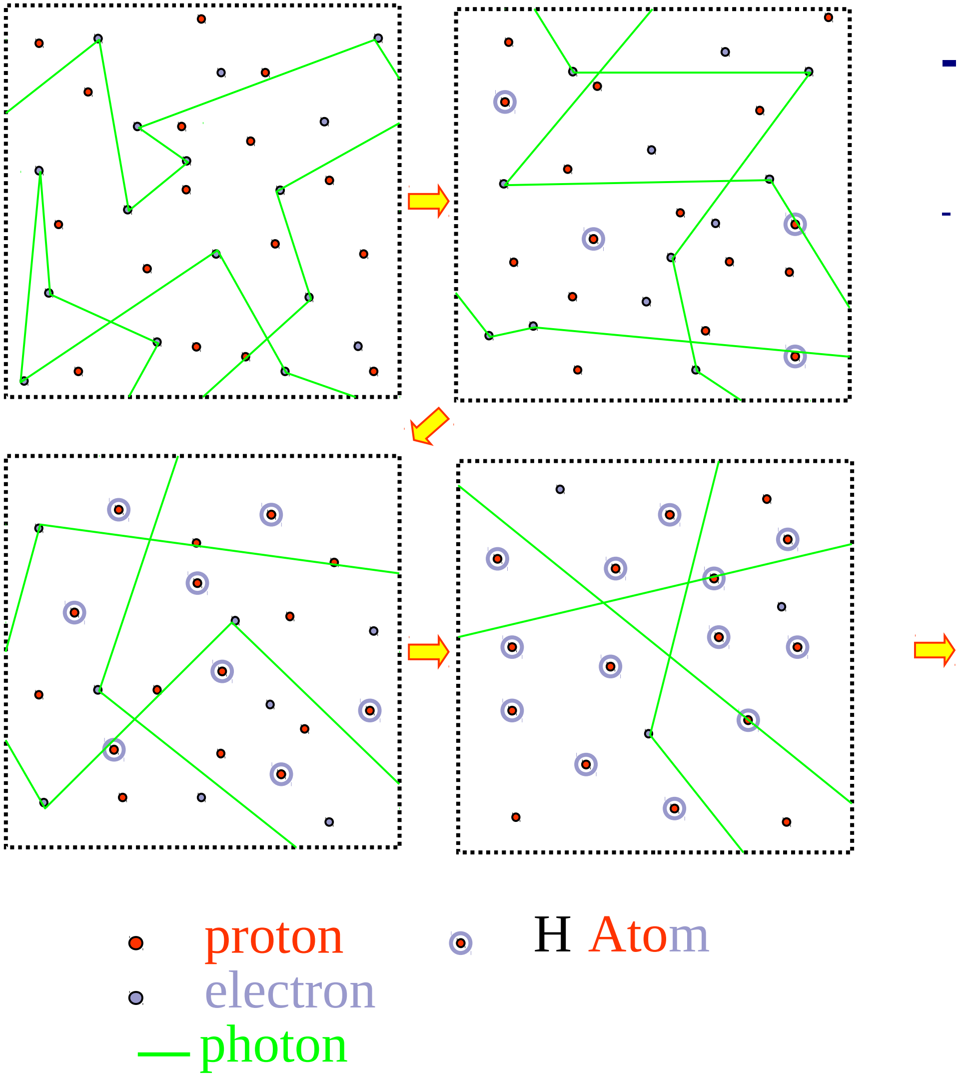


$$r_{\rm em} = \int_{t_{\rm em}}^{t_{\rm reception}} \frac{{\rm d}\tau}{a(\tau)}$$

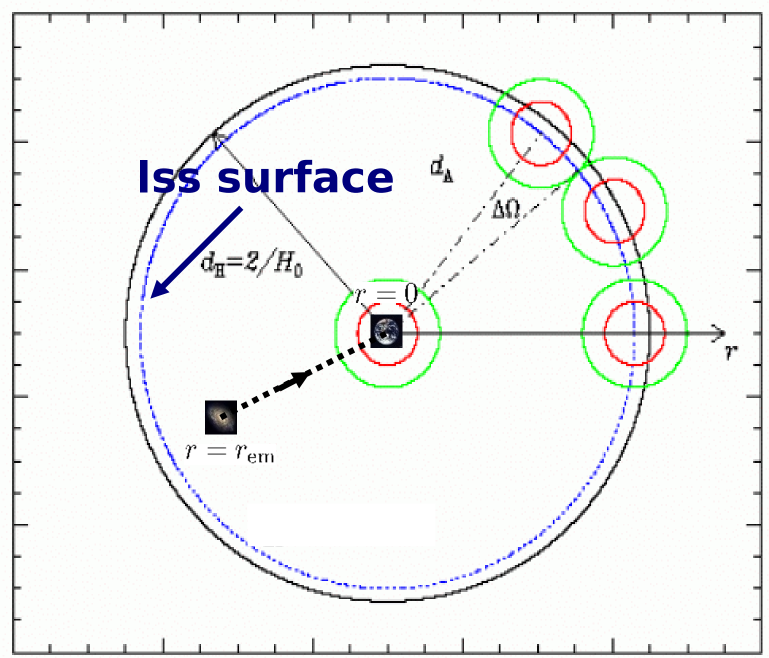
t_em=0="Big-Bang"

$$d_{\rm H}(t_{\rm reception}) = a(t_{\rm reception}) \int_0^{t_{\rm reception}} \frac{{\rm d}\tau}{a(\tau)}$$

Reminder: the lss



- At the lss, z~1100, the Universe became transparent
- Therefore, this is the furthest event we can "see"



Q3: what is the angular diameter of the horizon at the lss?

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right)$$

$$D = a(t_{\rm em})r(t_{\rm em})\Delta\Omega$$

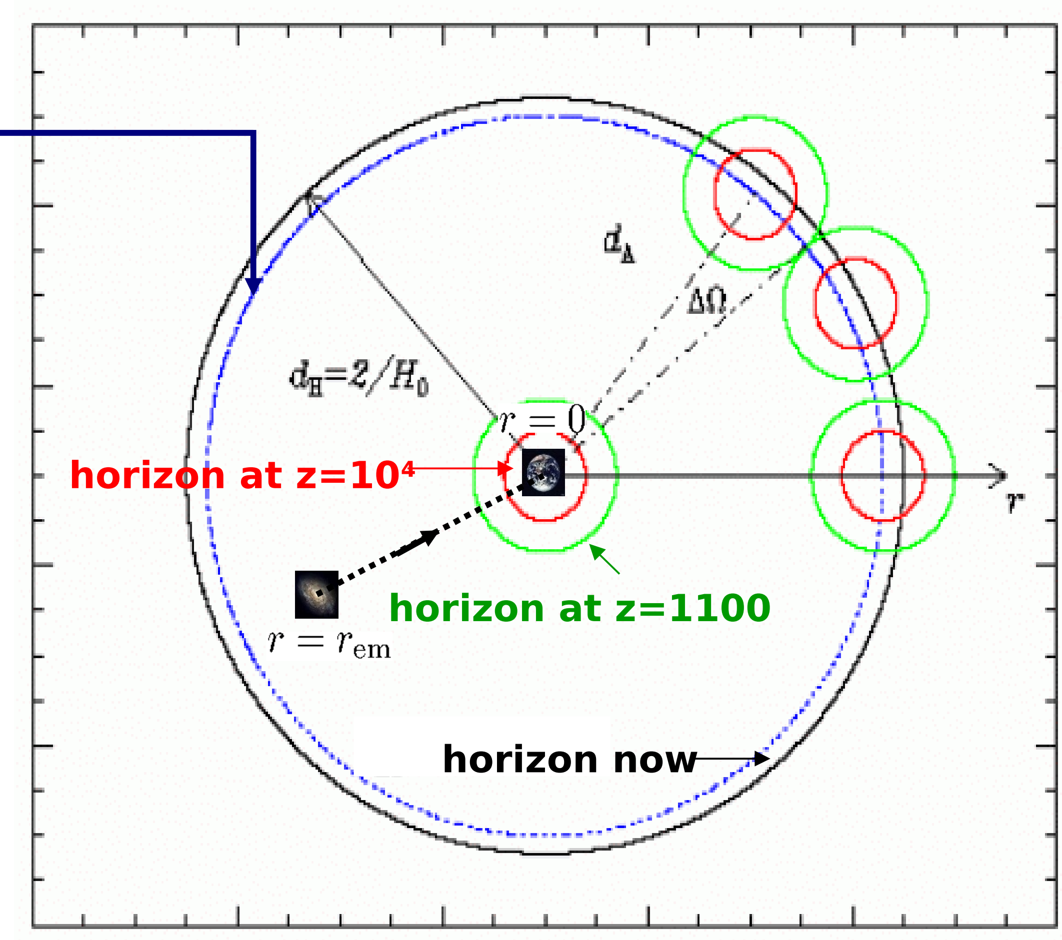
$$d_{\rm H}(t_{\rm reception})$$

with t_reception=t_ lss

$$r_{\rm em} = \int_{t_{\rm lss}}^{t_{\rm now}} \frac{\mathrm{d}\tau}{a(\tau)}$$

emitted at the lss
received now

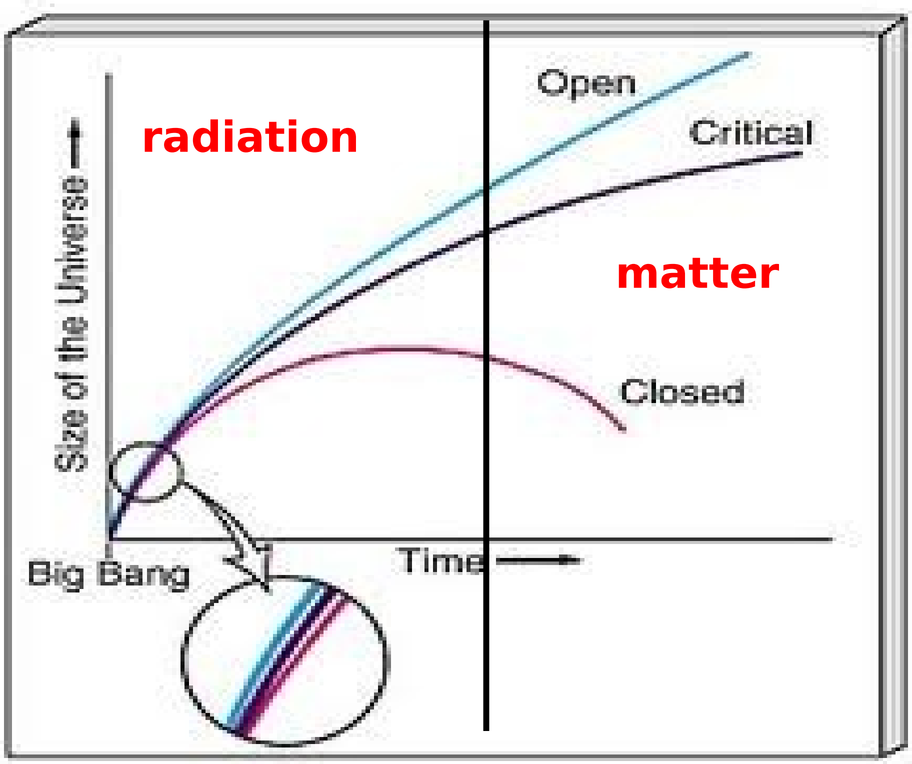
$$d_{\rm H}(t_{\rm reception}) = a(t_{\rm reception})\int_0^{t_{\rm reception}} \frac{\mathrm{d}\tau}{a(\tau)}$$



$$\Delta\Omega = \left[\int_0^{t_{\rm lss}} \frac{\mathrm{d}\tau}{a(\tau)}\right] \times \left[\int_{t_{\rm lss}}^{t_{\rm now}} \frac{\mathrm{d}\tau}{a(\tau)}\right]^{-1}$$

- For the case of a radiation dominated era followed by a matter dominated era, the
calculation is easy

- NB: we identify the lss with the equivalence time (but leads to a small error)

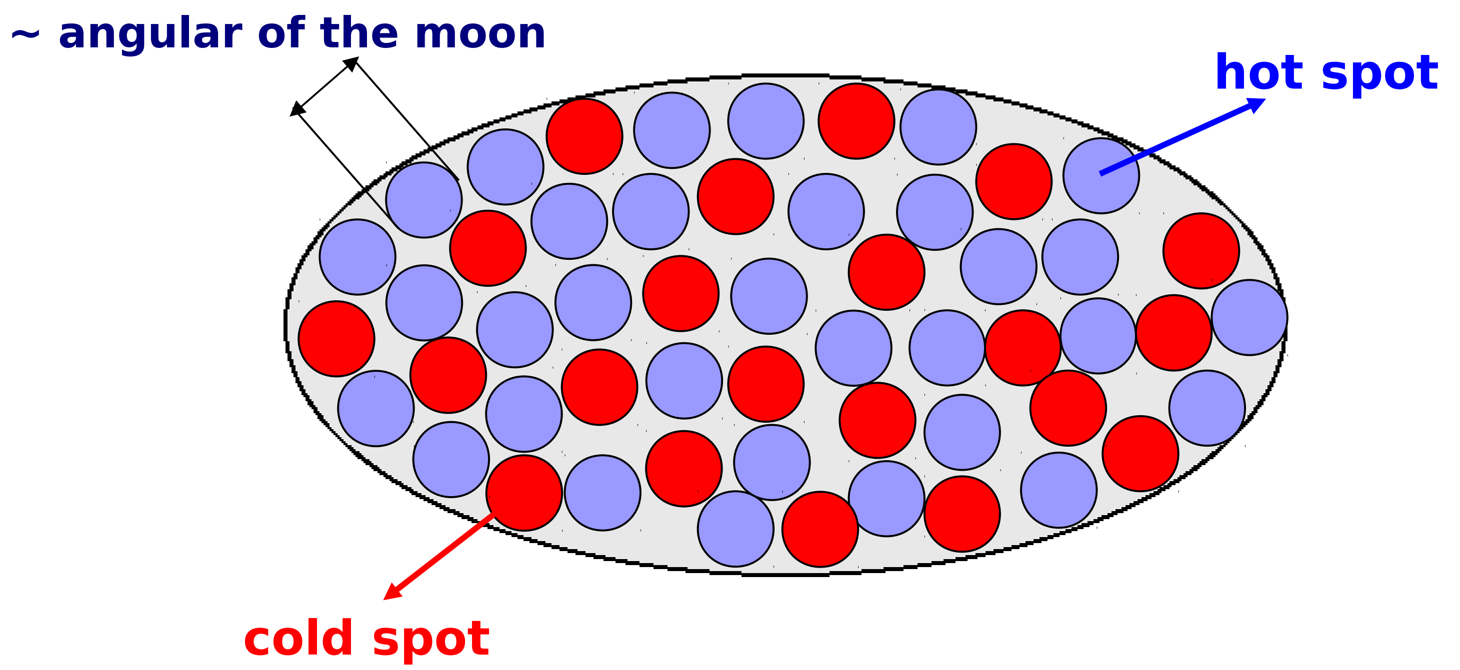


$$\Delta\Omega = \frac{1}{2}\left[1-(1+z_{\rm lss})^{-1/2}\right]^{-1}(1+z_{\rm lss})^{-1/2}$$

$$\Delta\Omega \simeq 0.5\times(1+z_{\rm lss})^{-1/2} \simeq 0.85^\circ$$

Size of the moon!

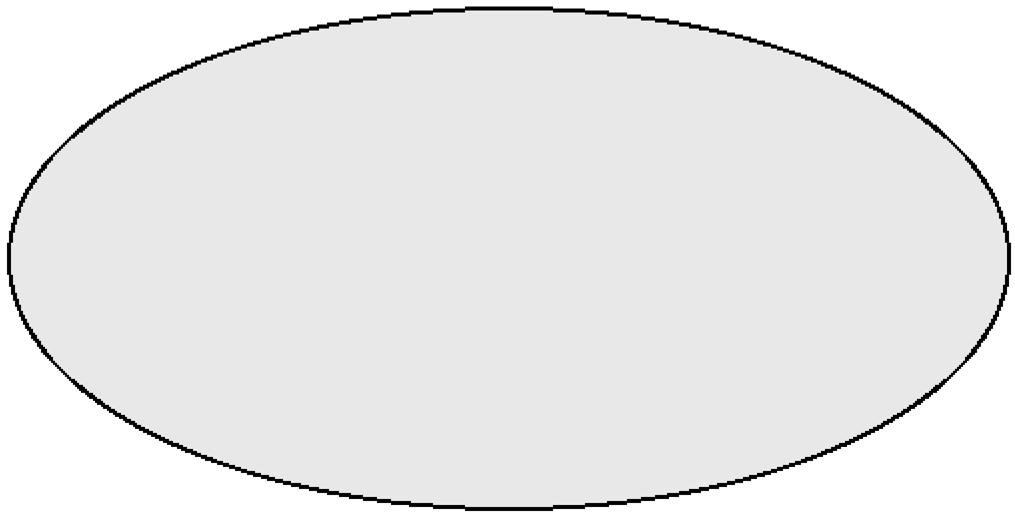
What we have just computed …



$$\frac{\delta T}{T} \simeq 1$$

Low contrast map

What we observe …



Low contrast map

Flatness problem

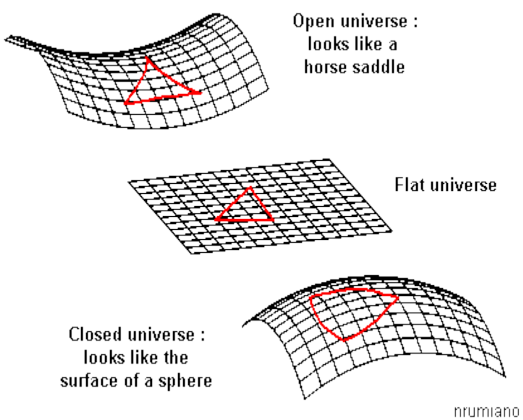
Despite its impressive achievements, we therefore see that the standard models has “isssues”

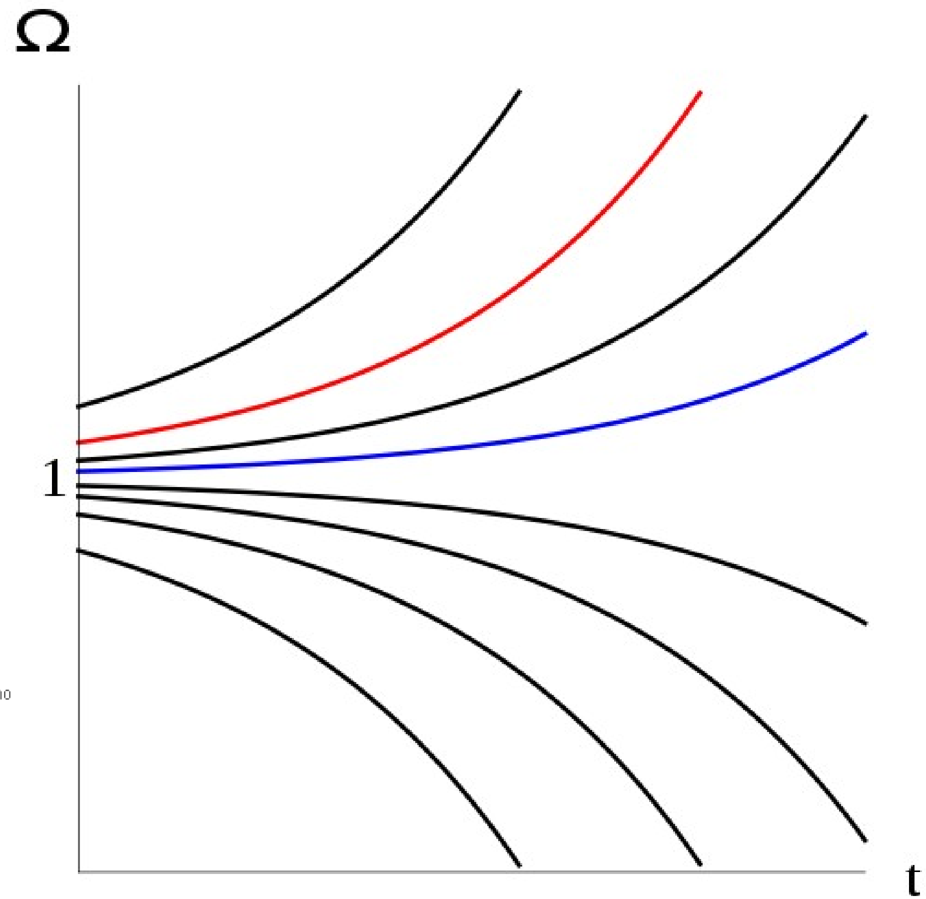
Why is the Universe so spatially flat??

$$\Omega < 1$$

$$\Omega = 1$$

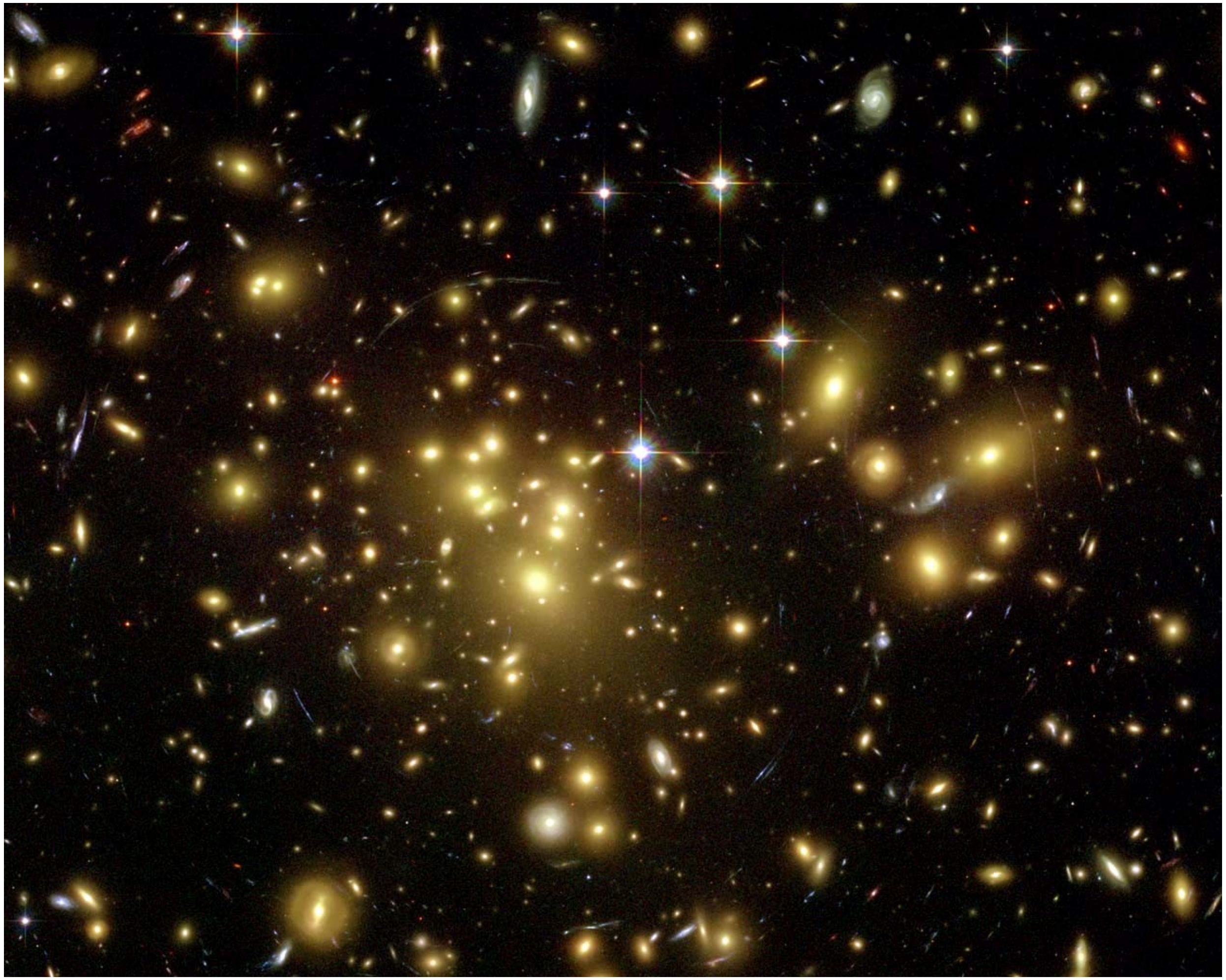
$$\Omega > 1$$





Beyond the cosmological principle

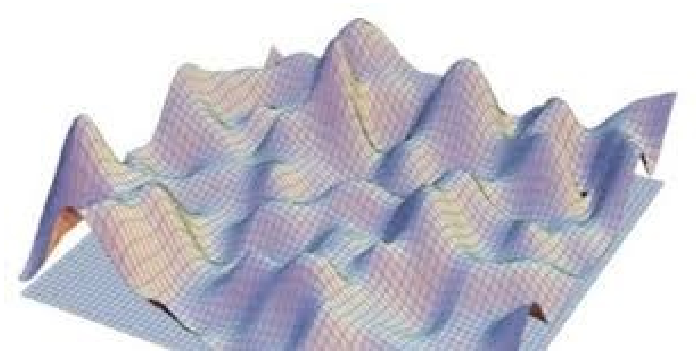
The real Universe (ie on « small » scales) is of course not homogeneous and isotropic!



- The real Universe is not homogeneous and isotropic!

- The mechanism amplifying the perturbations is gravitational instability.

- Today the inhomogeneities are large but, in the early Universe, they were small. One can therefore work with a linear theory and study the various scales independently in Fourier space.



$$g_{\mu\nu}(t) + \delta g_{\mu\nu}(t,\vec{x})$$

$$\rho(t) + \delta\rho(t,\vec{x})$$

The perturbations obey the perturbed Einstein equations

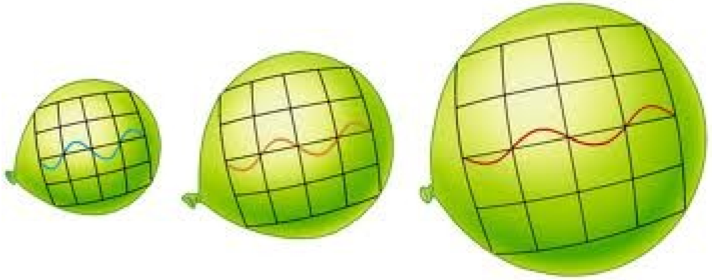
$$\delta G_{\mu\nu} = \frac{8\pi G}{c^4}\sum_{i=1}^{N}\delta T_{\mu\nu}^{(i)}$$

- But the real Universe is not homogeneous and isotropic!
- The mechanism amplifying the perturbations is gravitational instability.
- Today the inhomogeneities are large but, in the early Universe, they were small. One can therefore work with a linear theory and study the various scales independently in Fourier space.

$$\delta g_{\mu\nu}(t,\vec{x}) = \int \mathrm{d}\vec{k}\, \delta g_{\mu\nu}\left(t,\vec{k}\right)\, \mathrm{e}^{i\vec{k}\cdot\vec{x}}$$

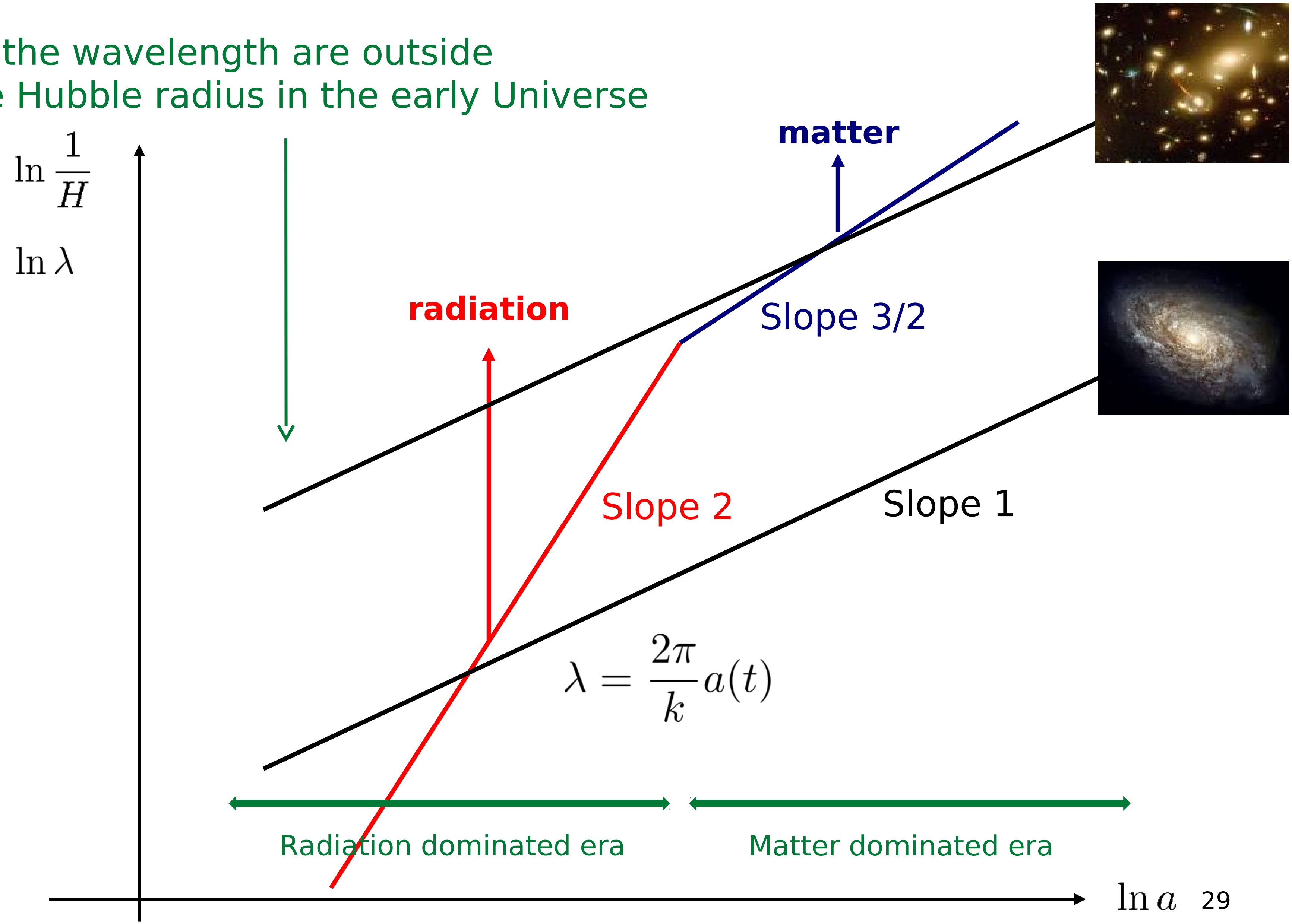
The wavelength of each mode is increasing because the Universe is expanding:

$$\lambda = \frac{2\pi}{k} a(t)$$



Fourier transform=analyzing “scale by scale”

All the wavelength are outside
the Hubble radius in the early Universe



- But what is the source of the fluctuations??

- Observationally, we know that if the initial power spectrum (two point correlation function) is close to scale invariance, then one can reproduce the observations.

$$\left\langle \delta g_{\mu\nu}\left(t_{\rm ini},\vec{k}\right)\delta g_{\mu\nu}\left(t_{\rm ini},\vec{k}\right)\right\rangle \sim k^{0}$$

→ Scale outside the Hubble radius

- In the standard model, this is just postulated.

- But why is it so??

Lecture II: The inflationary solution

Puzzles of the standard model

The standard model, despite its impressive achievements, suffers from a number of troubling puzzles

- Horizon problem

- Flatness problem

- Origin of the inhomogeneities in our Universe

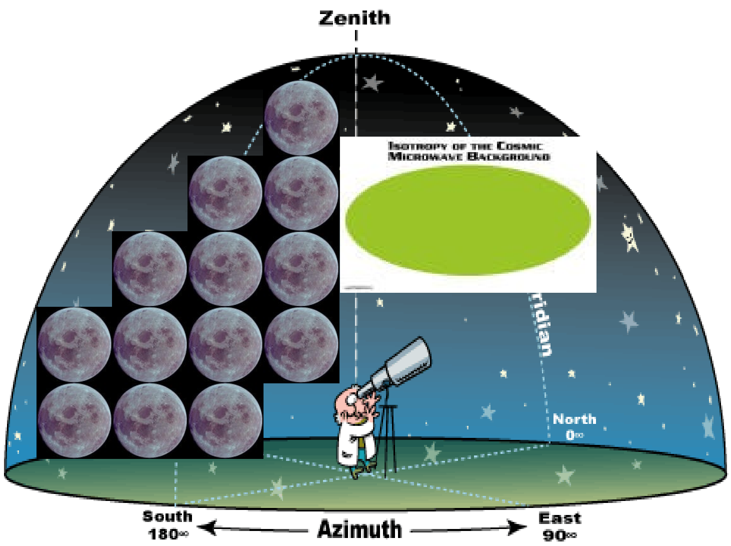
- etc …

All this issues are related to the initial conditions

Inflation is a phase of accelerated expansion taking place in the very early Universe. The scale factor is such that

$$\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} > 0$$

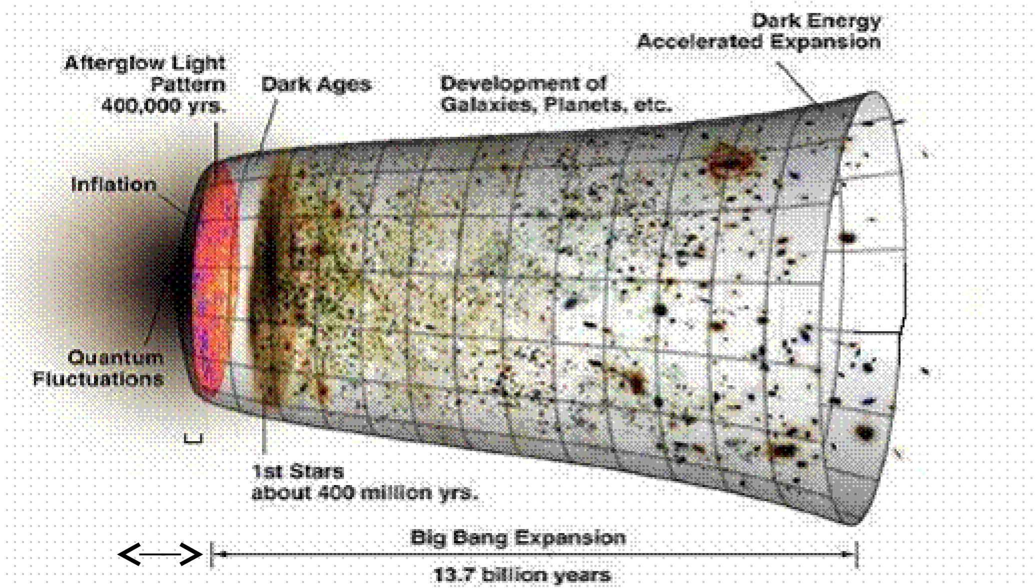
This assumption allows us to solve several problems of the standard hot Big Bang model:



- **Horizon problem**
- **Flatness**
- **Monopoles problem ...**

Inflationary energy scale

- **Inflation does not replace the Hot Big Bang model. It is a new ingredient which completes the standard model. It takes place before the Hot Big Bang phase**



Inflation

$\rho_{\mathrm{now}} \simeq 10^{-47}\mathrm{GeV}^4$

$\rho_{\mathrm{LHC}} \simeq (100\mathrm{GeV})^4$

Inflation

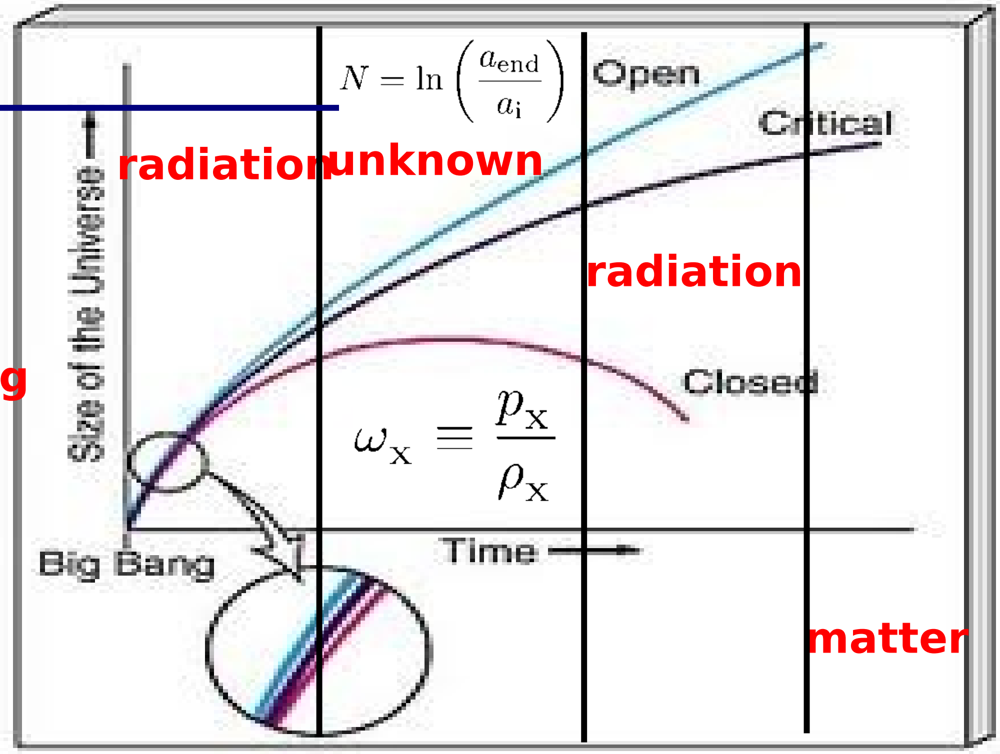
$\rho_{\mathrm{inf}} \simeq \left(10^{16}\mathrm{GeV}\right)^4$

- **The energy scale of inflation is poorly constrained**

A simple model …

New phase driven by an unknown fluid. Can be switched off by putting N to zero …

$$a(t) = a_{\rm i}\left[\frac{3}{2}(1+\omega_{\rm X})H_{\rm i}(t-t_{\rm i})+1\right]^{2/[3(1+\omega_{\rm X})]}$$



$$\Delta\Omega = \frac{1}{2}\left[1-(1+z_{\rm lss})^{-1/2}\right]^{-1}(1+z_{\rm lss})^{-1/2}\left\{1+\frac{1-3\omega_{\rm X}}{1+3\omega_{\rm X}}\frac{1+z_{\rm lss}}{1+z_{\rm end}}\left[1-{\rm e}^{-N(1+3\omega_{\rm X})/2}\right]\right\}$$

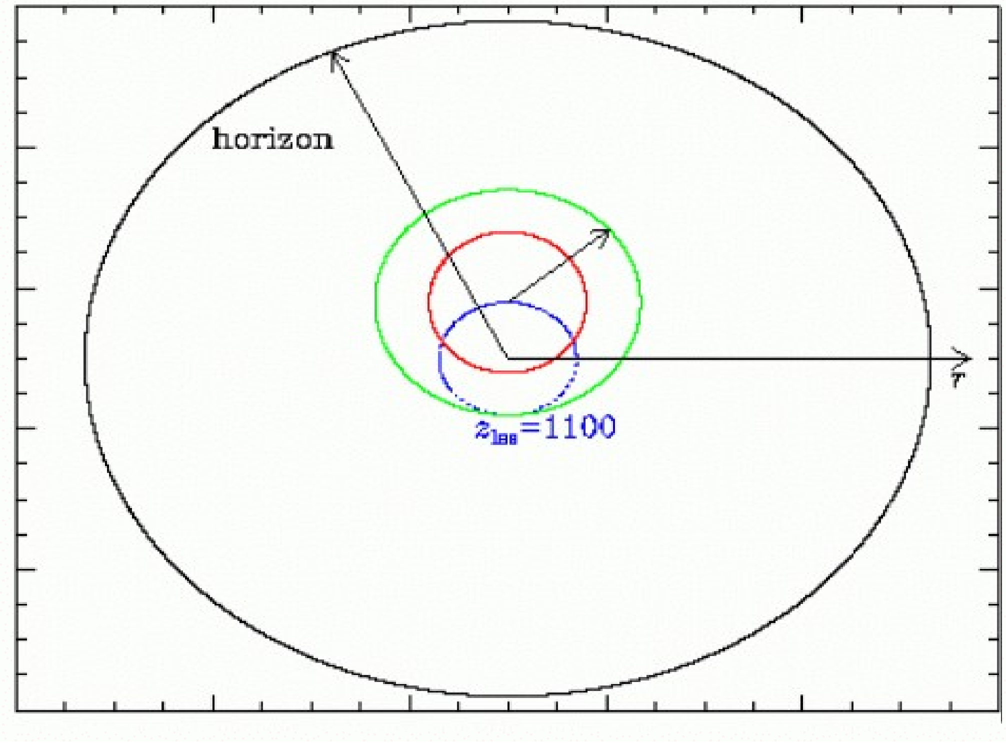
We "just" need a solid angle larger than 4π

$$\Delta\Omega = \frac{1}{2}\left[1-(1+z_{\rm lss})^{-1/2}\right]^{-1}(1+z_{\rm lss})^{-1/2}\left\{1+\frac{1-3\omega_{\rm X}}{1+3\omega_{\rm X}}\frac{1+z_{\rm lss}}{1+z_{\rm end}}\left[1-{\rm e}^{-N(1+3\omega_{\rm X})/2}\right]\right\}$$

$$1+3\omega_{\rm X}<0$$

$$N > -4+\ln z_{\rm end}$$

We need a fluid with a negative pressure such that p<-ρ/3



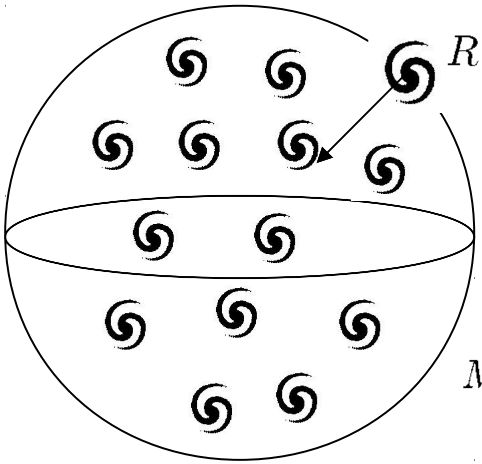
Negative pressure means accelerated expansion, ie inflation!

$$1 + 3\omega_X < 0 \longrightarrow p_X < -\frac{\rho_X}{3}$$

$$\left.\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{\kappa}{3}\rho \\ -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) &= \kappa p \end{aligned}\right\} \quad \boxed{\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3p)}$$

$$\ddot{a} > 0$$

We have acceleration if
the pressure is **<u>negative</u>!**



$$R(t) = a(t)r$$

$$M(r) = \frac{4\pi a^3 r^3}{3}\rho$$

$$m\ddot{R} = -\frac{GM(r)m}{R^2}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\rho$$

Einstein equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3}\rho$$

$$-\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = \kappa p$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3p)$$

“Every form of energy weighs in General Relativity”

Implementing inflation

1- At high energies, field theory is the correct framework to describe matter

2- The Universe is homogeneous and isotropic: spin 0 particle

The action of a self interacting scalar field is given by

$$S = -\int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + V\left(\Phi\right)\right]$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}$$

$$T^{\mu}{}_{\nu} = g^{\mu\lambda}\partial_\lambda\Phi\partial_\nu\Phi - \delta^{\mu}{}_{\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi + V\left(\Phi\right)\right]$$

$$\begin{cases} \rho = \dfrac{\dot{\Phi}^2}{2} + V\left(\Phi\right) \\ p = \dfrac{\dot{\Phi}^2}{2} - V\left(\Phi\right) \end{cases}$$

- The potential energy must dominate

- The potential must be flat in order to have inflation

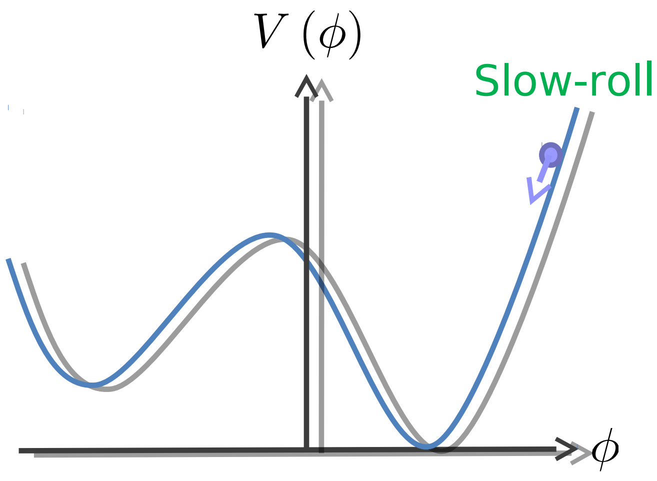
1- At high energies, field theory is the correct framework to describe matter

2- The Universe is homogeneous and isotropic: spin 0 particle

3- Energy density & pressure

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi)$$

$$p = \frac{\dot{\phi}^2}{2} - V(\phi)$$



The potential must be flat

Implementing inflation

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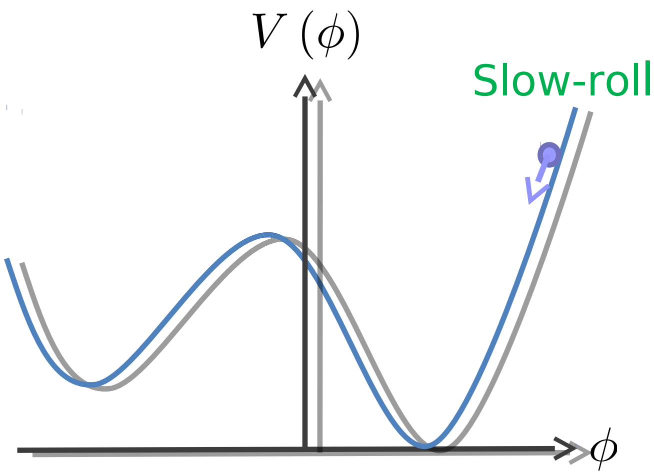


4- Inflation stops when the potential is no longer flat enough, ie when $\epsilon_1 = 1$

$$\frac{\ddot{a}}{a} = H^2(1-\epsilon_1) \quad \text{with} \quad \epsilon_1 \equiv -\frac{\dot{H}}{H^2} \simeq \frac{M_{\rm Pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2$$

Inflation is a quasi exponential expansion of spacetime

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi)$$

$$p = \frac{\dot{\phi}^2}{2} - V(\phi)$$

$$\Downarrow$$

$$p \simeq -\rho$$

$$\Downarrow$$

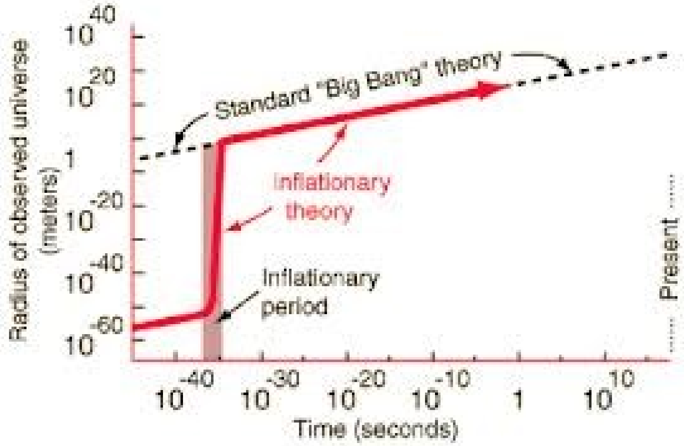
$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + 3H(\rho + p) = 0 \simeq \frac{\mathrm{d}\rho}{\mathrm{d}t}$$

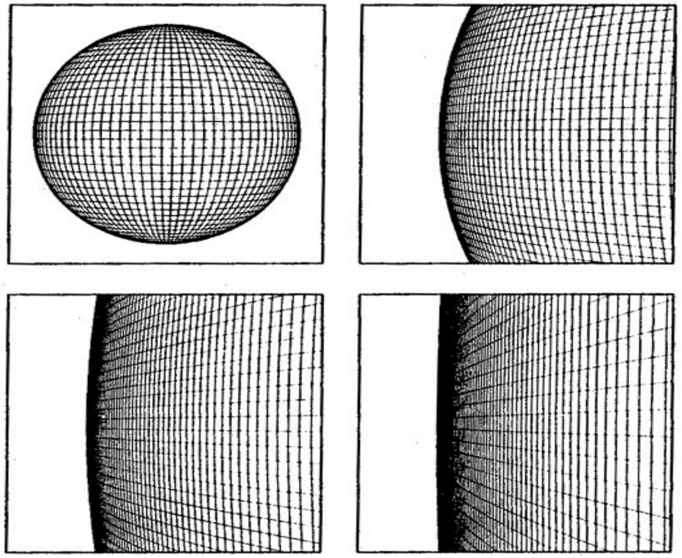
$$\Downarrow$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3M_{\mathrm{Pl}}^2}$$

$$\Downarrow$$

$$a(t) \sim \mathrm{e}^{Ht}$$





Flatness problem solved!

Inflation is a quasi exponential expansion of spacetime

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi)$$

$$p = \frac{\dot{\phi}^2}{2} - V(\phi)$$

$$\Downarrow$$

$$p \simeq -\rho$$

$$\Downarrow$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + 3H(\rho + p) = 0 \simeq \frac{\mathrm{d}\rho}{\mathrm{d}t}$$

$$\Downarrow$$

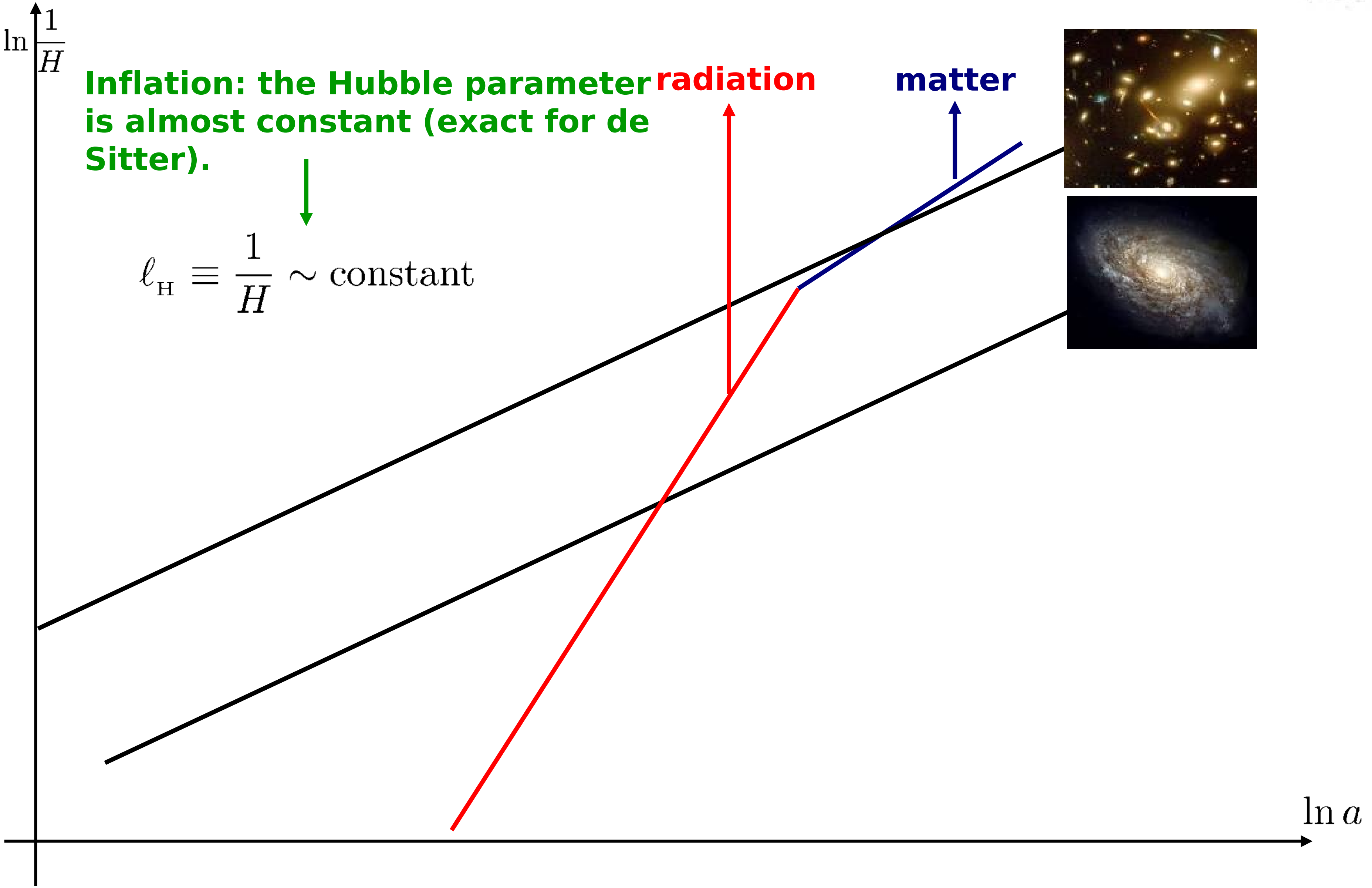
$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3M_{\mathrm{Pl}}^2}$$

$$\Downarrow$$

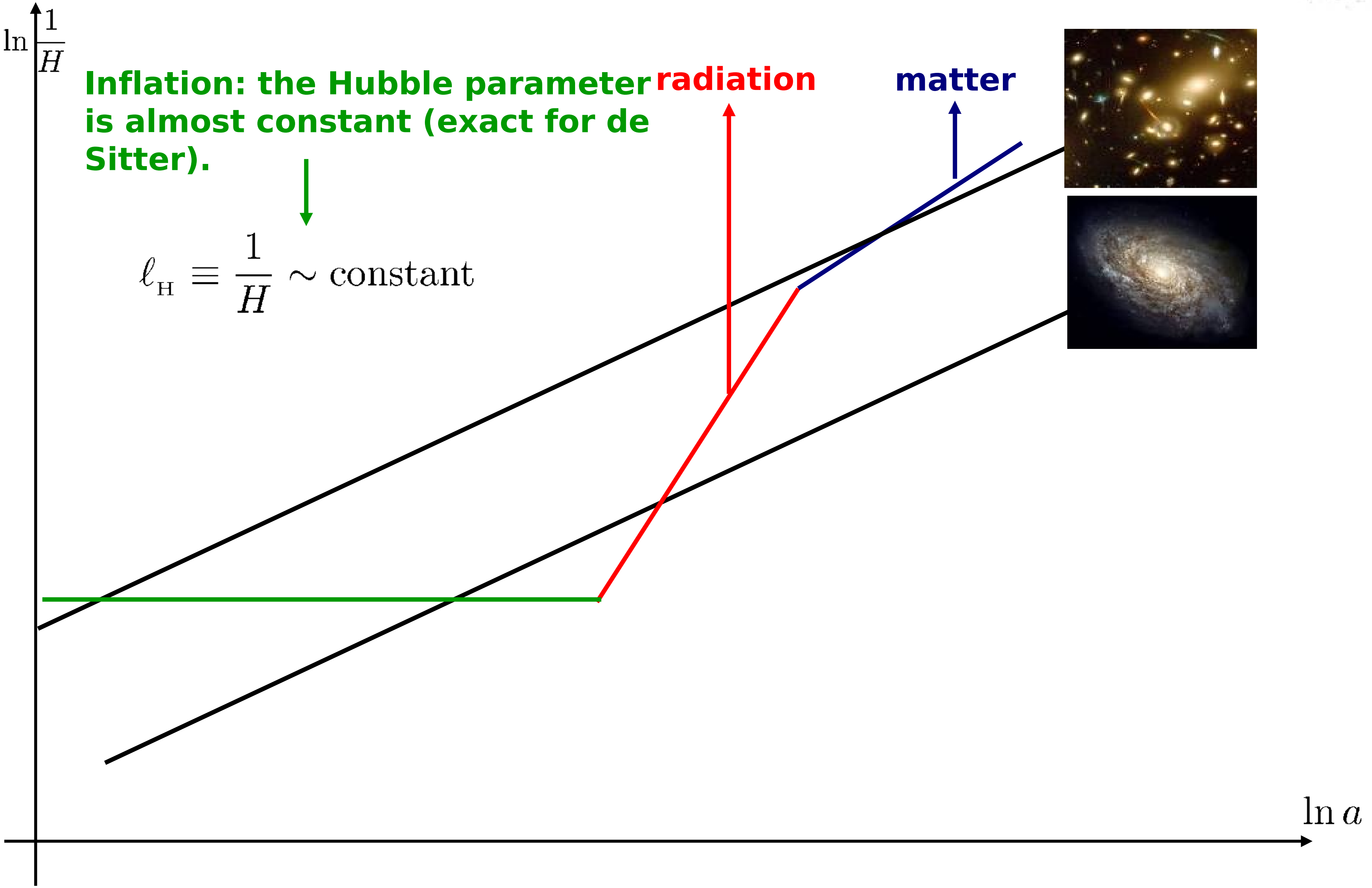
$$a(t) \sim \mathrm{e}^{Ht}$$

The Hubble radius 1/H is almost constant during inflation

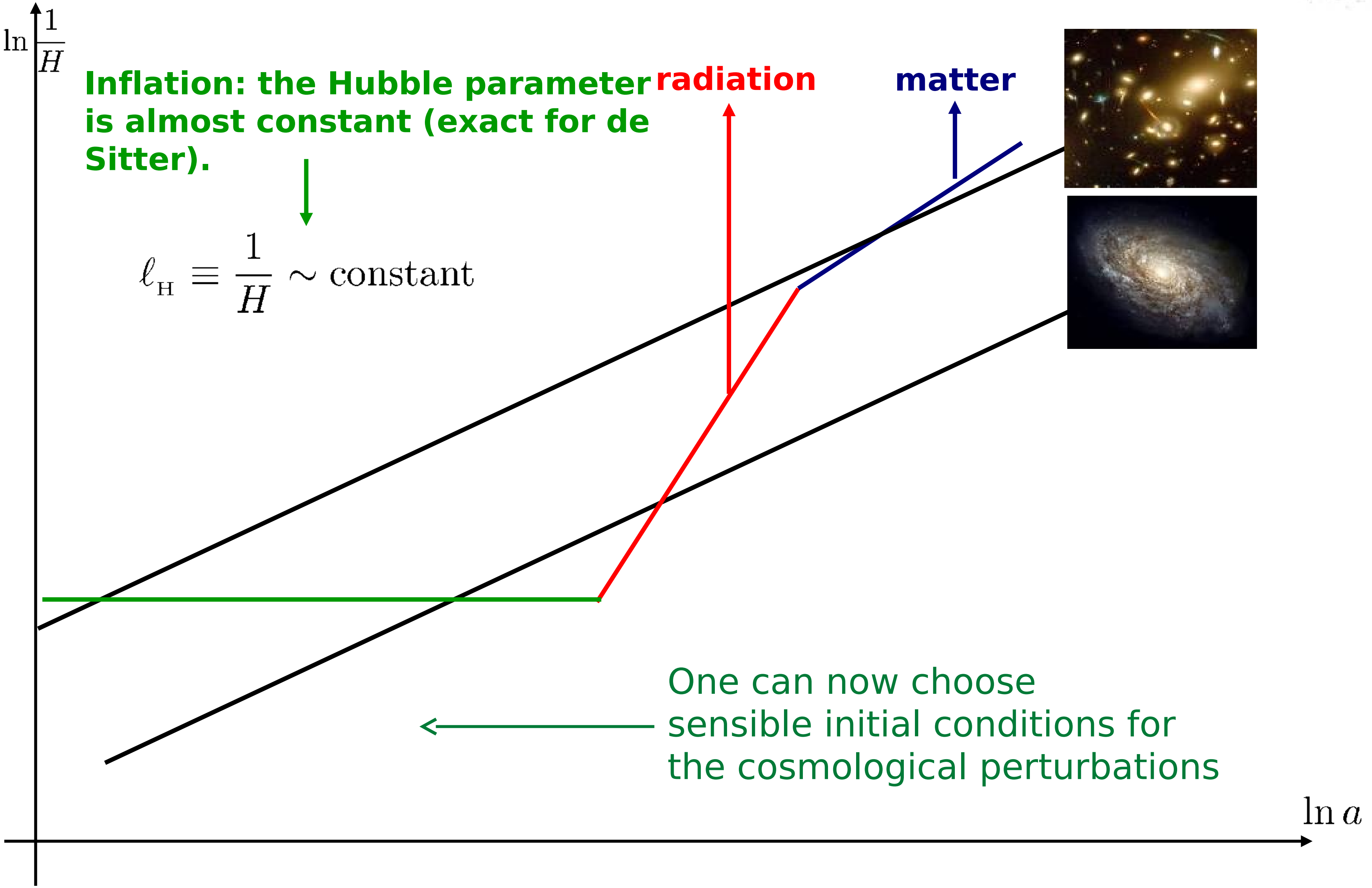
Fourier transform=analyzing “scale by scale”



Fourier transform=analyzing "scale by scale"

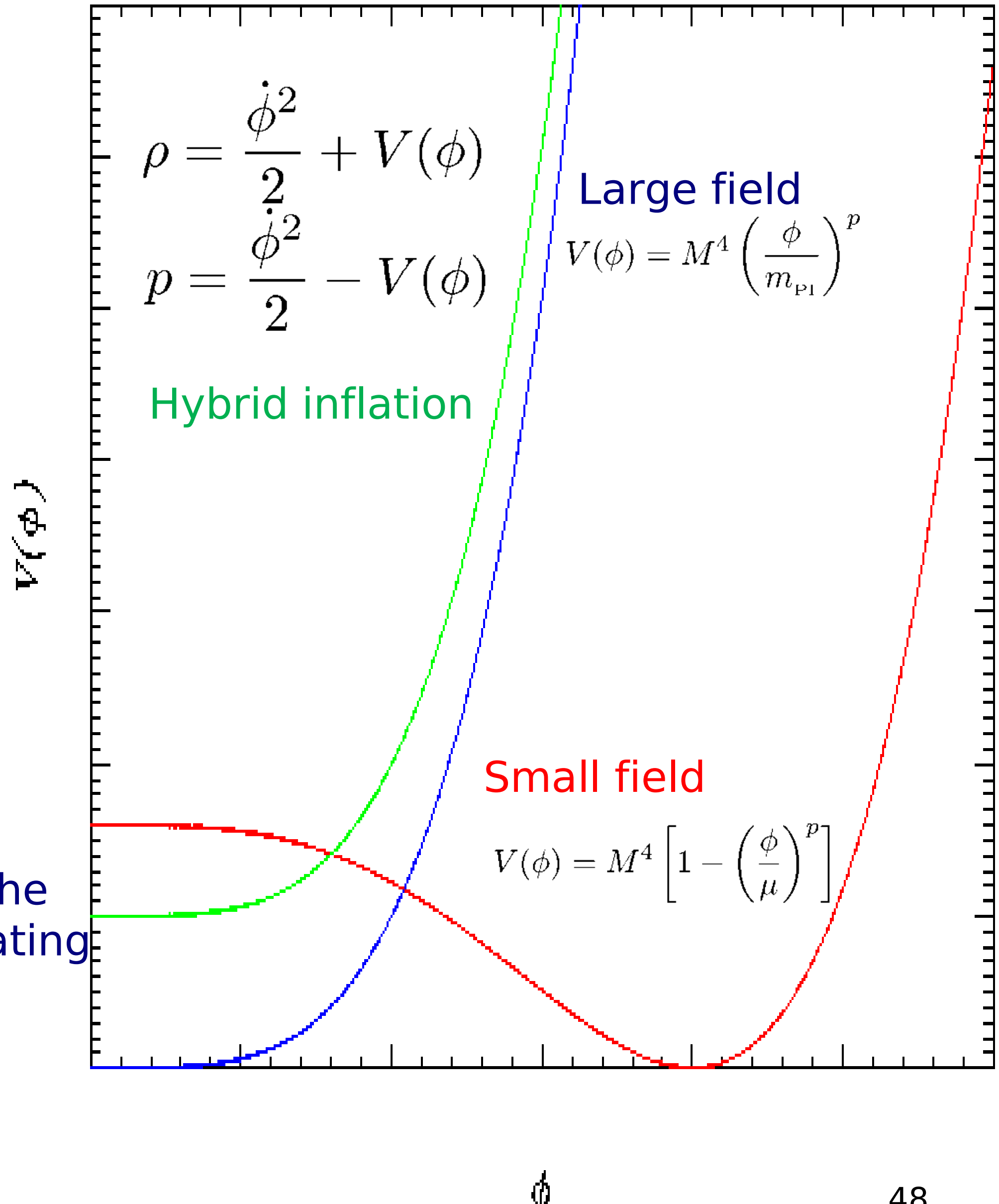


Fourier transform=analyzing "scale by scale"

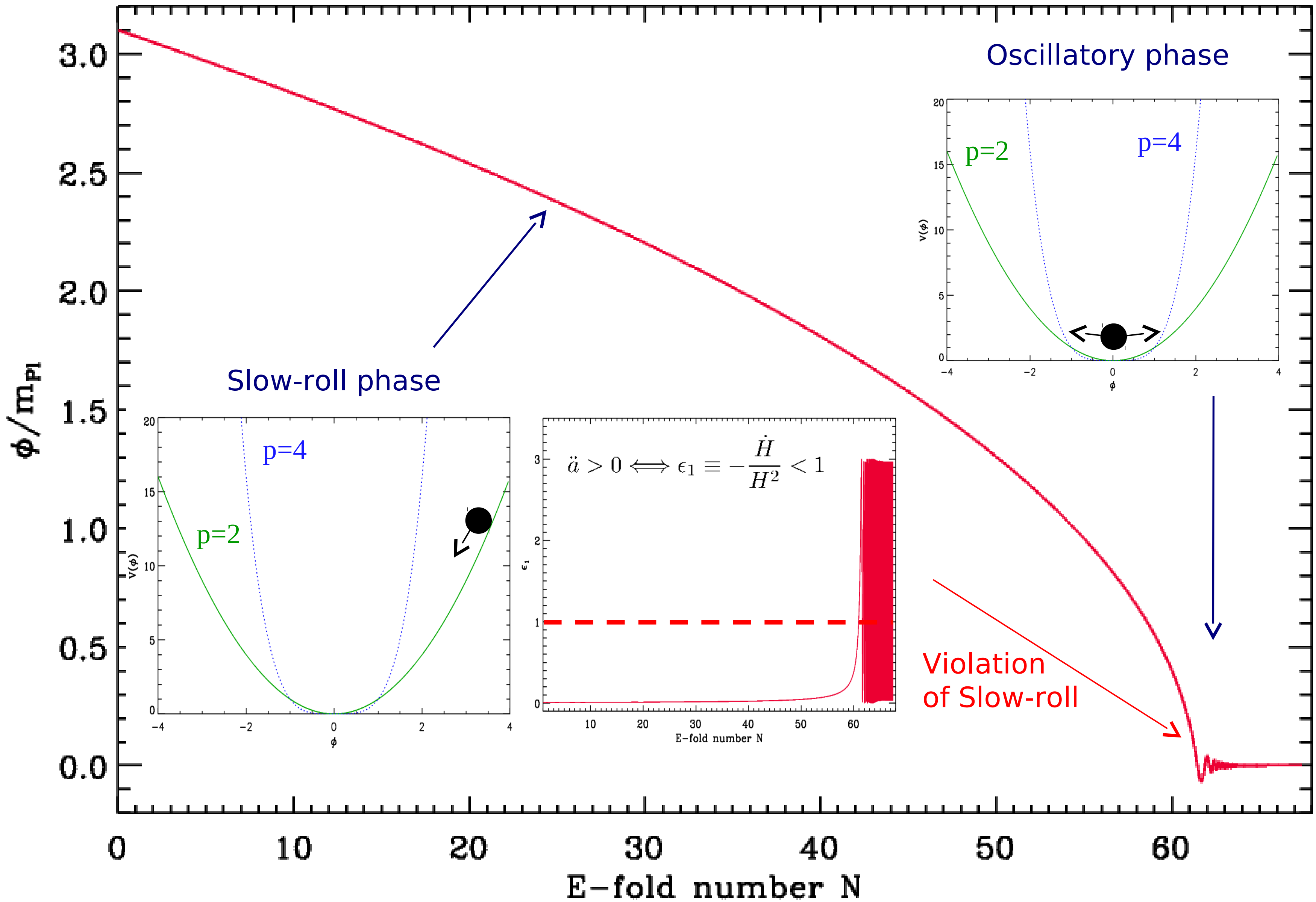


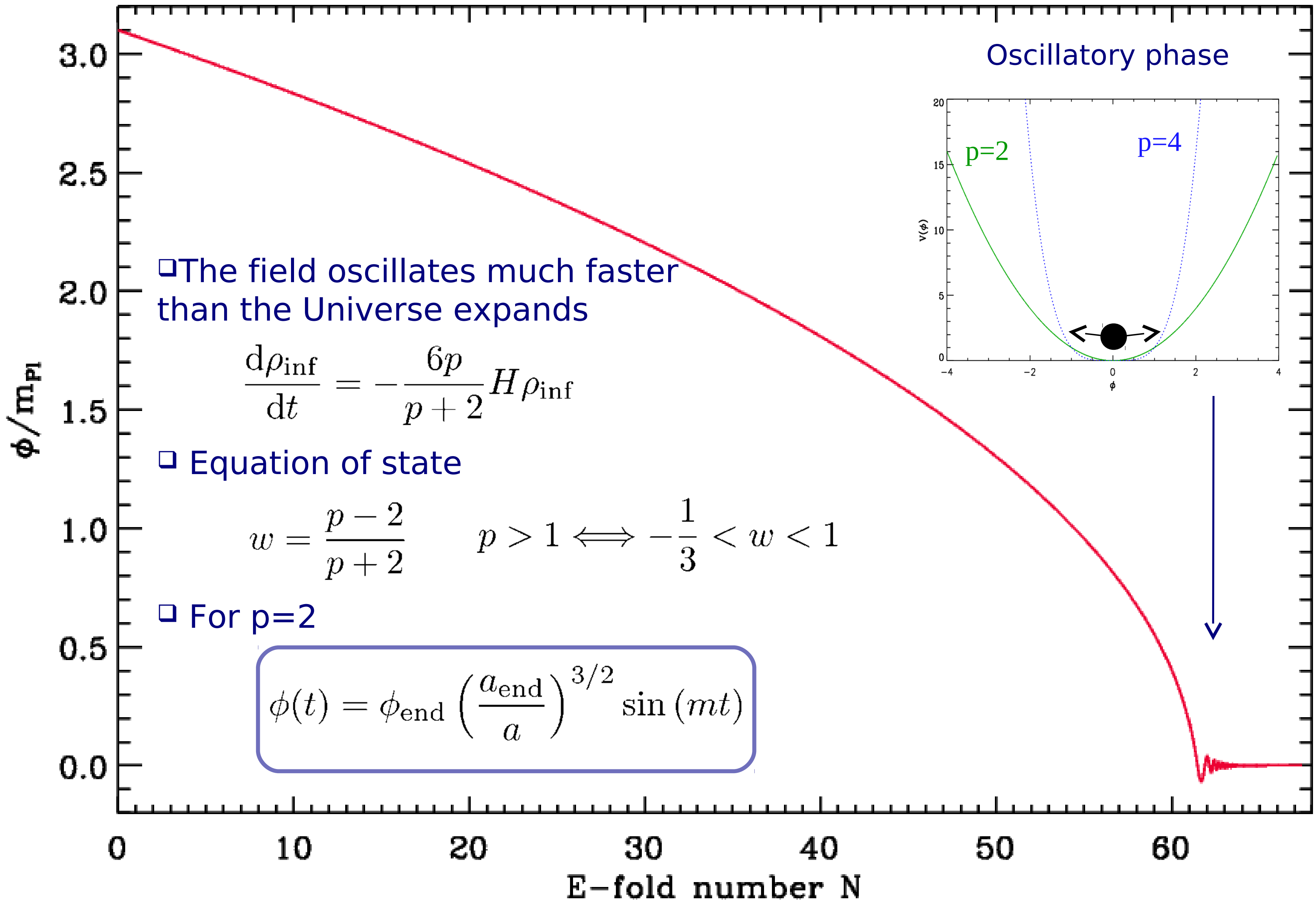
Inflation in a nutshell

- Field theory is the correct description at high energies.
- A natural realization is a scalar field slowly rolling down its flat potential
- Inflation ends by violation of the slow-roll conditions or by instability
- After inflation, the field oscillates at the bottom of its potential: this is the reheating



End of Inflation (I)





- The field oscillates much faster than the Universe expands

$$\frac{\mathrm{d}\rho_{\mathrm{inf}}}{\mathrm{d}t} = -\frac{6p}{p+2} H \rho_{\mathrm{inf}}$$

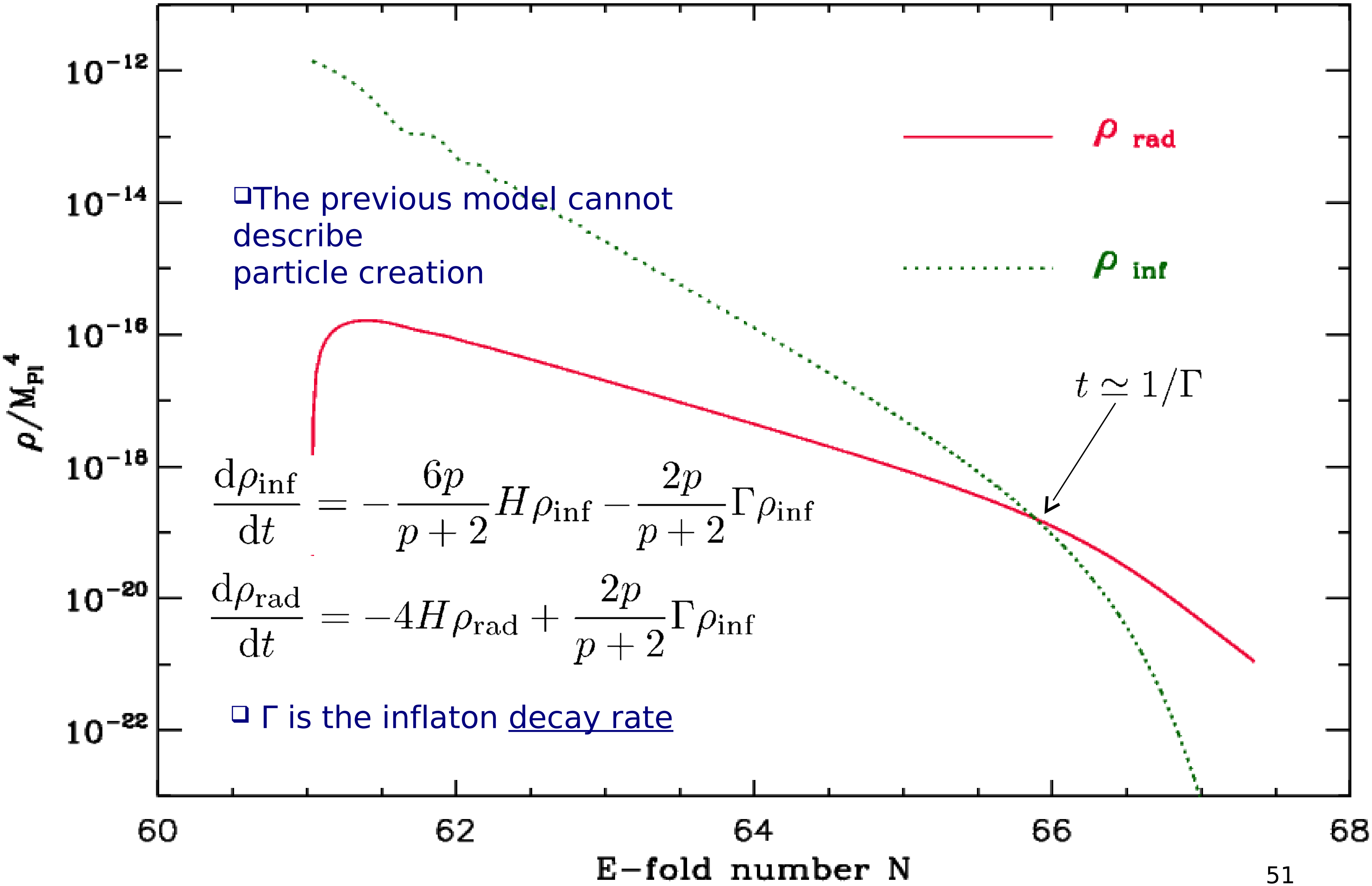
- Equation of state

$$w = \frac{p-2}{p+2} \qquad p > 1 \Longleftrightarrow -\frac{1}{3} < w < 1$$

- For p=2

$$\phi(t) = \phi_{\mathrm{end}} \left(\frac{a_{\mathrm{end}}}{a}\right)^{3/2} \sin\left(mt\right)$$

End of Inflation (III)



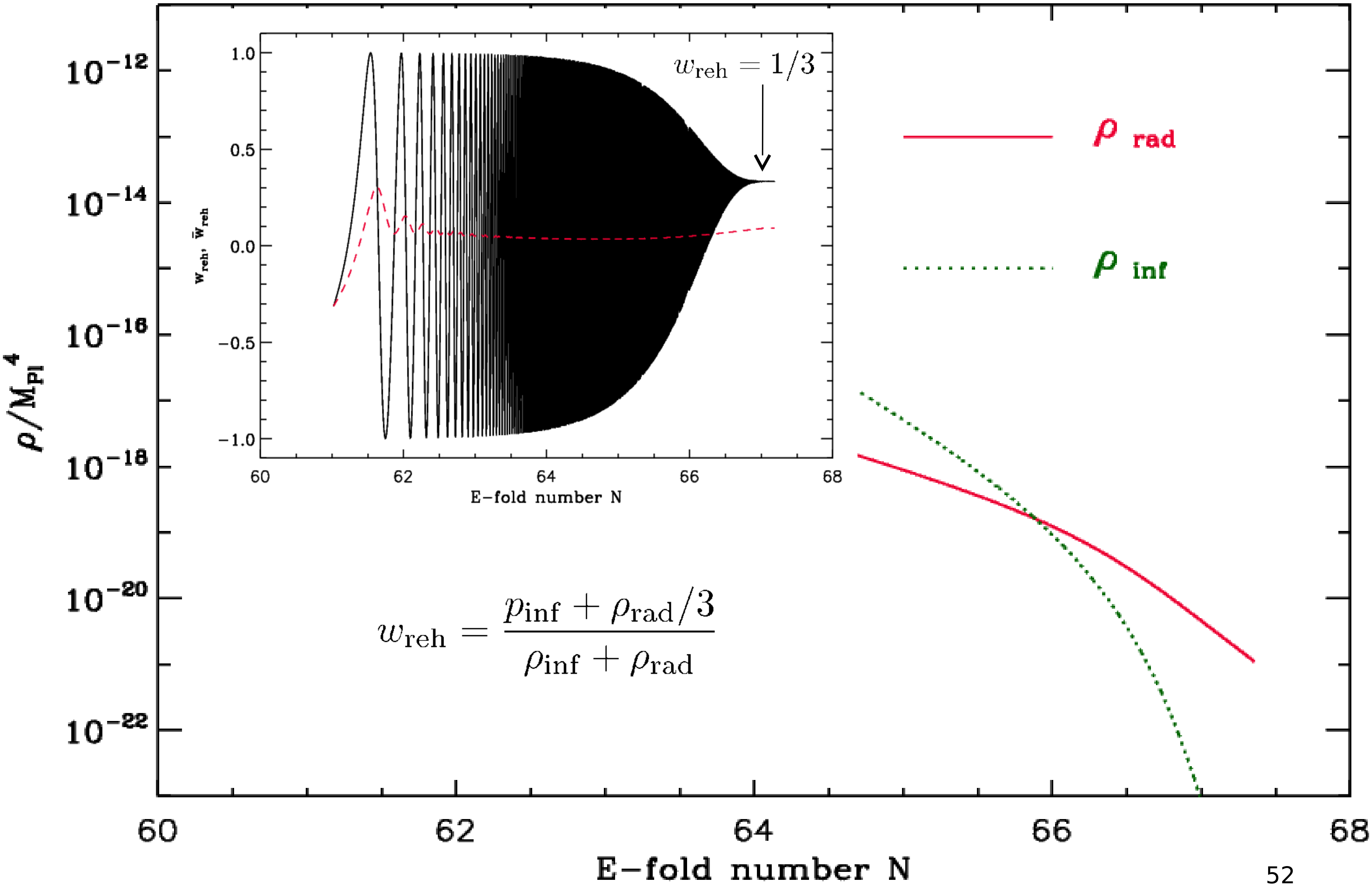
- The previous model cannot describe particle creation

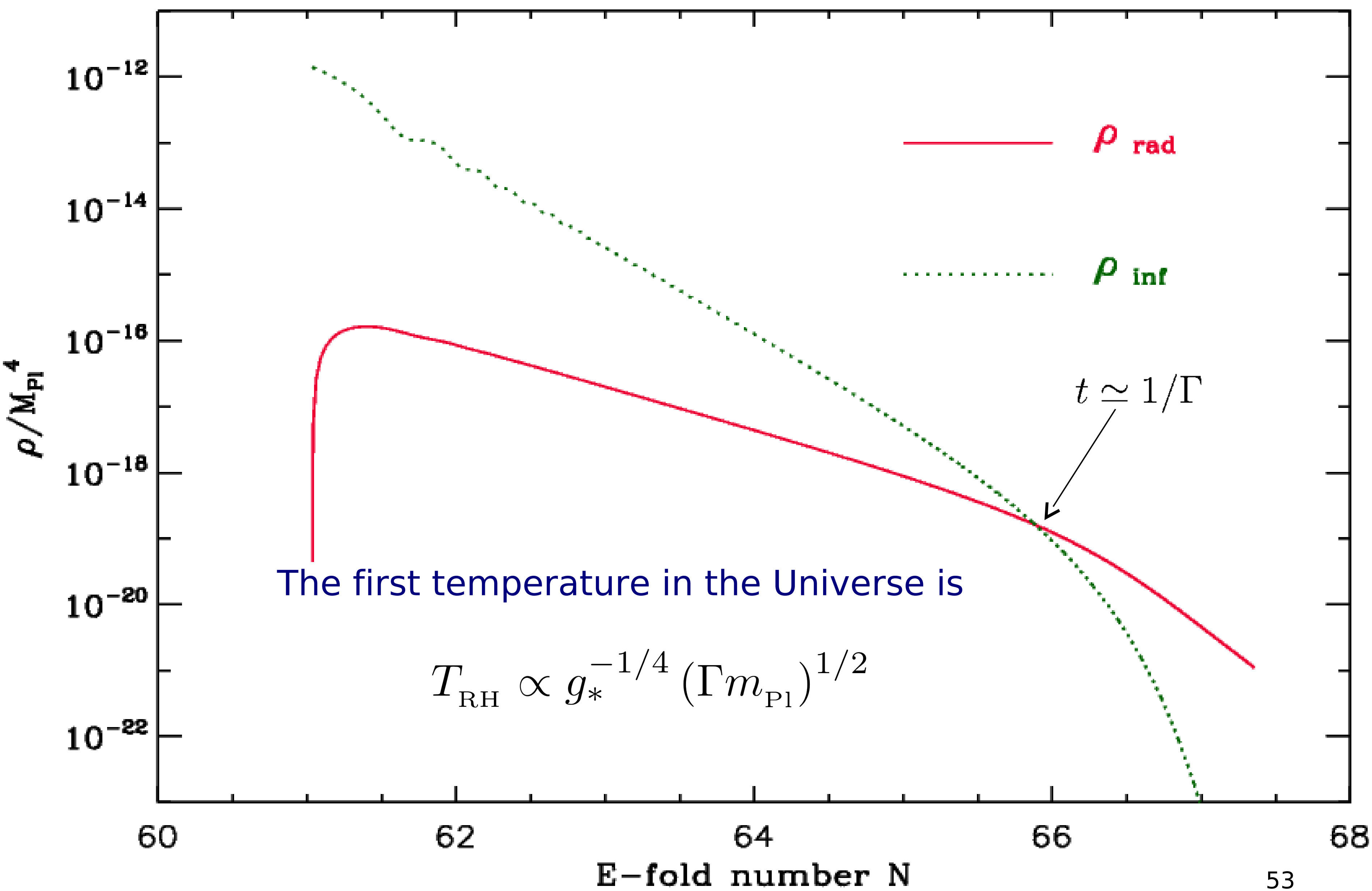
$$\frac{d\rho_{\rm inf}}{dt} = -\frac{6p}{p+2} H\rho_{\rm inf} - \frac{2p}{p+2}\Gamma\rho_{\rm inf}$$

$$\frac{d\rho_{\rm rad}}{dt} = -4H\rho_{\rm rad} + \frac{2p}{p+2}\Gamma\rho_{\rm inf}$$

- Γ is the inflaton decay rate

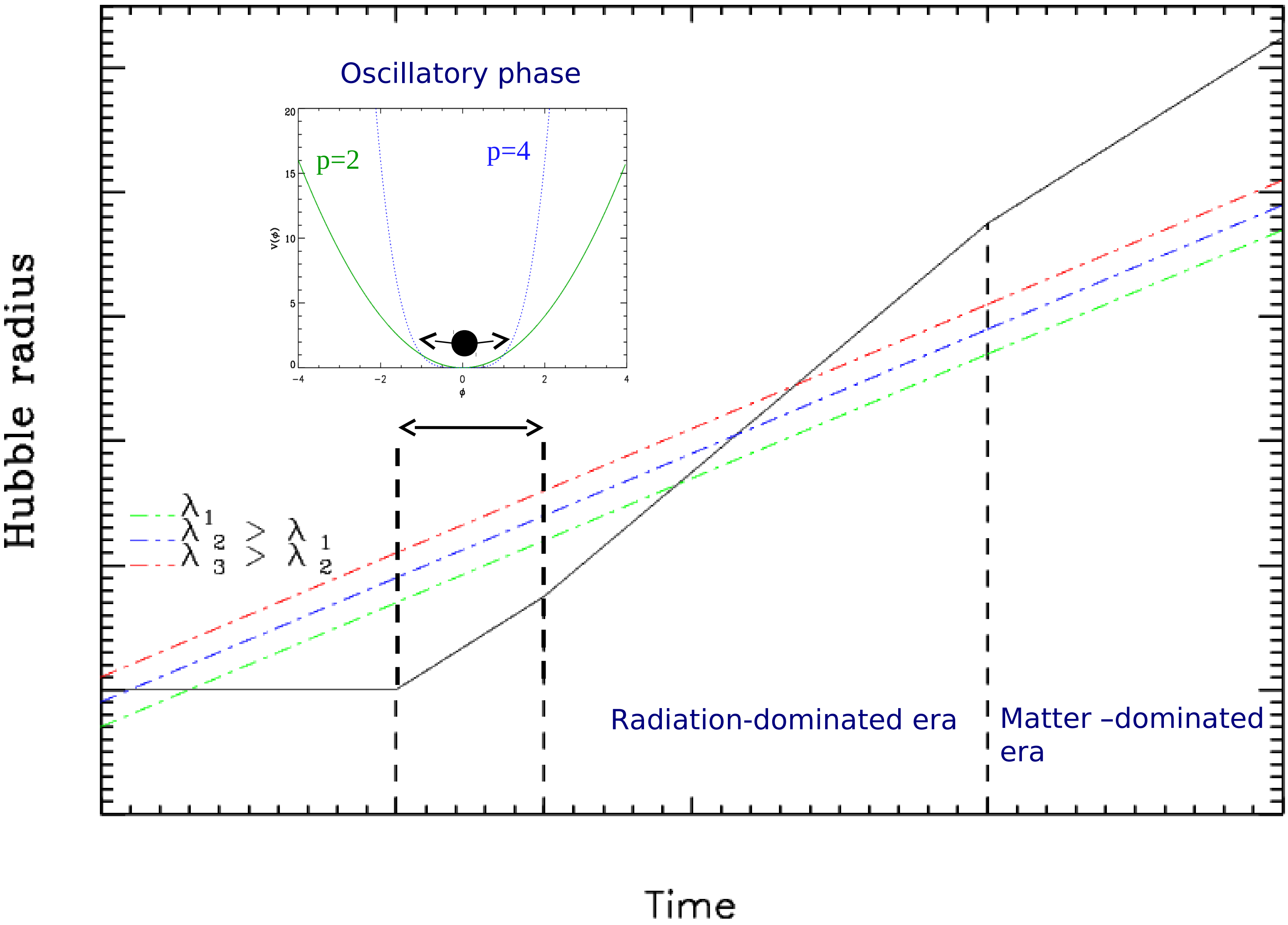
End of Inflation (IV)





The first temperature in the Universe is

$$T_{\rm RH} \propto g_*^{-1/4}\left(\Gamma m_{\rm Pl}\right)^{1/2}$$


Oscillatory phase
p=2
p=4
V(ϕ)
ϕ
λ1
λ2 > λ1
λ3 > λ2
Hubble radius
Radiation-dominated era
Matter –dominated era
Time

Consequences of reheating

- So far we do not know so much on the reheating temperature, ie (can be (improved – the upper bound- if gravitinos production is taken into account)

$$\rho_{end} > \rho_{reh} > \rho_{BBN}$$

- The previous description is a naive description of the infaton/rest of the world coupling. It can be much more complicated.

    - Theory of preheating, thermalization etc …

- How does the reheating affect the inflationary predictions?

    - It modifies the relation between the physical scales now and the number of e-folds at which perturbations left the Hubble radius

    - Can the oscillations of the inflaton affect the behaviour of the perturbations?

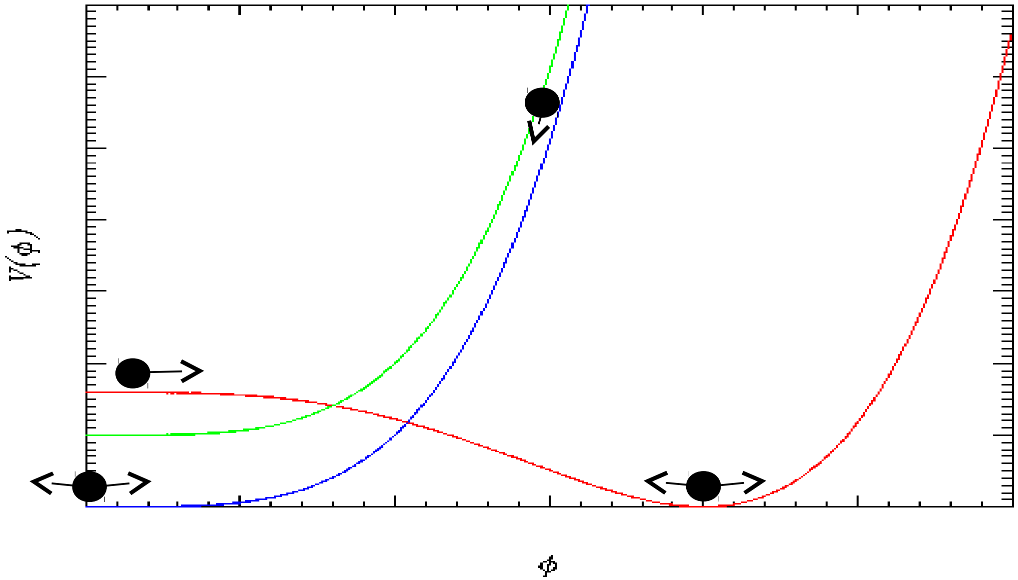
Implementing Inflation

- The common way to realize inflation is to assume that there is a scalar field (or several scalar fields) dominating in the early Universe

- The common way to realize inflation is to assume that there is a scalar field (or several scalar fields) dominating in the early Universe.

- There are plenty of different models

1- Single field inflation with standard kinetic term



Different models are characterized by different potentials

$$\mathcal{L}(\dot{\phi},\phi) = \frac{\dot{\phi}^2}{2} - V(\phi)$$

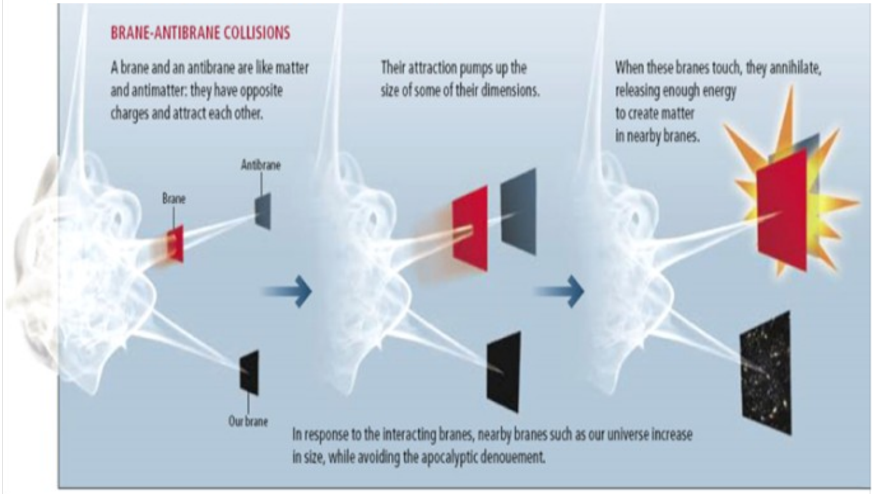
Implementing Inflation (III)

- The common way to realize inflation is to assume that there is a scalar field (or several scalar fields) dominating in the early Universe.

- There are plenty of different models

1- Single field inflation with standard kinetic term

2- Single field with non-standard kinetic term (K-inflation)



Different models are characterized by different potentials and different kinetic terms

$$\mathcal{L}_{\rm DBI}\left(X,\phi\right) = -T\left(\phi\right)\sqrt{1-\frac{2X}{T\left(\phi\right)}}+T\left(\phi\right)-V\left(\phi\right)$$

$$X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

Implementing Inflation (IV)

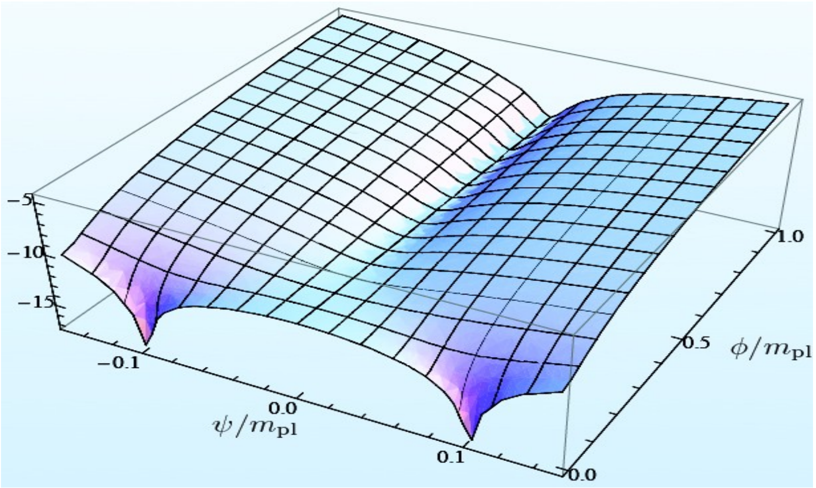
- The common way to realize inflation is to assume that there is a scalar field (or several scalar fields) dominating in the early Universe.

- There are plenty of different models

1- Single field inflation with standard kinetic term

2- Single field with non-standard kinetic term (K-inflation)

3- Multiple field inflation



Different models are characterized by different potentials; the inflationary trajectory can be complicated

$$\mathcal{L}_{\rm STAND}(\dot\phi,\dot\psi,\phi,\psi)=\frac{\dot\phi^2}{2}+\frac{\dot\psi^2}{2}-V(\phi,\psi)$$

Lecture III: inflationary perturbations of quantum-mechanical origin

Puzzles of the standard model

The standard model, despite its impressive achievements, suffers from a number of troubling puzzles

- Horizon problem

- Flatness problem

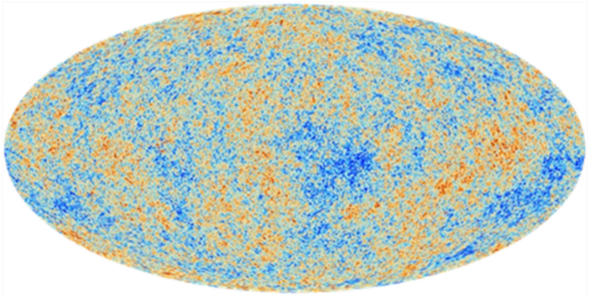
- Origin of the inhomogeneities in our Universe

- etc ...

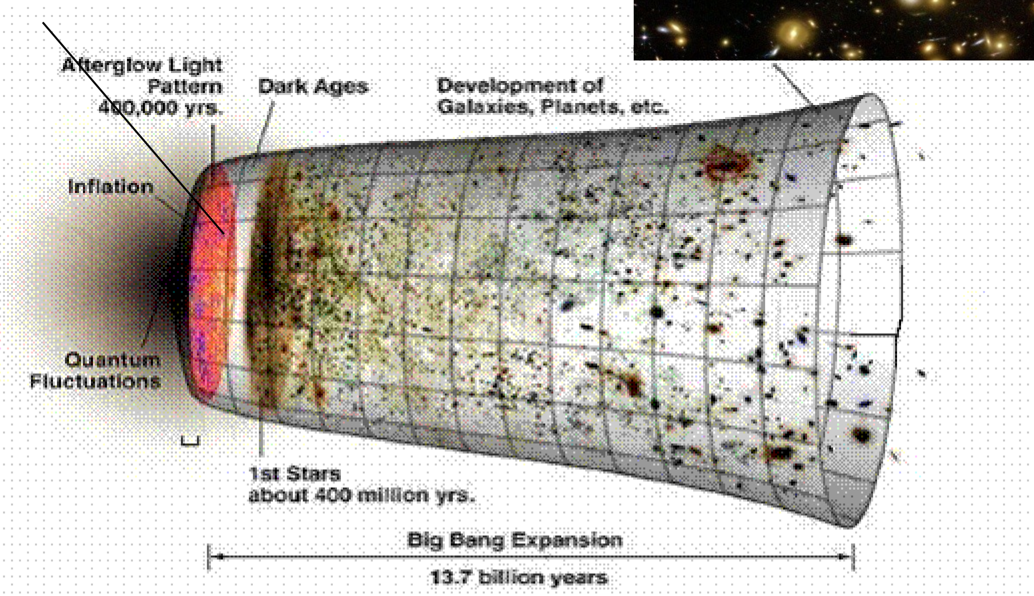
All this issues are related to the initial conditions

The mechanism amplifying the perturbations is gravitational instability

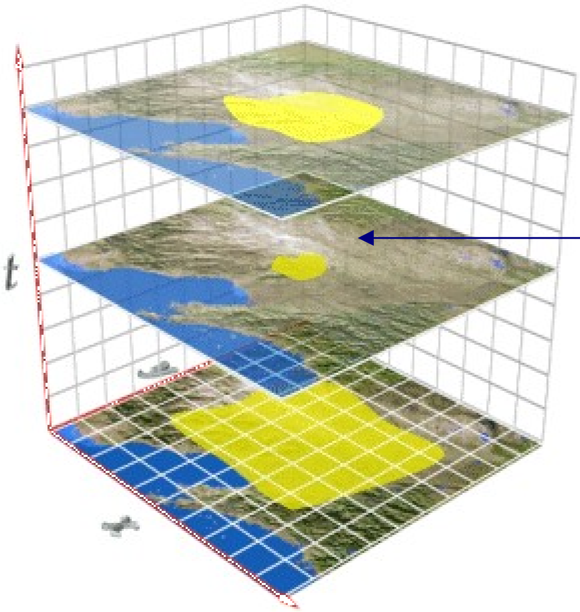
What is the source??

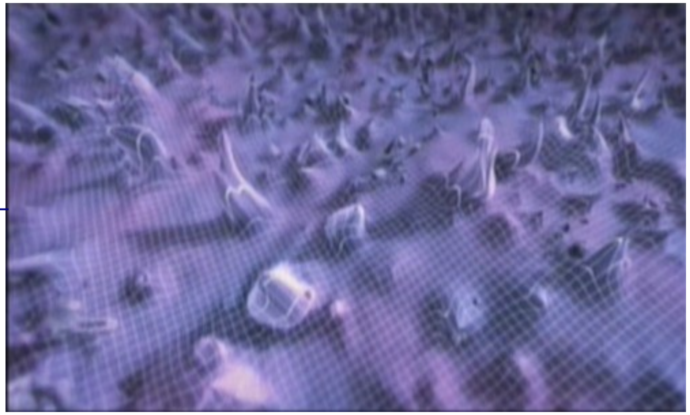


Gravitational collapse



Primordial fluctuations: main ideas





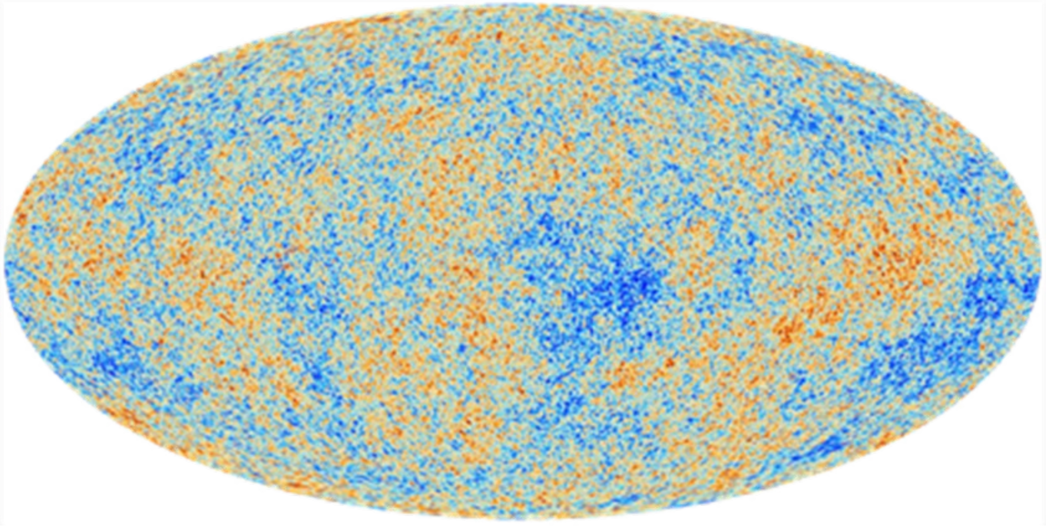
small fluctuations of the geometry and matter on top of the FLRW Universe

- In order to have a more realistic description of the (early) universe (CMB, structure formation …) one <u>must go beyond the cosmological principle.</u>

- In the early universe, the deviations are small since $\delta T/T$» 10^{-5}. This allows us to use a <u>linear theory</u>

- The source of these fluctuations will be the <u>unavoidable</u> quantum fluctuations of the coupled gravitational field and matter.

- This leads to inflation's main success: the production of a <u>scale invariant spectrum</u>.



$$\frac{\delta T}{T}(\vec{e}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\vec{e}) \longrightarrow \left\langle \frac{\delta T}{T}(\vec{e}_1)\frac{\delta T}{T}(\vec{e}_2)\right\rangle = \sum_{\ell=2}^{+\infty} \frac{2\ell+1}{4\pi} C_\ell P_\ell(\cos\theta)$$

with

$$C_\ell = \langle a_{\ell m} a_{\ell m}^* \rangle = \int_0^{+\infty} \frac{\mathrm{d}k}{k} \underbrace{j_\ell^2\left(k r_{\rm lss}\right)}\, \underbrace{T\left(k;\theta_{\rm stand}\right)}\, \underbrace{\mathcal{P}_\zeta\left(k;\theta_{\rm reh},\theta_{\rm inf}\right)}$$

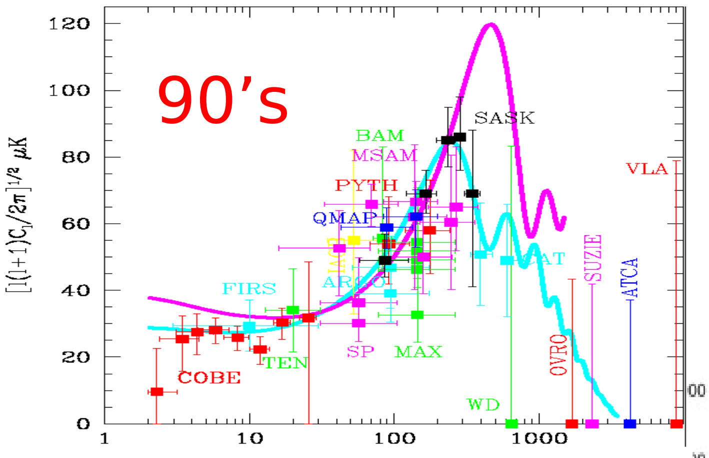
Translate 3d into 2d

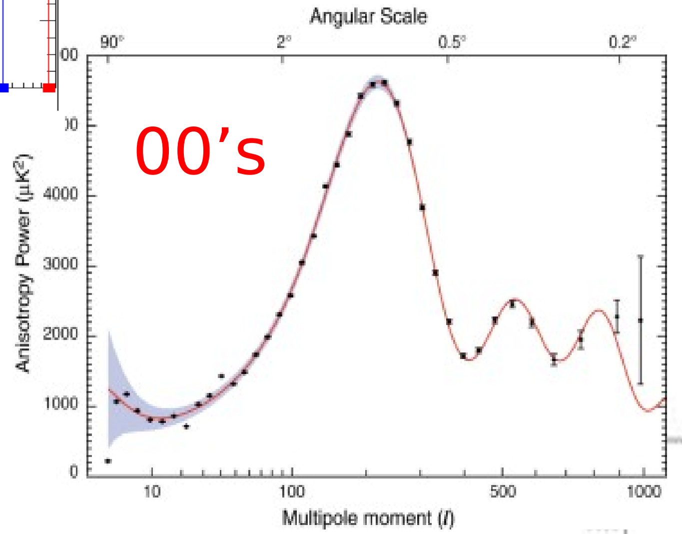
Describes the evolution of the pertubations when they re-enter the Hubble radius

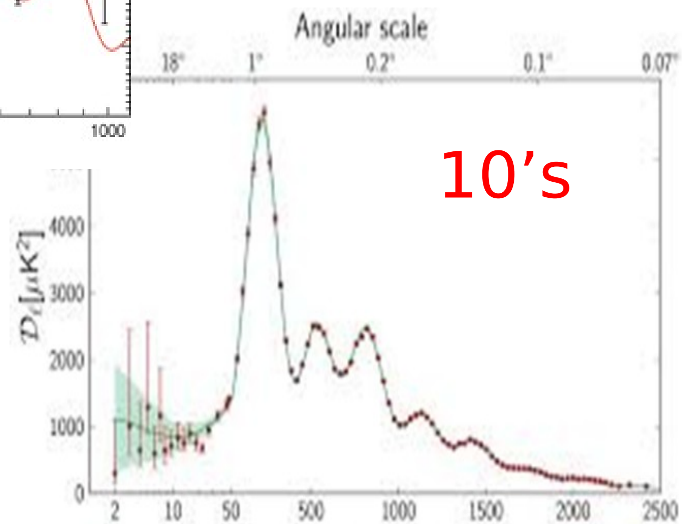
Inflationary power spectrum

$$k^3 \left|\zeta_{\vec{k}}\right|^2$$

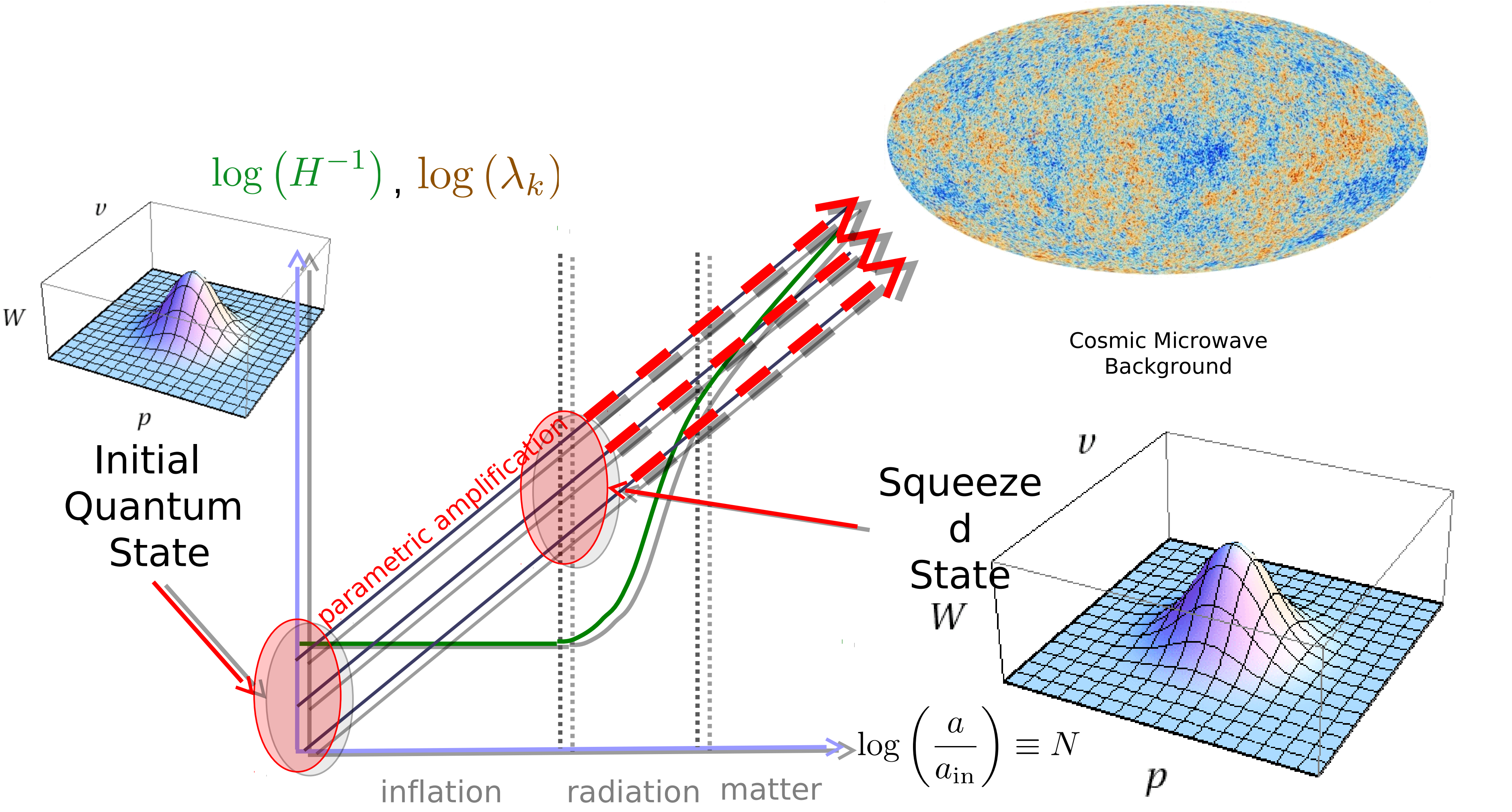
From COBE to Planck ...



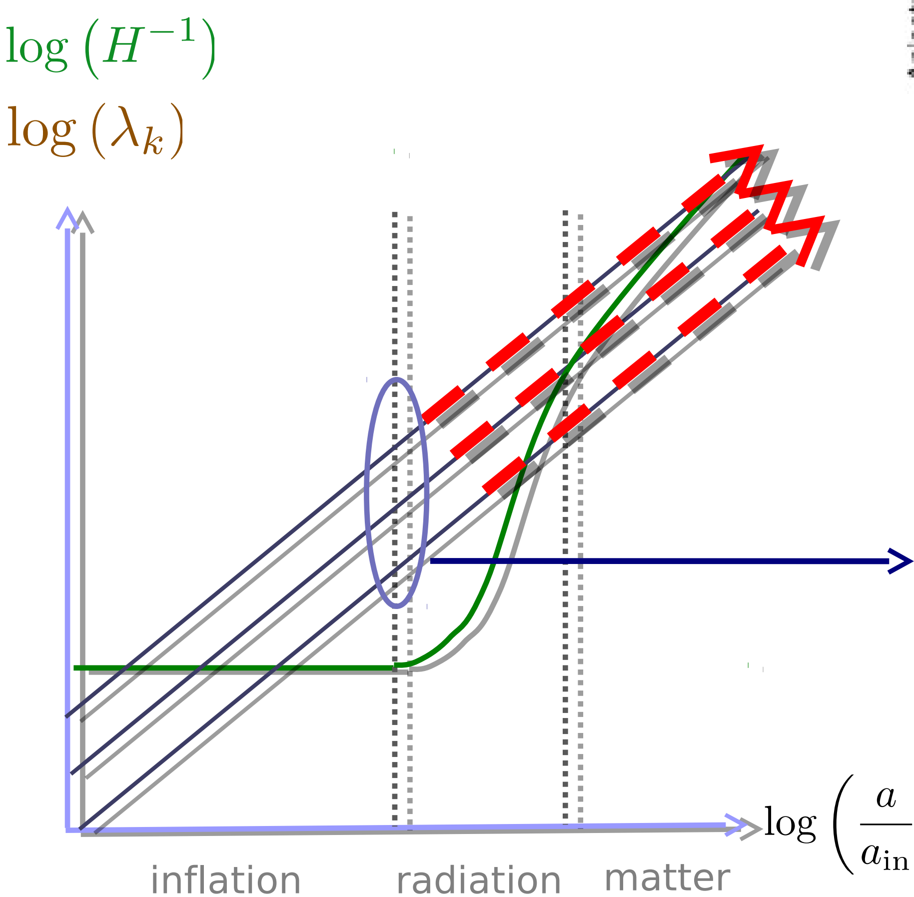




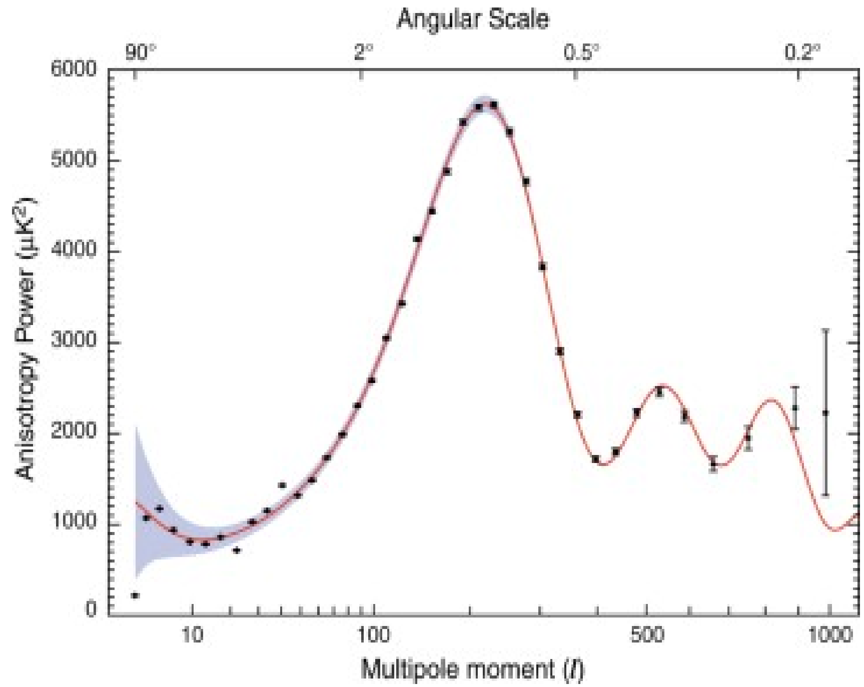
Quantum fluctuations as seeds of CMB anisotropy and large scale structures



Inflationary power spectrum



$\log\left(\frac{a}{a_{\rm in}}\right) \equiv N$



Large scales

Intermediate scales

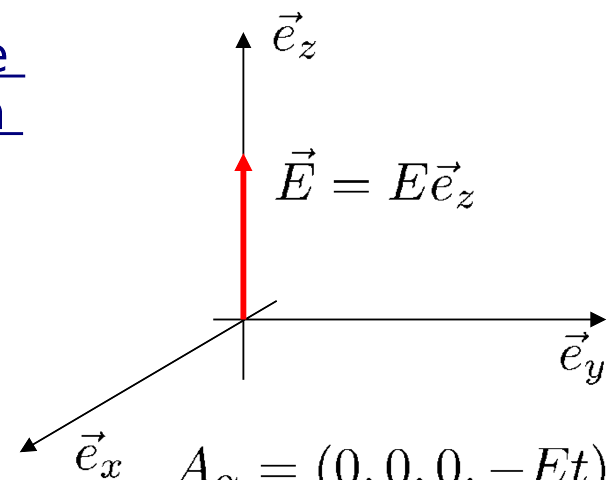
Small scales

$$\mathcal{P}_\zeta\left(k;\theta_{\rm reh},\theta_{\rm inf}\right) \sim k^{n-1}$$

Must be (almost) scale invariant n~1

The Schwinger Effect

Production of cosmological perturbations in the Early universe is very similar to pair creation in a static electric field E



$$S = -\int \mathrm{d}^4x \left(\frac{1}{2}\eta^{\alpha\beta}\mathcal{D}_\alpha\phi\mathcal{D}_\beta\phi^* + \frac{1}{2}m^2\phi\phi^*\right)$$

$$A_\alpha = (0,0,0,-Et)$$

$$\mathcal{D}_\alpha\phi = \partial_\alpha\phi + iq\phi A_\alpha$$

J. Martin, Lect. Notes Phys. 738: 193-241, 2008, arXiv:0704.3540

One works in the Fourier space $\phi(t,\vec{x}) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3\vec{k}\,\phi_{\vec{k}}(t)\,\mathrm{e}^{i\vec{k}\cdot\vec{x}}$

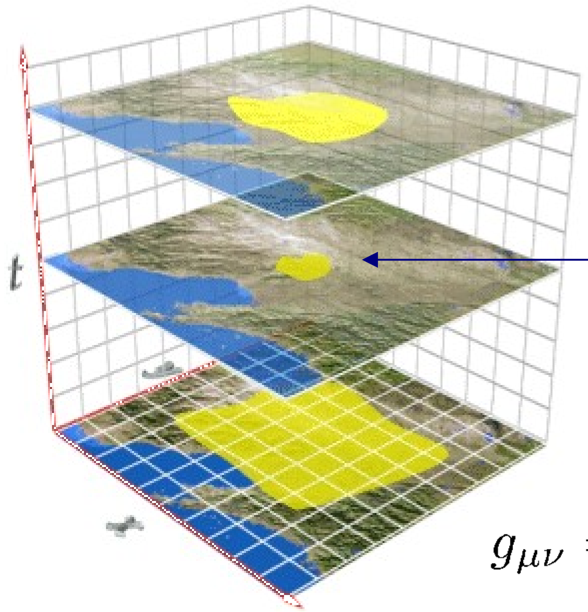
$$\frac{\mathrm{d}^2\phi_{\vec{k}}}{\mathrm{d}\tau^2} + \left(\Upsilon + \tau^2\right)\phi_{\vec{k}} = 0$$

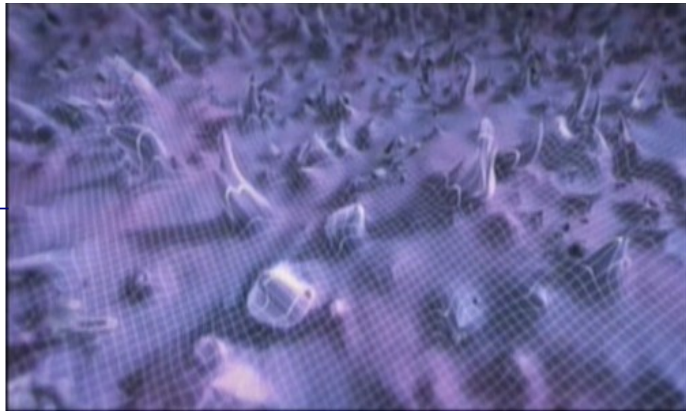
$$\Upsilon \equiv \frac{k_\perp^2 + m^2}{qE}$$

$$k_\perp^2 = k_x^2 + k_y^2$$

$$\tau \equiv \sqrt{qE}t - \frac{k_z}{\sqrt{qE}}$$

The frequency is time-dependent: one has to deal with a parametric oscillator





$$g_{\mu\nu} = g_{\mu\nu}^{\mathrm{FLRW}} + \delta g_{\mu\nu}(t, \vec{x})$$

The inflationary mechanism is a conservative one: similar to the Schwinger effect in QFT

Schwinger effect

- **Scalar field**
- **Classical electric field**
- **Amplitude of the effect controlled by E**

Inflationary cosmological perturbations

- **Perturbed metric**
- **Background gravitational field: scale factor**
- **Amplitude controlled by the Hubble parameter H**

What are the equations of motion?

Perturbed Einstein equations $\delta G_{\mu\nu} = \kappa \delta T_{\mu\nu}$

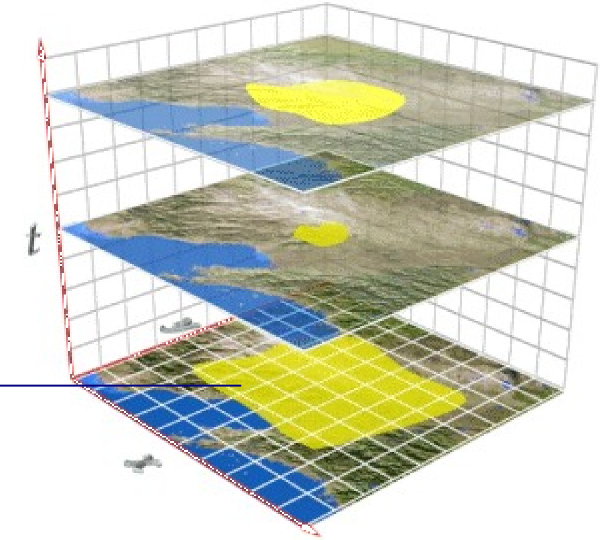
$$\delta\Gamma = \delta g(\partial g + \cdots) + g(\partial \delta g + \cdots)$$

$$\delta R = \partial\delta g - \partial\delta g + \delta\Gamma\Gamma + \Gamma\delta\Gamma + \cdots$$

$$\delta G = \delta R - \frac{1}{2}\delta R g - \frac{1}{2} R \delta g$$

$$\phi + \delta\phi$$

$$\delta T = \dot{\phi}\delta\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi}\delta\phi + \cdots$$



On top of the classical FLRW background, we have small quantum perturbations of the matter fields, and through Einstein equations, of the metric tensor. We have "gravitational phonons".

$$ds^2 = a^2(\eta)\left\{-(1-2\phi)\,\mathrm{d}\eta^2 + 2(\partial_i B)\,\mathrm{d}x^i\mathrm{d}\eta + [(1-2\psi)\,\delta_{ij} + 2\partial_i\partial_j E + h_{ij}]\,\mathrm{d}x^i\mathrm{d}x^j\right\}$$

scalar perturbations

$$\Phi_{\rm B} = \phi + \frac{1}{a}\left[a\left(B - E'\right)\right]'$$ **Bardeen potential**

$$v = a\left[\delta\phi^{(\rm gi)} + \phi'\frac{\Phi_{\rm B}}{\mathcal{H}}\right]$$ **Mukhanov-Sasaki variable**

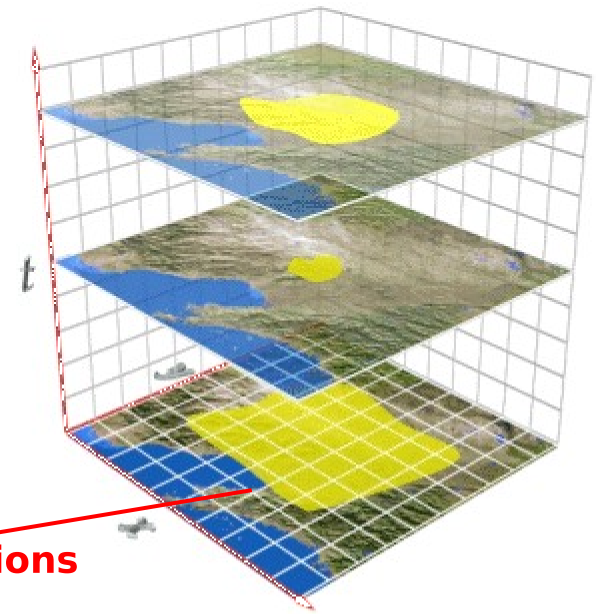
$$\zeta = -\frac{\mu}{2a\sqrt{\epsilon_1}}$$ **Curvature perturbations**

tensor perturbations or gravitational wave

$$\delta^{ij}h_{ij} = 0$$

$$\partial^i h_{ij} = 0$$

One uses the fact that the spacelike sections are flat and we Fourier transform the previous quantities



Scalar perturbations

$$v(\eta, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}\vec{k}\, \mu_{\mathrm{S}\,\vec{k}}(\eta) \mathrm{e}^{i\vec{k}\cdot\vec{x}}$$

Gravitational waves

$$h_{ij}(\eta, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \frac{1}{a(\eta)} \int \mathrm{d}\vec{k} \sum_{s=+,\times} p_{ij}^{s}\left(\vec{k}\right) \mu_{\mathrm{T}\,\vec{k}}^{s}(\eta) \mathrm{e}^{i\vec{k}\cdot\vec{x}}$$

$$\frac{\mathrm{d}^2 \mu_{\vec{k}}}{\mathrm{d}\eta^2} + \omega^2(k, \eta)\, \mu_{\vec{k}} = 0$$

Key result: The amplitudes of scalar and tensor pertubations in GR obey the equation of a parametric oscillator

- The effective frequency is controlled by the scale factor, ie by the background gravitational field

It is different for scalar and tensor. Hence one can expect a different result for S and T

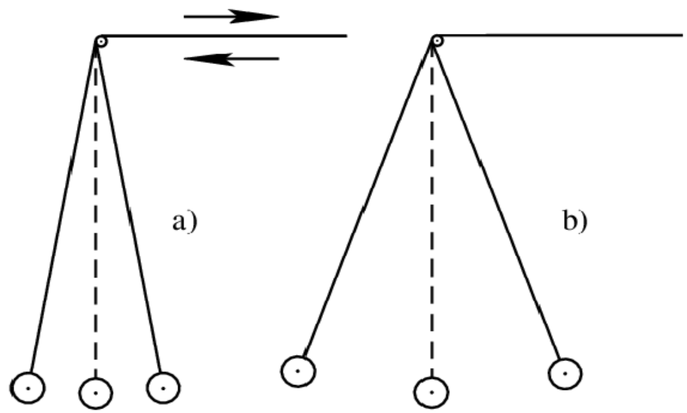


Fig. 1. Parametric amplification. a) variation of the length of the pendulum, b) increased amplitude of oscillations.

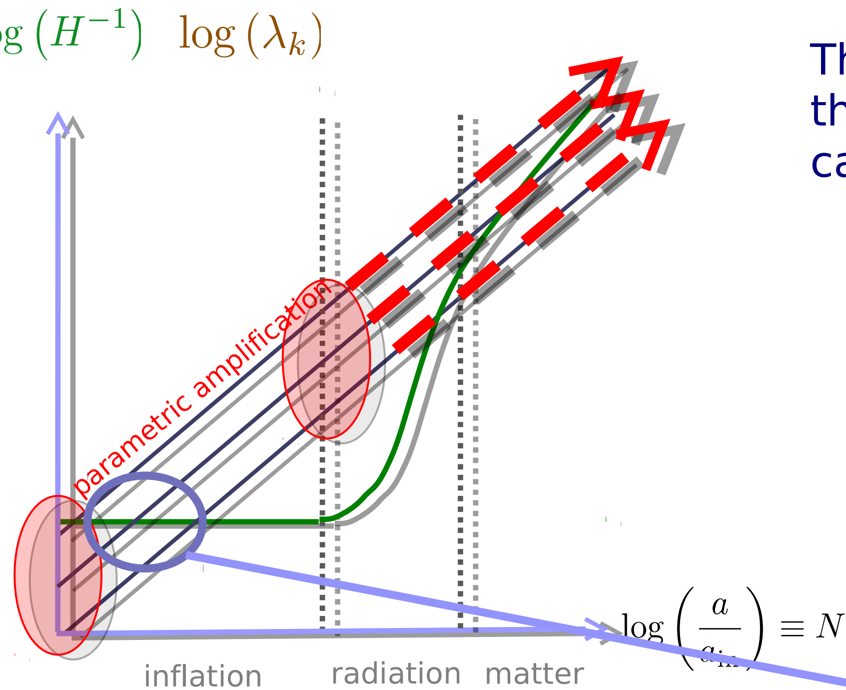
$$\frac{\mathrm{d}^2\mu_{\vec{k}}}{\mathrm{d}\eta^2} + \omega^2(k,\eta)\,\mu_{\vec{k}} = 0$$

Gravitational waves

$$\omega_{\mathrm{T}}^2(k,\eta) = k^2 - \frac{a''}{a}$$

scalar perturbations

$$\omega_{\mathrm{S}}^2(k,\eta) = k^2 - \frac{\left(a\sqrt{\epsilon_1}\right)''}{a\sqrt{\epsilon_1}}$$



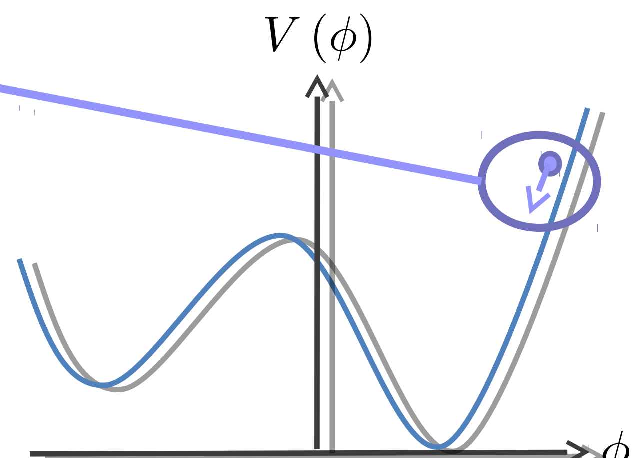
The slow-roll parameters are the "<u>small parameter</u>" of a perturbative calculation of the power spectrum

$$\epsilon_0 \propto H^{-1} \simeq \text{constant}$$

$$\epsilon_{n+1} = \frac{\mathrm{d}\ln\left|\epsilon_n\right|}{\mathrm{d}N}\,, \qquad n \geq 0$$

$$\epsilon_1 = \frac{M_{\rm Pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2$$

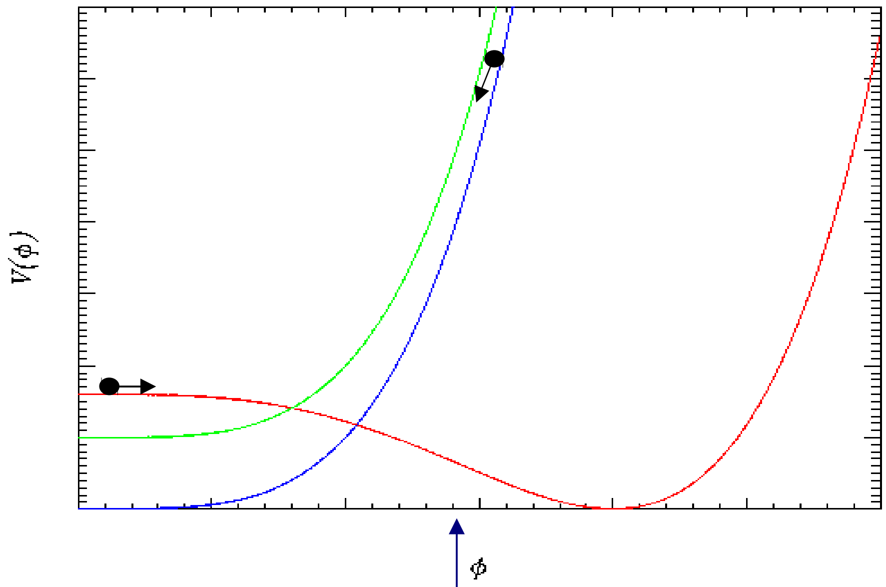
$$\epsilon_2 = 2M_{\rm Pl}^2\left[\left(\frac{V_\phi}{V}\right)^2 - \frac{V_{\phi\phi}}{V}\right]$$



What is the typical time-dependence of the frequency?

- The time-dependence if different from that of the Schwinger effect

- It encodes the microphysics of inflation



$$\omega_{\mathrm{S}}^2(k,\eta) = k^2 - \frac{2+3\epsilon_1+3\epsilon_2/2}{\eta^2}$$

$$\omega_{\mathrm{T}}^2(k,\eta) = k^2 - \frac{2+3\epsilon_1}{\eta^2}$$

The effective frequency depends on the detailed shape of the inflaton potential

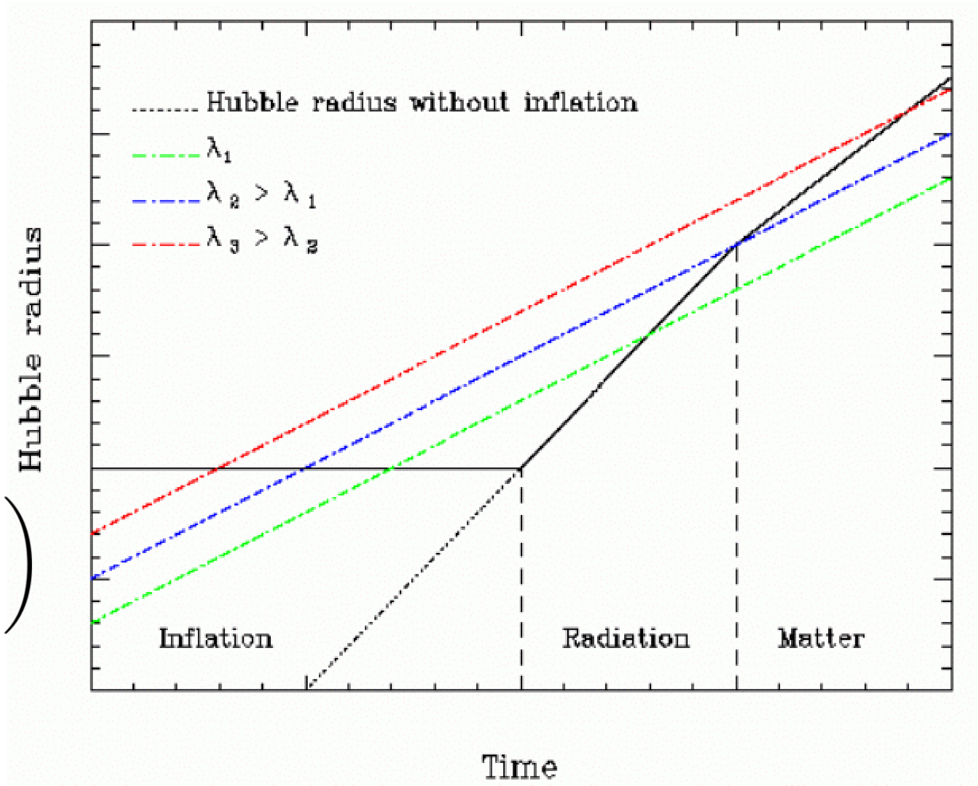
$$\frac{\omega_{\rm S}^2(k,\eta)}{a^2H^2}=\frac{k^2}{a^2H^2}-\left(2-\epsilon_1+\frac{3}{2}\epsilon_2\right)=\left(2\pi\frac{\ell_{\rm H}}{\lambda}\right)^2-\left(2-\epsilon_1+\frac{3}{2}\epsilon_2\right)$$

Small scales

$$\omega_{\rm S}^2(k,\eta)\sim k^2$$

Large scales

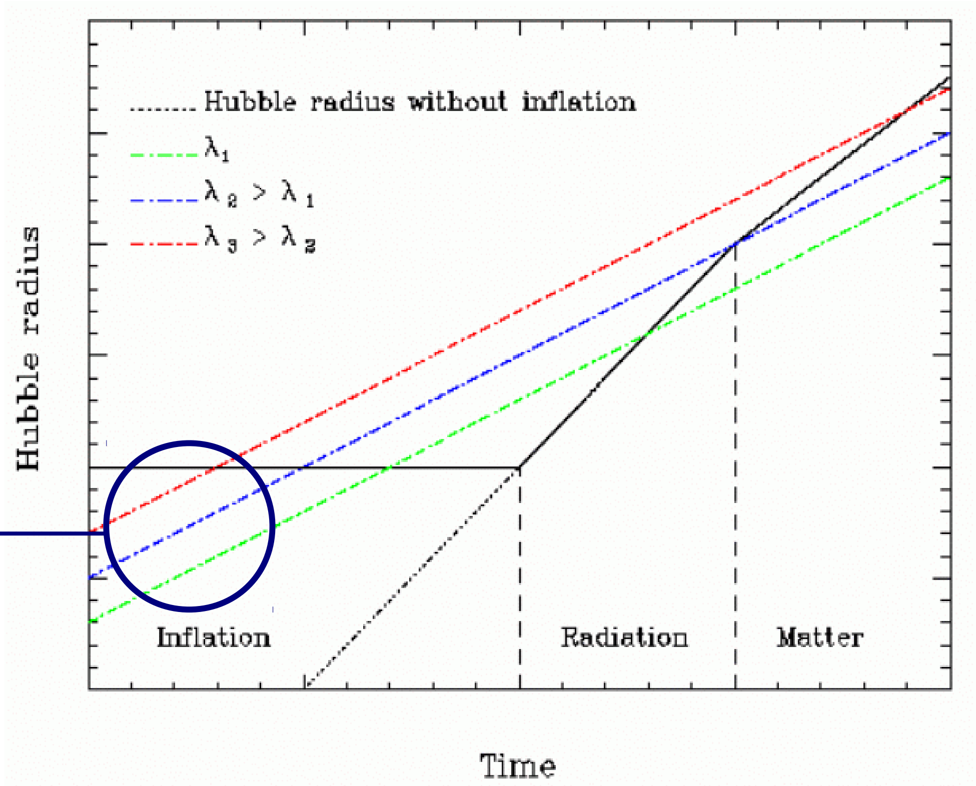
$$\omega_{\rm S}^2(k,\eta)\sim\frac{1}{\eta^2}\left(2+3\epsilon_1+\frac{3}{2}\epsilon_2\right)$$



Which initial conditions?

$$\mu''_{\vec{k}} + k^2 \mu_{\vec{k}} = 0$$

$$\mu_{\vec{k}}(\eta) \simeq \frac{\alpha_{\vec{k}}}{\sqrt{2k}} e^{-ik\eta} + \frac{\beta_{\vec{k}}}{\sqrt{2k}} e^{ik\eta}$$



Key assumption: one choose the adiabatic vacuum initially

$$\alpha_{\vec{k}} = 1\,, \quad \beta_{\vec{k}} = 0$$

$$\mu_{\vec{k}}(\eta) \simeq \frac{1}{\sqrt{2k}} e^{-ik\eta}$$

$$k^3 P_\zeta = \frac{H^2}{\pi \epsilon_1 m_{\rm Pl}^2}\left[1 - 2(C+1)\epsilon_1 - C\epsilon_2 - (2\epsilon_1 + \epsilon_2)\ln\frac{k}{k_{\rm P}}\right]$$

$$k^3 P_h = \frac{16 H^2}{\pi m_{\rm Pl}^2}\left[1 - 2(C+1)\epsilon_1 - 2\epsilon_1 \ln\frac{k}{k_{\rm P}}\right]$$

- The amplitude is controlled by H

- For the scalar modes, the amplitude also depends on ε $_1$

The power spectra are scale-invariant plus logarithmic corrections the amplitude of which depend on the sr parameters, ie on the microphysics of inflation

The ratio of dp to gw amplitudes is given by

$$r = \frac{\mathcal{P}_h}{\mathcal{P}_\zeta} = 16\epsilon_1$$

Gravitational waves are subdominant

The spectral indices are given by

$$n_{\rm S} - 1 \equiv \frac{{\rm d}\ln \mathcal{P}_\zeta}{{\rm d}\ln k}\,, \quad n_{\rm T} \equiv \frac{{\rm d}\ln \mathcal{P}_h}{{\rm d}\ln k}$$

$$n_{\rm S} - 1 = -2\epsilon_1 - \epsilon_2\,, \quad n_{\rm T} = -2\epsilon_1$$

The running, i.e. the scale dependence of the spectral indices, of dp and gw are

$$\alpha_{\rm S} \equiv \frac{{\rm d}^2\ln \mathcal{P}_\zeta}{{\rm d}(\ln k)^2} \qquad \alpha_{\rm T} \equiv \frac{{\rm d}^2\ln \mathcal{P}_h}{{\rm d}(\ln k)^2} \qquad \alpha_{\rm S} = \mathcal{O}\left(\epsilon^2, \cdots\right) \quad \alpha_{\rm T} = \mathcal{O}\left(\epsilon^2, \cdots\right)$$

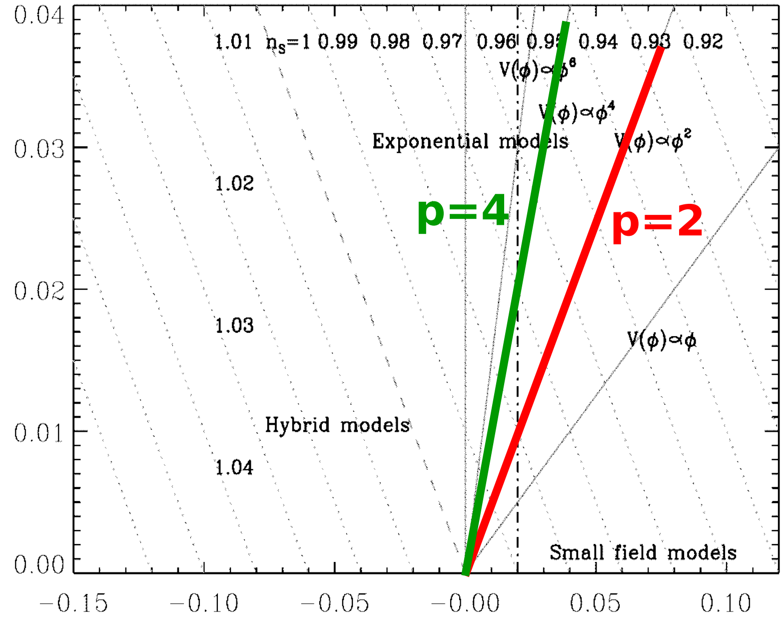
Slow-roll predictions: the example of LF

$$V(\phi) = M^4 \left(\frac{\phi}{m_{\rm Pl}}\right)^p$$

$$\epsilon_1 = \frac{p^2}{2}\frac{M_{\rm Pl}^2}{\phi^2}$$

$$\epsilon_2 = 2p\frac{M_{\rm Pl}^2}{\phi^2}$$

$$\epsilon_1 = \frac{p}{4}\epsilon_2$$



$$\left\{\begin{aligned} \epsilon_1 &= \frac{M_{\rm Pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2 \\ \epsilon_2 &= 2M_{\rm Pl}^2\left[\left(\frac{V_\phi}{V}\right)^2 - \frac{V_{\phi\phi}}{V}\right] \end{aligned}\right.$$

So we get a trajectory in the sr parameters plane

Slow-roll predictions: the example of LFI2

The slow-roll parameters should be evaluated at (pivot scale) Hubble radius crossing

$$\Delta N_* = -\frac{1}{M_{\rm Pl}^2}\int_{\phi_*}^{\phi_{\rm end}} \frac{V}{V_\phi}\,\mathrm{d}\phi$$

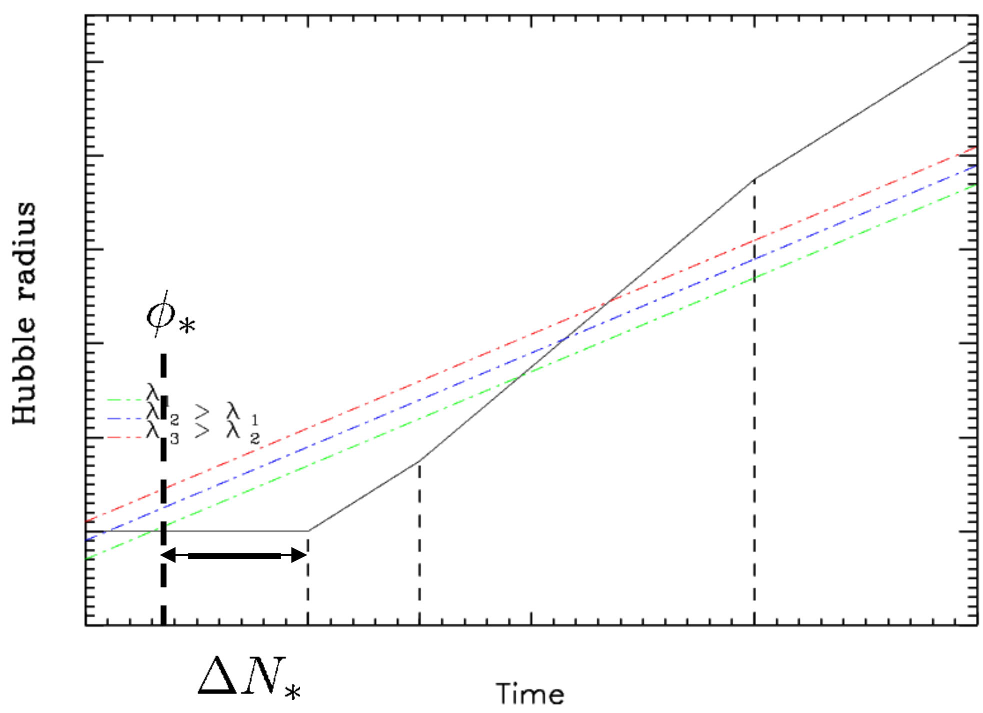
↓

$$\frac{\phi_*^2}{M_{\rm Pl}^2} = 4\left(\Delta N_* + \frac{1}{2}\right)$$

↓

$$\epsilon_1 = \frac{1}{2(\Delta N_* + 1/2)}$$

$$\epsilon_2 = \frac{1}{(\Delta N_* + 1/2)}$$



Slow-roll predictions: the example of LFI2

The slow-roll parameters should be evaluated at (pivot scale) Hubble radius crossing

$$\epsilon_1 = \frac{1}{2(\Delta N_* + 1/2)} \qquad \epsilon_2 = \frac{1}{(\Delta N_* + 1/2)}$$

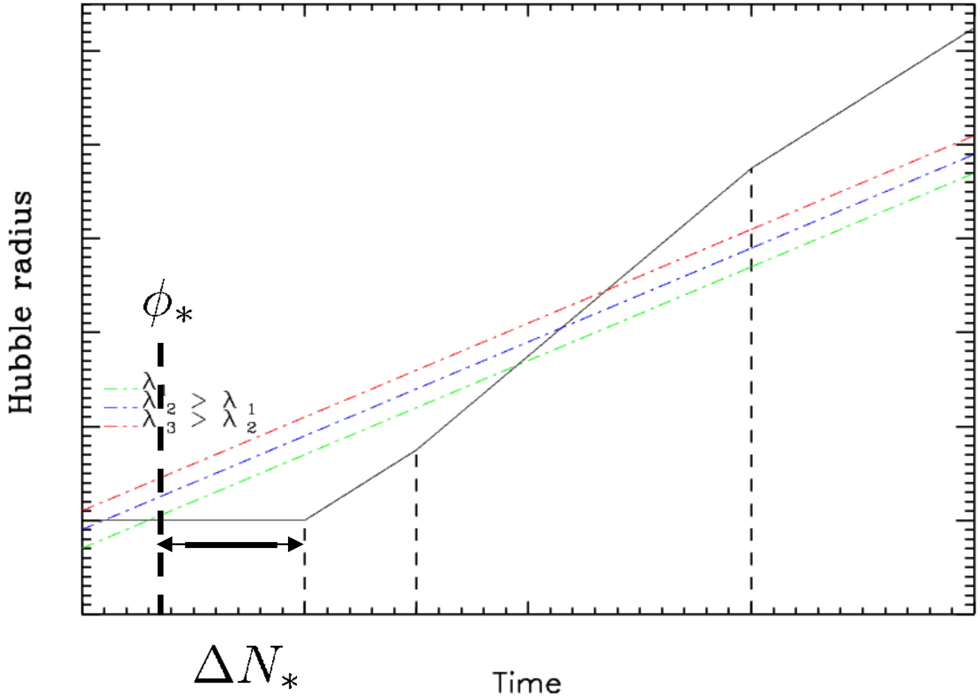
1- Changing the expansion during reheating, ie a_reh/a_end will change ΔN_*

2- But the possible variation of a_reh/a_end are limited because

$\rho_{end} > \rho_{reh} > \rho_{BBN}$

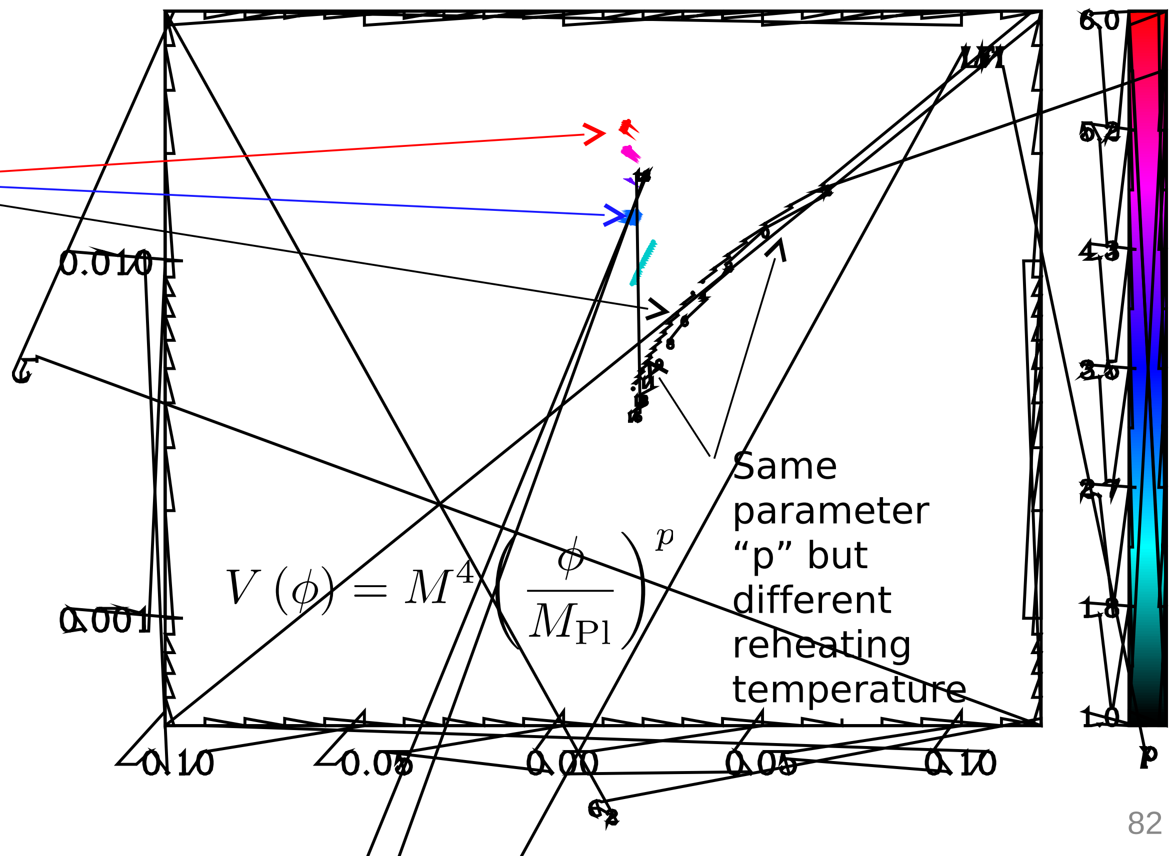
3- Hence the variations of ΔN_* are also limited, typically $40 < \Delta N_* < 70$

4- A given value of ΔN_* corresponds to a fixed value of the reheating temperature and of the equation of state during reheating



The inflationary predictions can be represented in the slow-roll plane

Different values of the parameter "p"



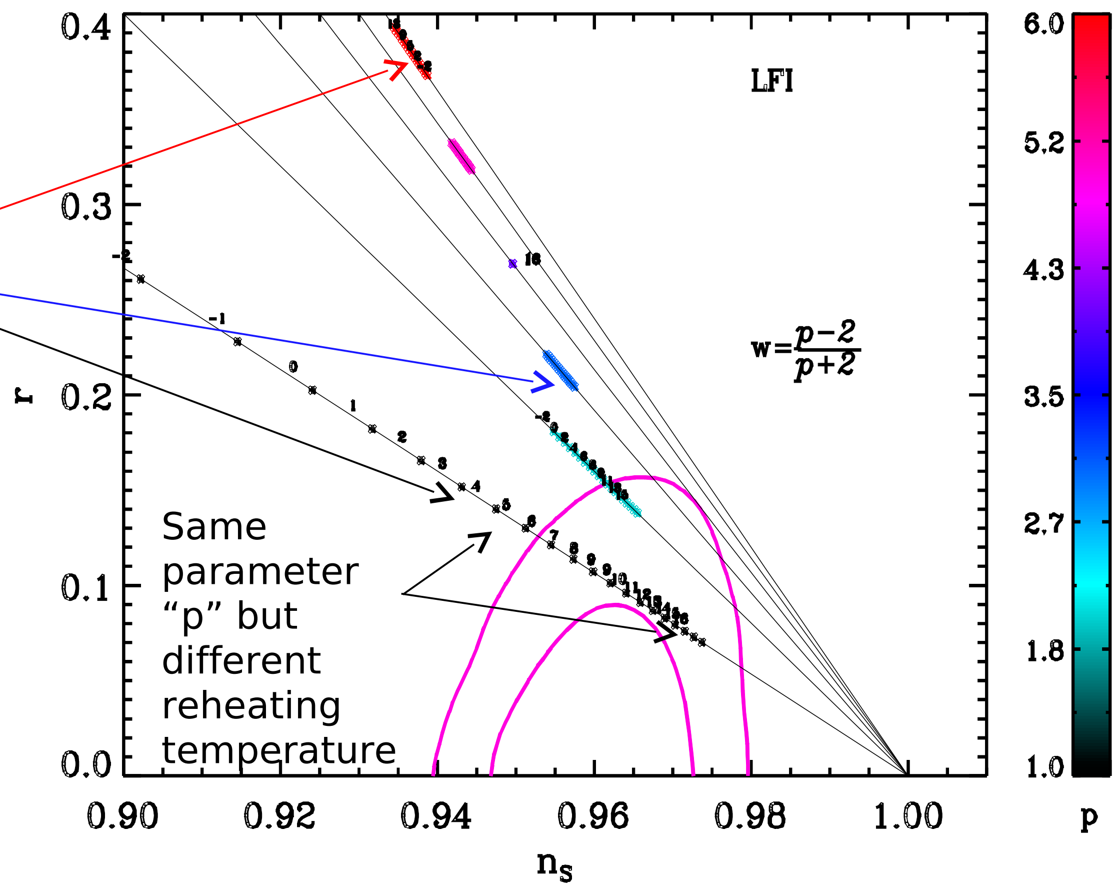
Inflationary predictions: the observable space

- Instead of working in the slow-roll plane, one can also work in the observable plane

$$r = 16\epsilon_1$$

$$n_S - 1 = -2\epsilon_1 - \epsilon_2$$

Different values of the parameter "p"



The amplitude of the fluctuations determine the mass scale of the potential

$$C_\ell = \langle a_{\ell m} a^*_{\ell m} \rangle = \int_0^{+\infty} \frac{\mathrm{d}k}{k}\, j_\ell^2\left(k r_{\mathrm{lss}}\right) T\left(k; \theta_{\mathrm{stand}}\right) \mathcal{P}_\zeta\left(k; \theta_{\mathrm{reh}}, \theta_{\mathrm{inf}}\right)$$

~1 on large scales

$$\mathcal{P}_\zeta \simeq \frac{H^2}{\pi \epsilon_1 m_{\mathrm{Pl}}^2}$$

$$C_\ell \simeq \frac{2H^2}{25\epsilon_1 m_{\mathrm{Pl}}^2} \frac{1}{\ell(\ell+1)}$$

$$C_2 = \mathcal{O}(1)\left(\Delta N_* + \frac{1}{2}\right)^2 \left(\frac{m}{m_{\mathrm{Pl}}}\right)^2 \longrightarrow m \simeq 10^{-5} m_{\mathrm{Pl}}$$

NB: $C_2 = \frac{4\pi}{5}\frac{Q^2}{T^2}$ with $Q = 18 \times 10^{-6}\, K$, $T = 2.7\, K$

Lecture IV: inflation after Planck

Planck results in brief:

$$100\,\Omega_{\mathcal{K}} = -0.05^{+0.65}_{-0.66}$$

$$\alpha^{(2,2500)}_{\mathcal{R}CDI} \in [-0.093, 0.014]$$

$$n_{\rm S} = 0.9603 \pm 0.0073$$

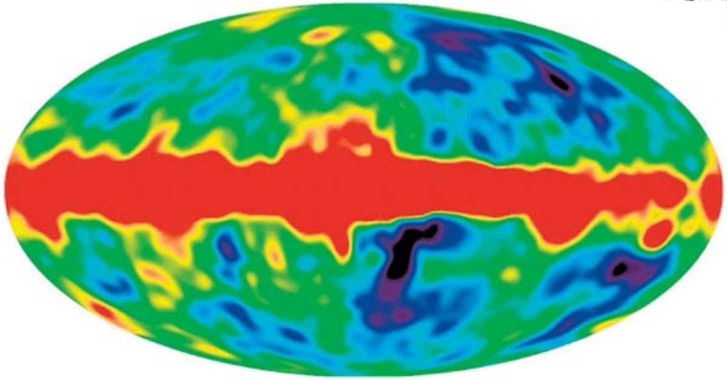
$$\frac{\mathrm{d}n_{\rm S}}{\mathrm{d}\ln k} = -0.0134 \pm 0.009$$

$$f_{\rm NL}^{\rm loc} = 2.7 \pm 5.8$$

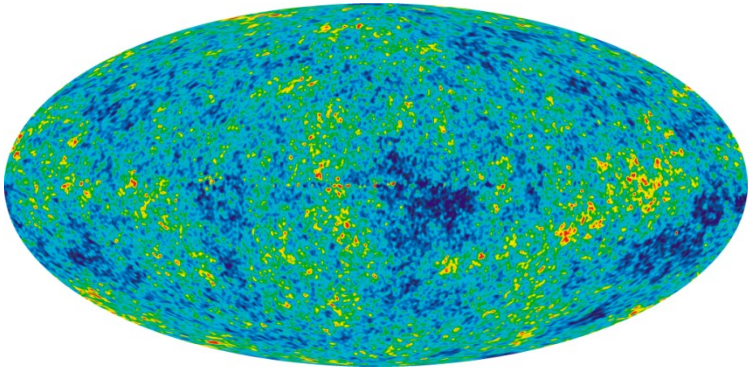
$$f_{\rm NL}^{\rm eq} = -42 \pm 75$$

$$f_{\rm NL}^{\rm ortho} = -25 \pm 39$$

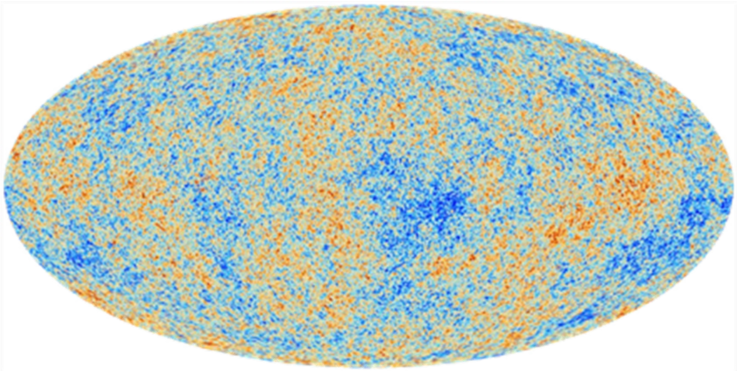
Flat universe with adiabatic, Gaussian and almost scale invariant fluctuations



COBE (1992)



WMAP (2003)

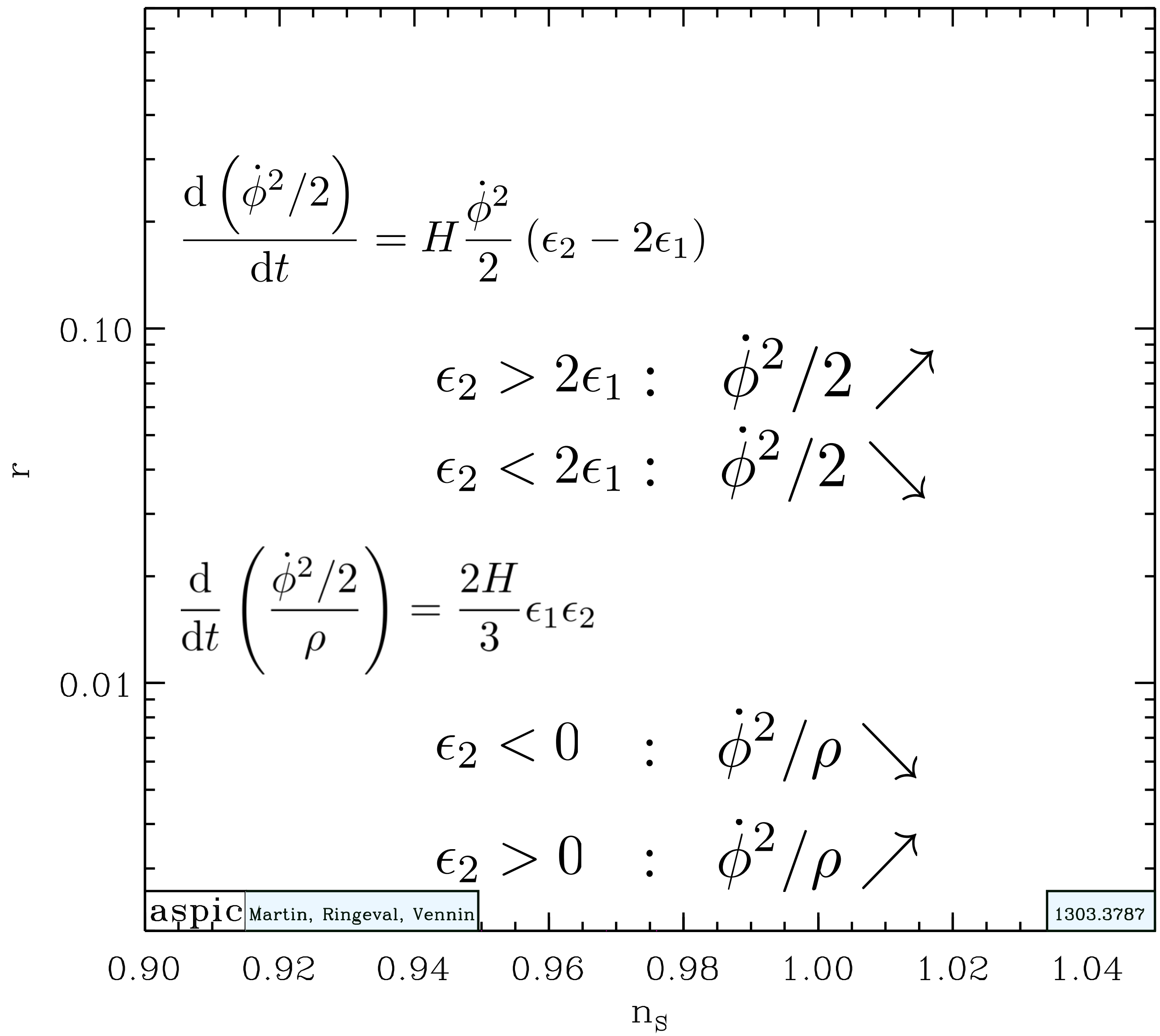


Planck (2013)

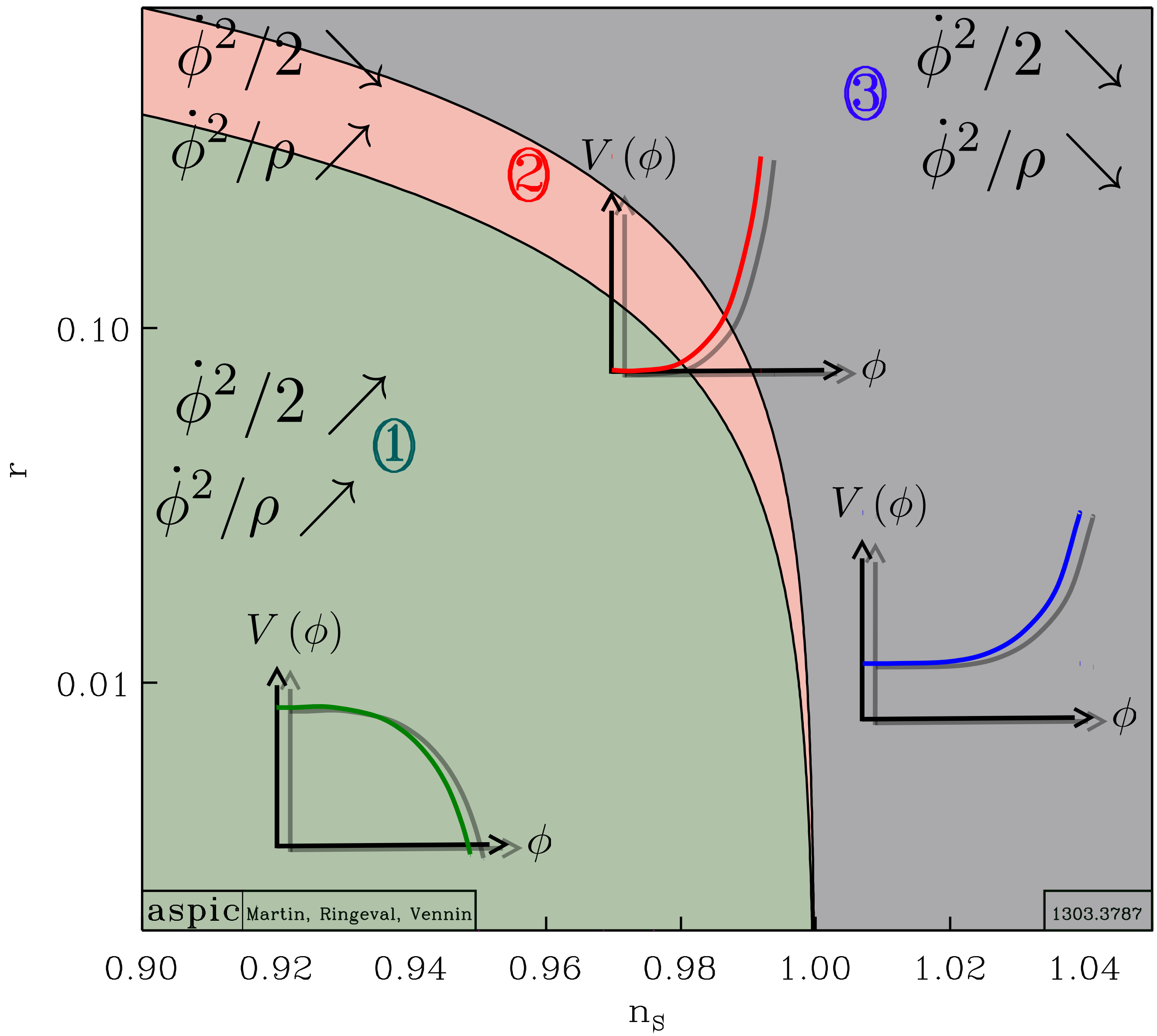
Observables \ Physical Models	Single Field slow-roll	Single Field with Features (ie non slow-roll)	Single Field with non-canonical kinetic terms	Multi field	
Scalar power spectrum $n_S \sim 1$ $\alpha_S \sim 0$	✓	DANGER	✓	DANGER	…
Entropic & adiabatic perturbations $\mathcal{I} \ll \mathcal{R}$	✓	✓	✓	DANGER	…
Gravity waves $r < 1$	✓	✓	✓	✓	…
Non-Gaussianities compatible with zero	✓	DANGER	DANGER	DANGER	…

Inflationary categories

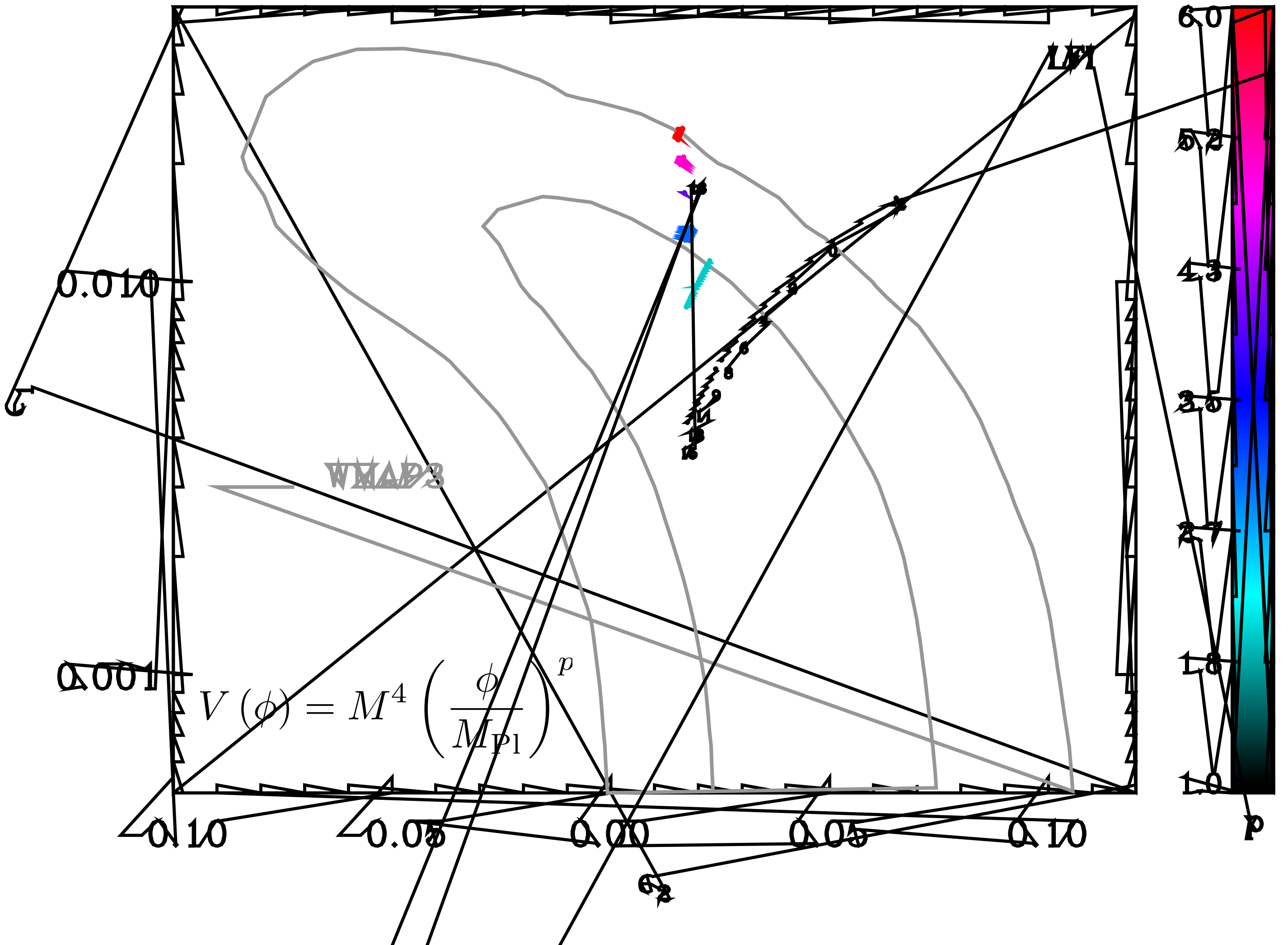
Understanding the (n_s,r) space



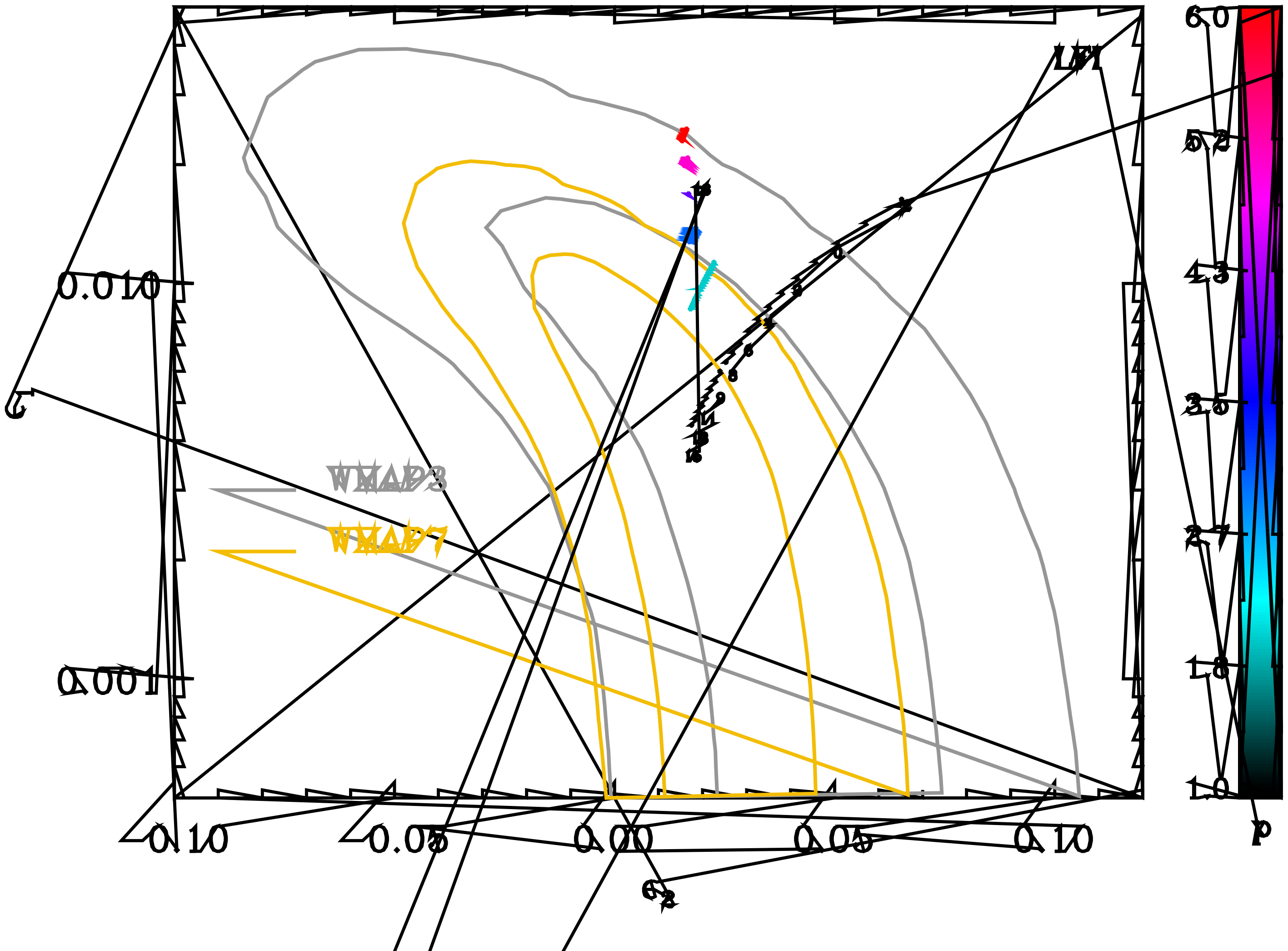
Understanding the (n_s,r) space



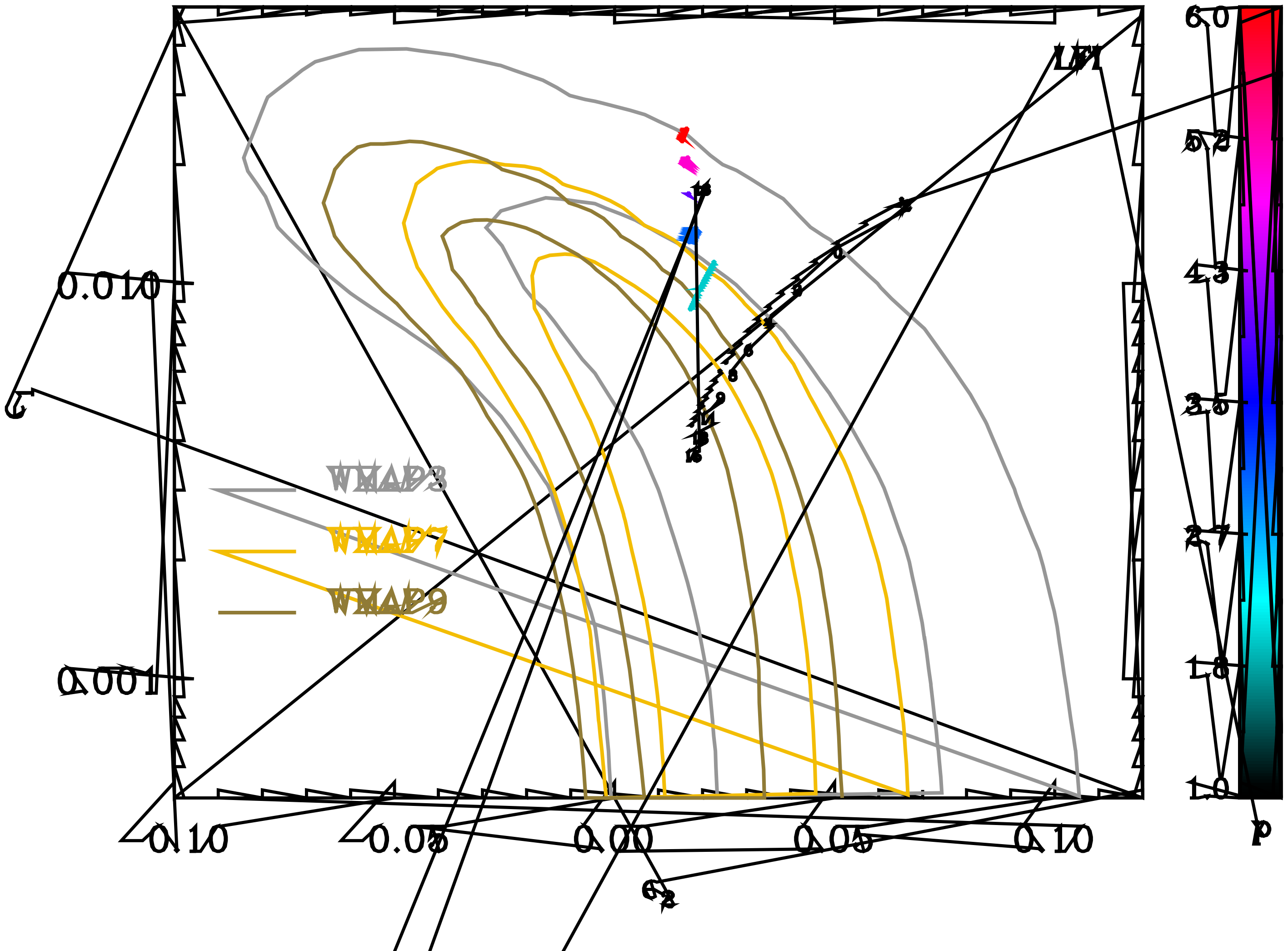
Constraining models: from WMAP to Planck



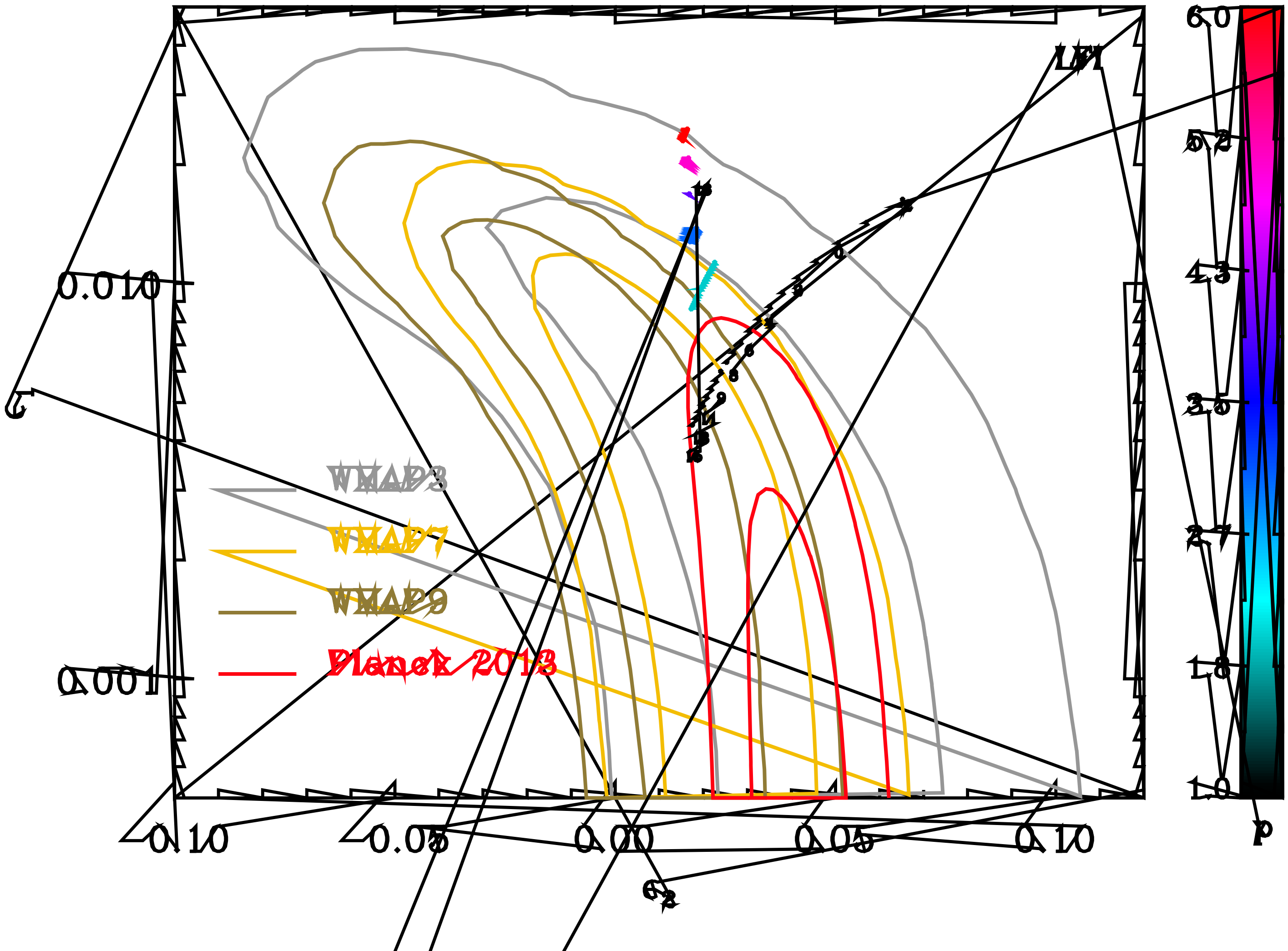
Constraining models: from WMAP to Planck



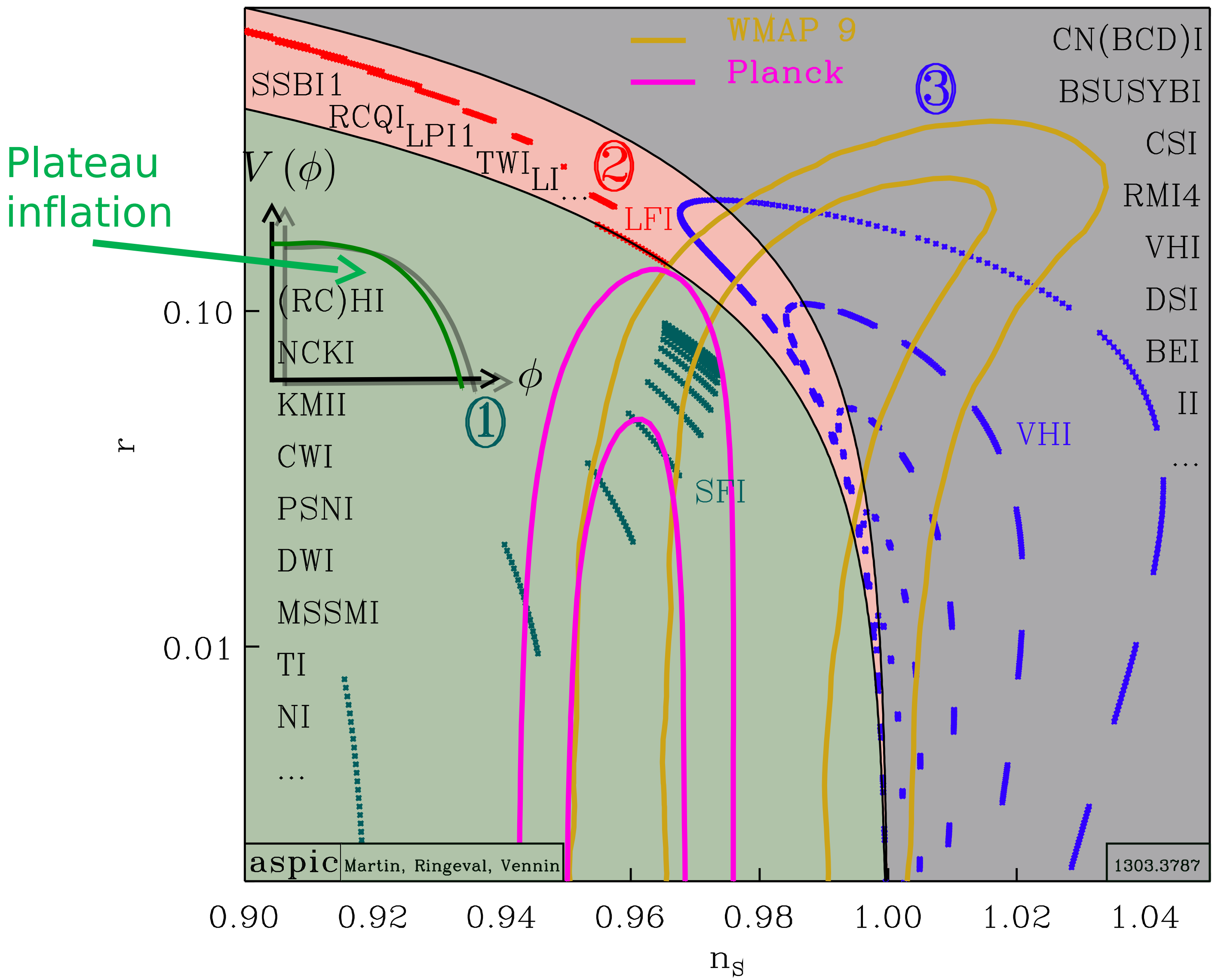
Constraining models: from WMAP to Planck



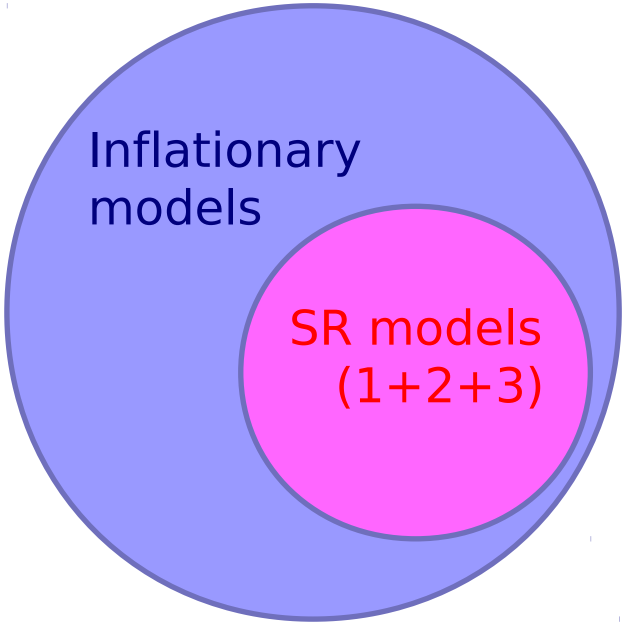
Constraining models: from WMAP to Planck



Category 1 is the category chosen by Planck



So … where do we stand?



- Single field slow-roll models is the favored class of models given the Planck data and the data prefers category 1.

- But this still leaves us with hundreds of scenarios and this does not tell us what is THE best model among those scenarios?

- In order to find the best model, we have to

  - Define “model 1 is better than model 2”: Bayesian evidence.

  - Apply this definition to the complete slow-roll landscape, ie we have to scan all single field slow-roll models, one by one, in an industrial way and study their predictions and how they perform: Planck data =big data era

  - Establish a complete ranking of all these models: model comparison

arXiv:1303.3787

≈ 74 models

≈ 700 slow roll formulas

≈ 365 pages

Encyclopædia Inflationaris

The encyclopedia contains the slow-roll treatment and comparison to the Planck data for all slow-roll models : <u>this is not a review paper!</u>

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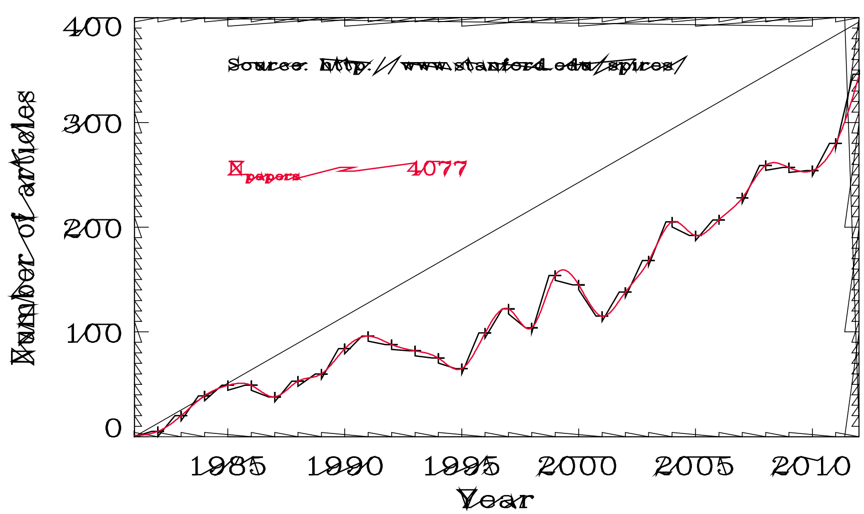
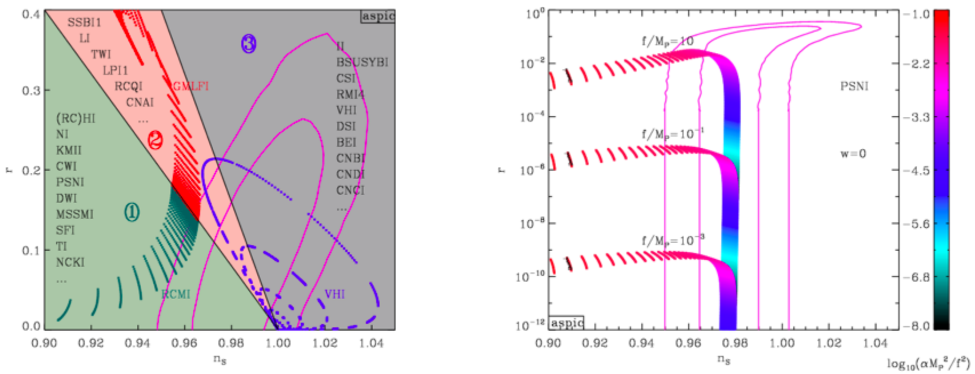
Keywords: Cosmic Inflation, Slow-Roll, Reheating, Cosmic Microwave Background, Aspic



theory.physics.unige.ch/~ringeval/aspic.html



Reheating consistent slow-roll predictions for a subset of inflationary models supported by **aspic** (left). The right panel features the Pseudo Natural Inflation (PSNI) predictions. The annotated values show the logarithmic energy scale, log(Ereh/GeV), at which a matter dominated reheating ends (arXiv:1303.3787).

Aspic is a collection of fast modern fortran routines for computing various observable quantities used in Cosmology from definite single field inflationary models. It is distributed as a scientific library and aims at providing an efficient, extendable and accurate way of comparing theoretical inflationary predictions with cosmological data. **Aspic** currently supports 64 models of inflation, and more to come!

By observable quantities, we currently refer to as the Hubble flow functions, up to second order in the slow-roll approximation, which are in direct correspondence with the spectral index, the tensor-to-scalar ratio and the running of the primordial power spectrum. The **aspic** library also provides the field potential, its first and second derivatives, the energy density at the end of inflation, the energy density at the end of reheating, and the field value (or e-fold value) at which the pivot scale crossed the Hubble radius during inflation. All these quantities are computed in a way which is consistent with the existence of a reheating phase.

The code is released as a GNU software which compiles itself into both a static and shared library. As the list of inflationary models is always increasing, you are encouraged to add support for any model that would n

Please, check the MAN file for a complete documentation a

download the source file.

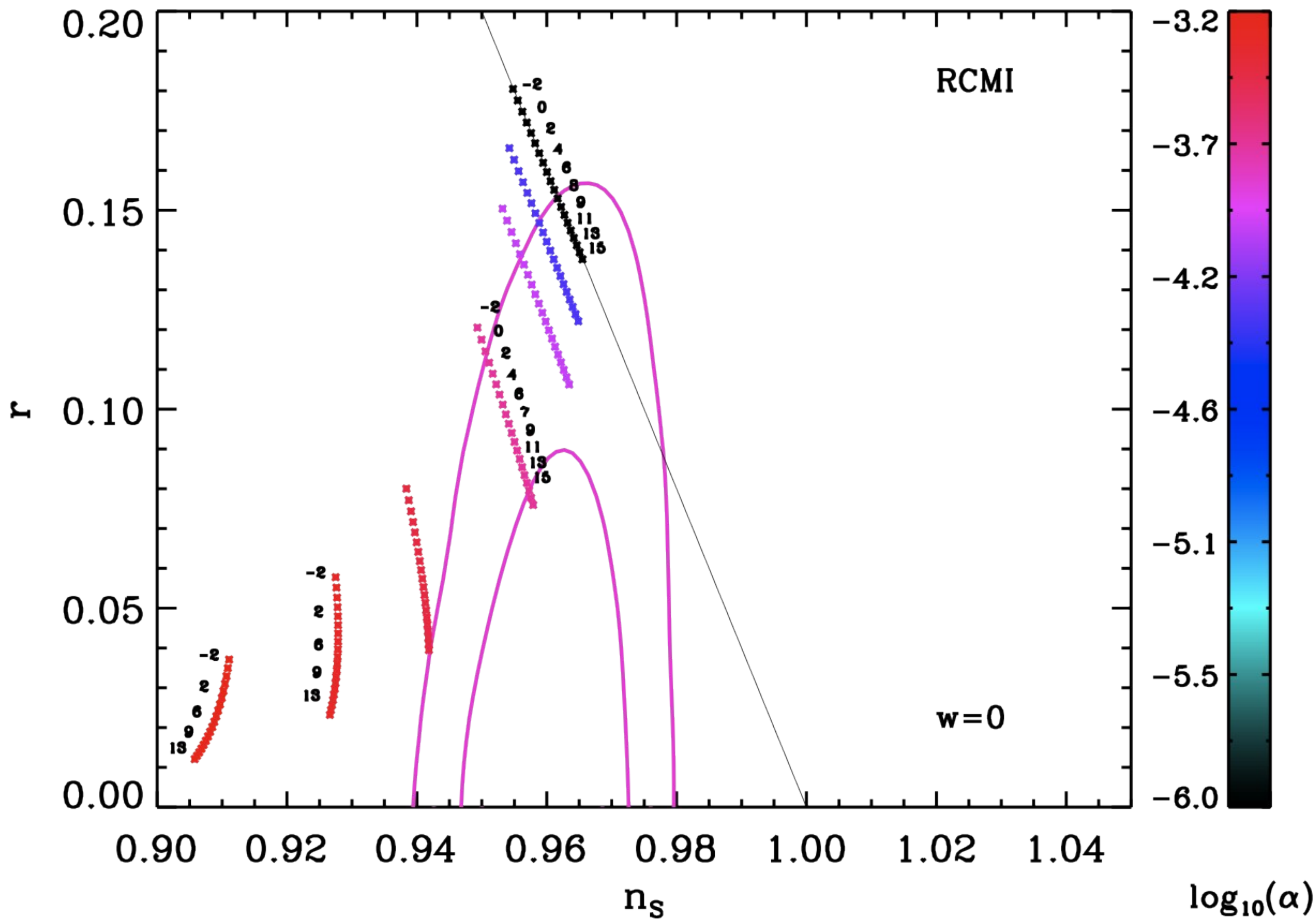
For details, please read the original paper arXiv:1303.3787

For an exact integration of any inflationary models, without assuming slow-roll, checkout the **fieldinf** code and library.

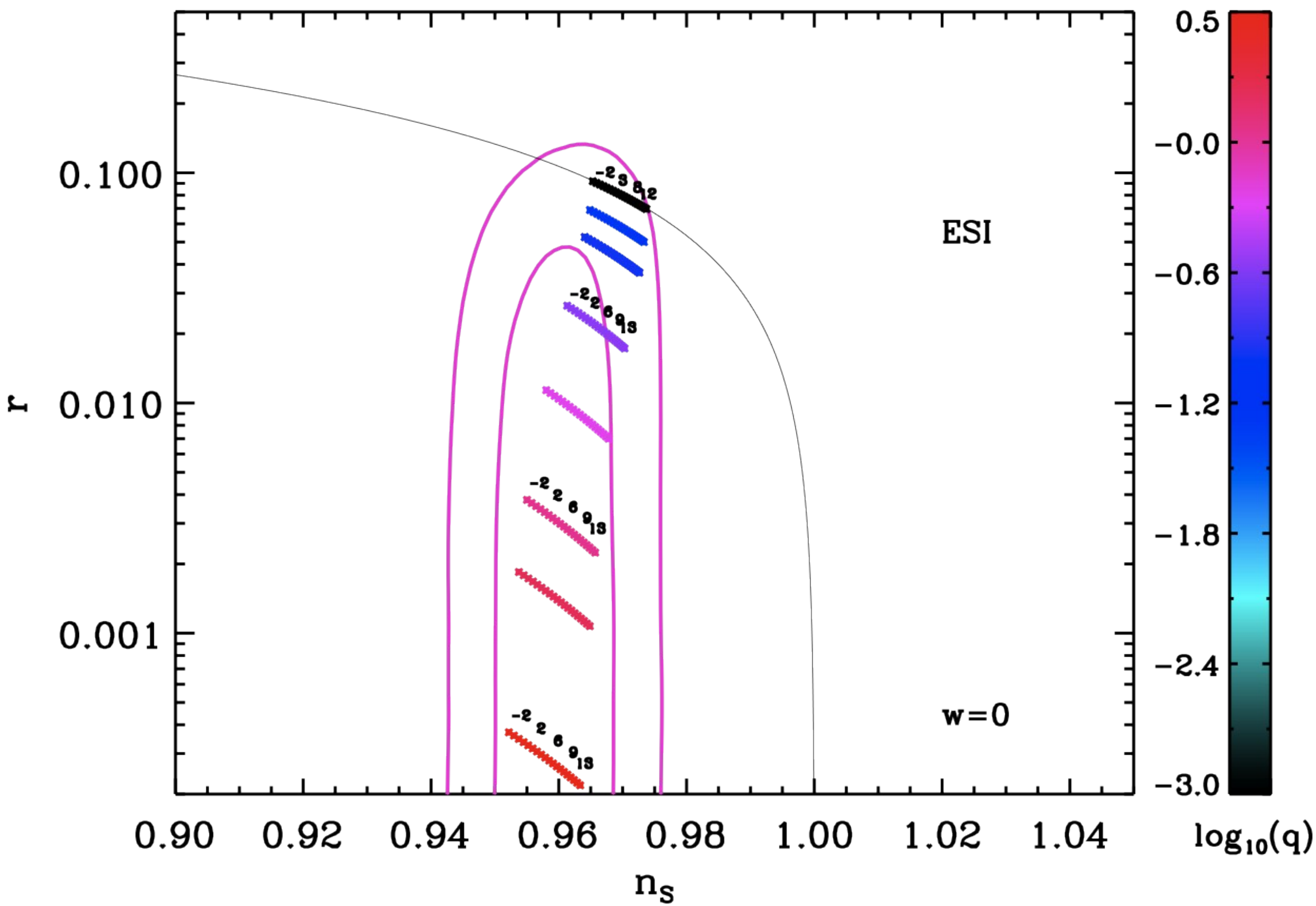
Last modif 03/2013

The ASPIC library provides all the numerical codes for all models

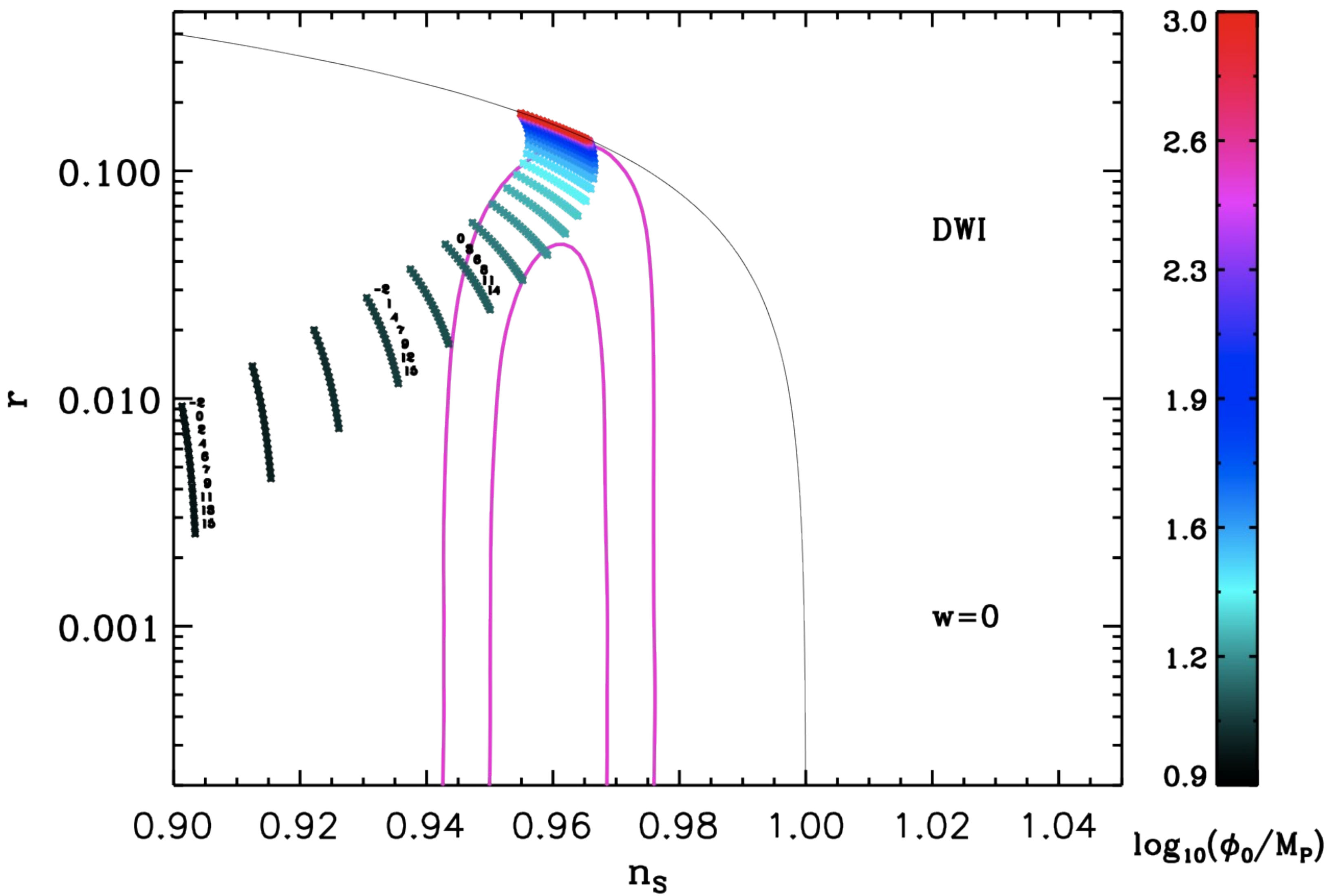
A few examples



A few examples



A few examples



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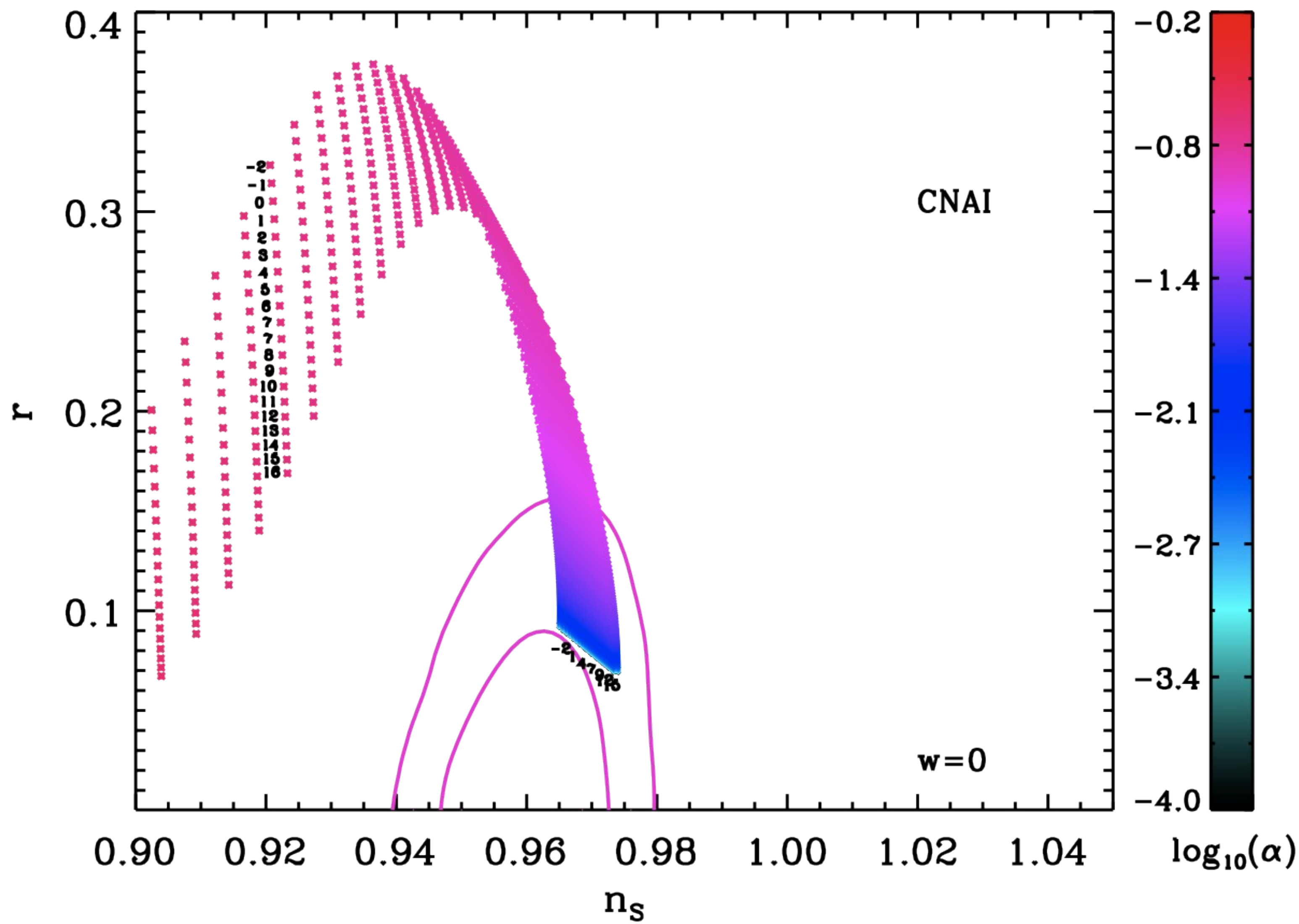

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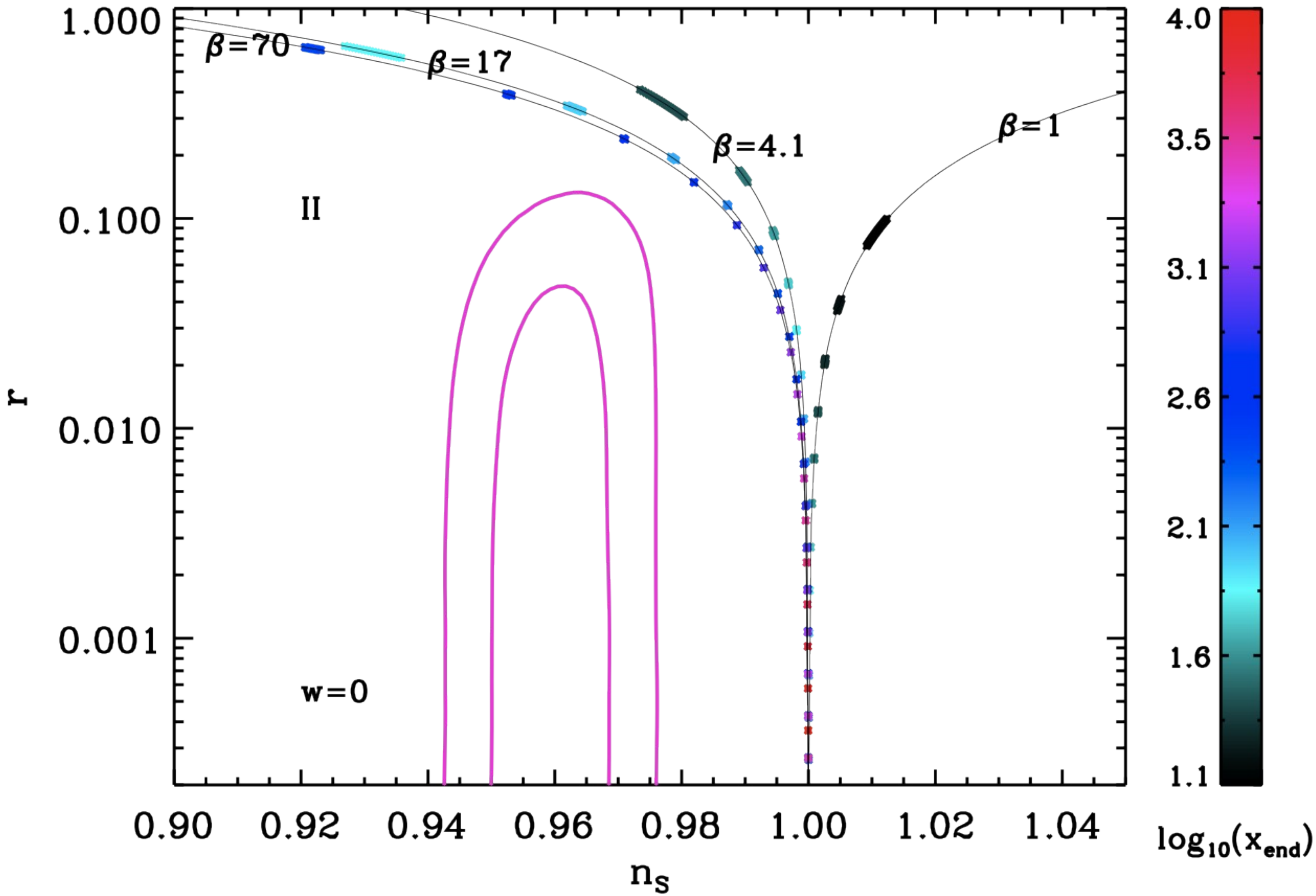
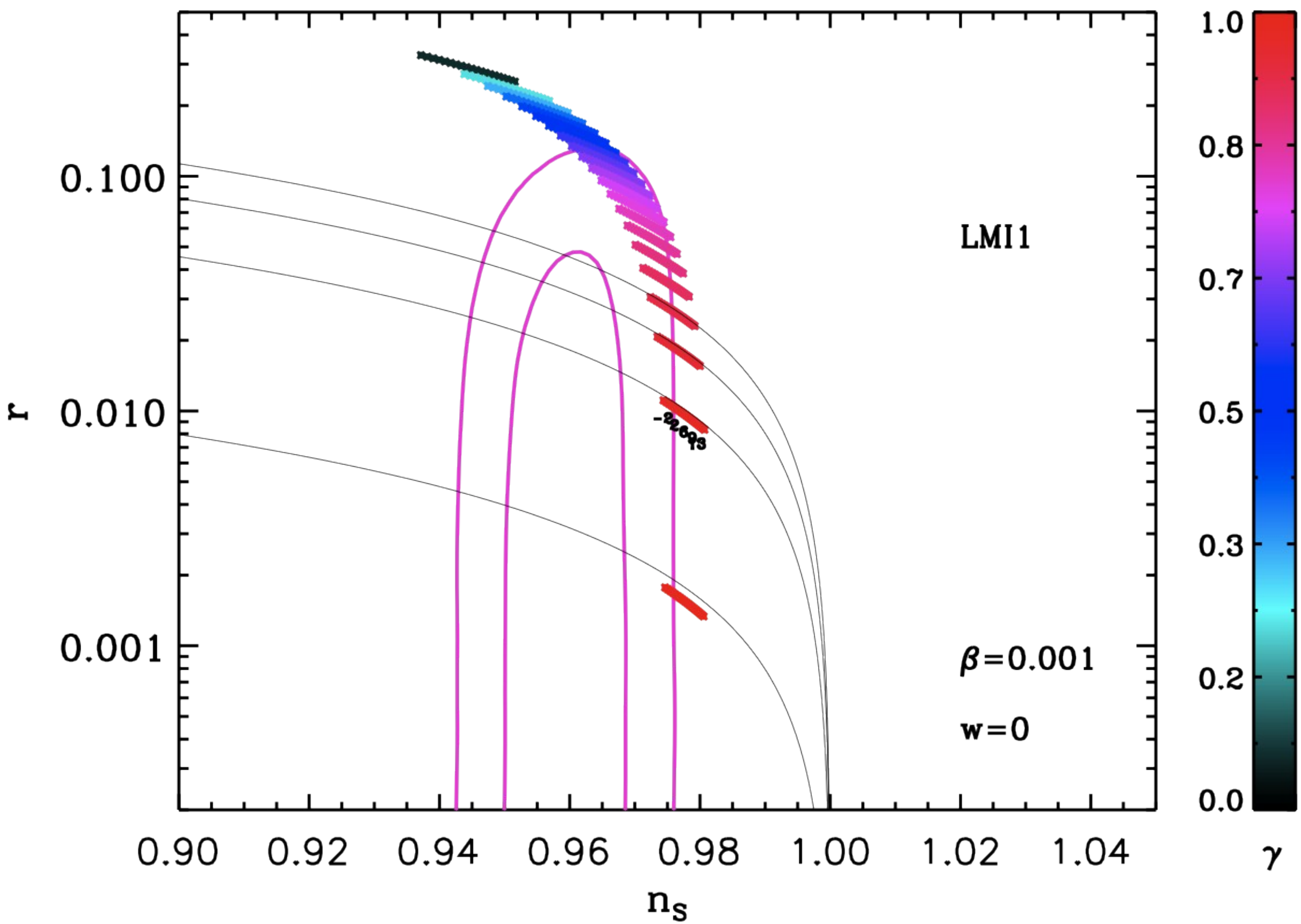


A few examples



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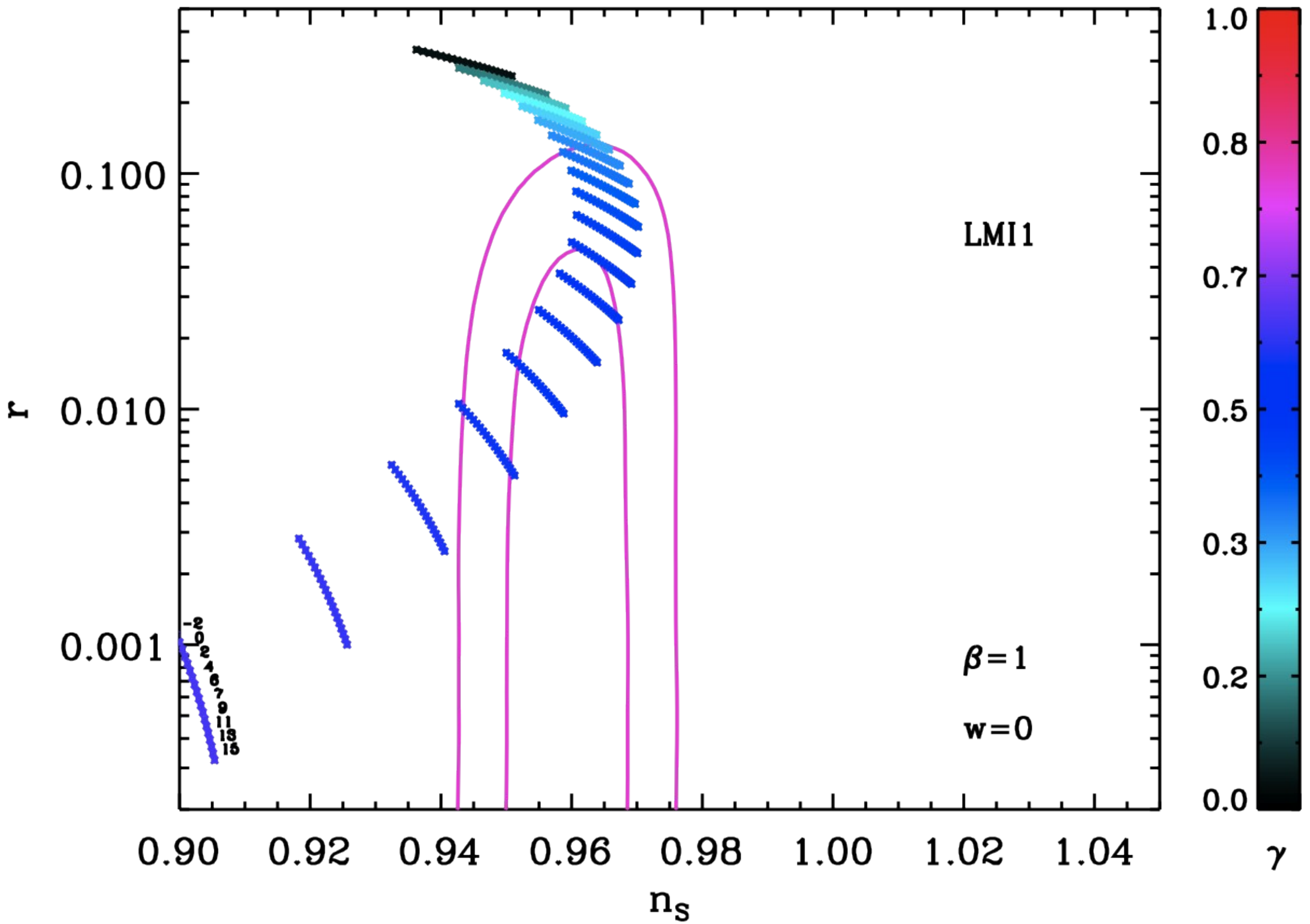
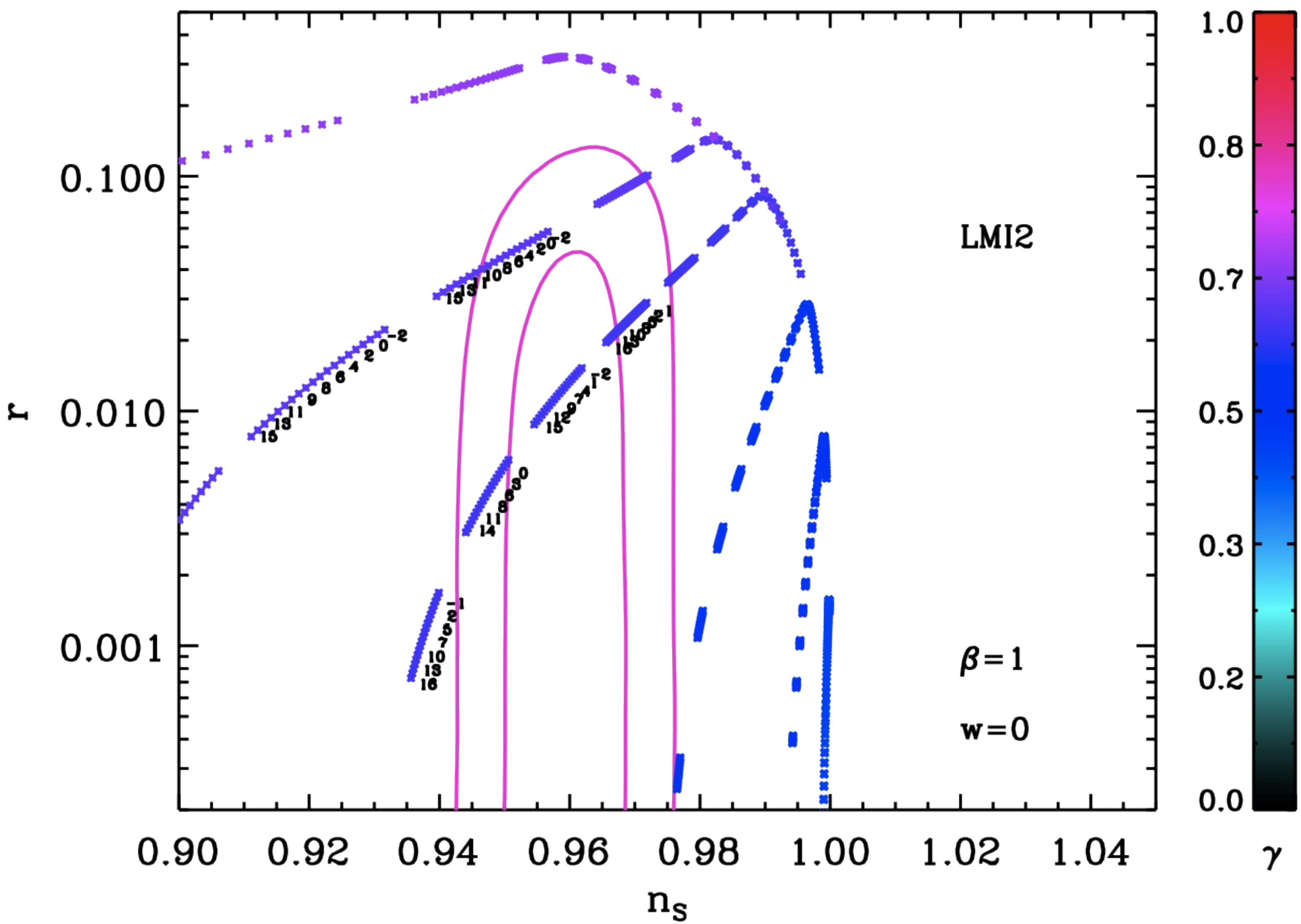
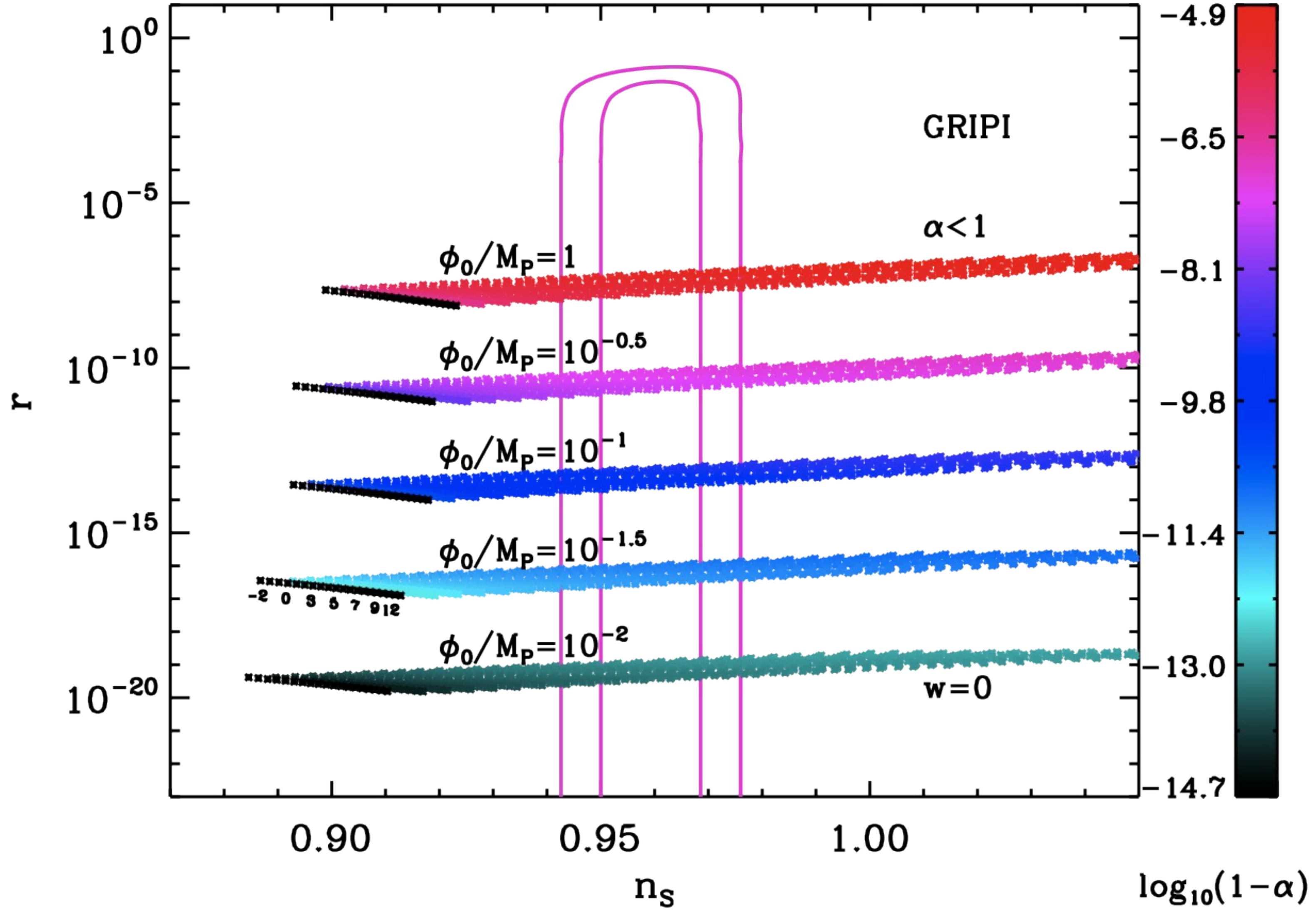


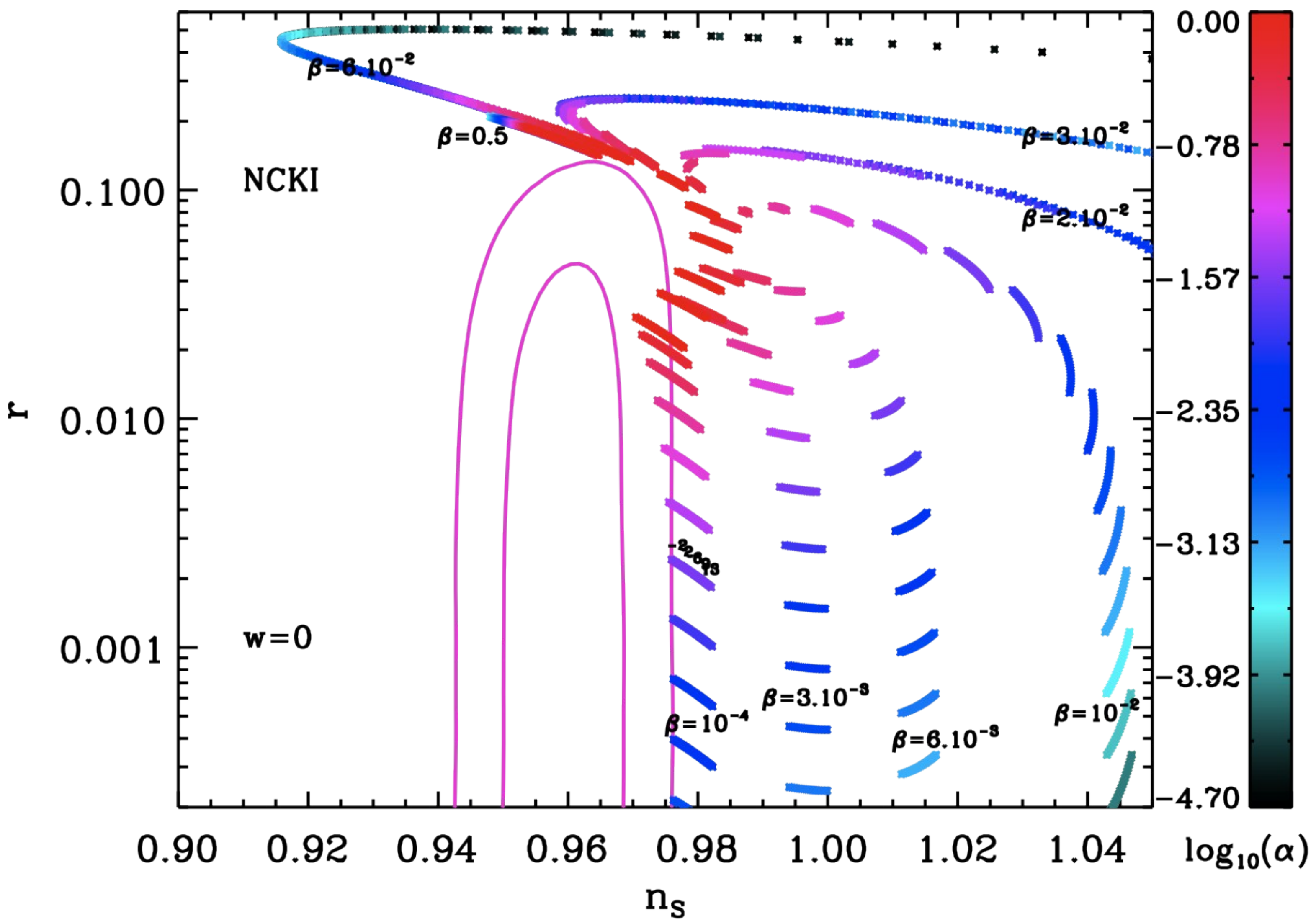
A few examples



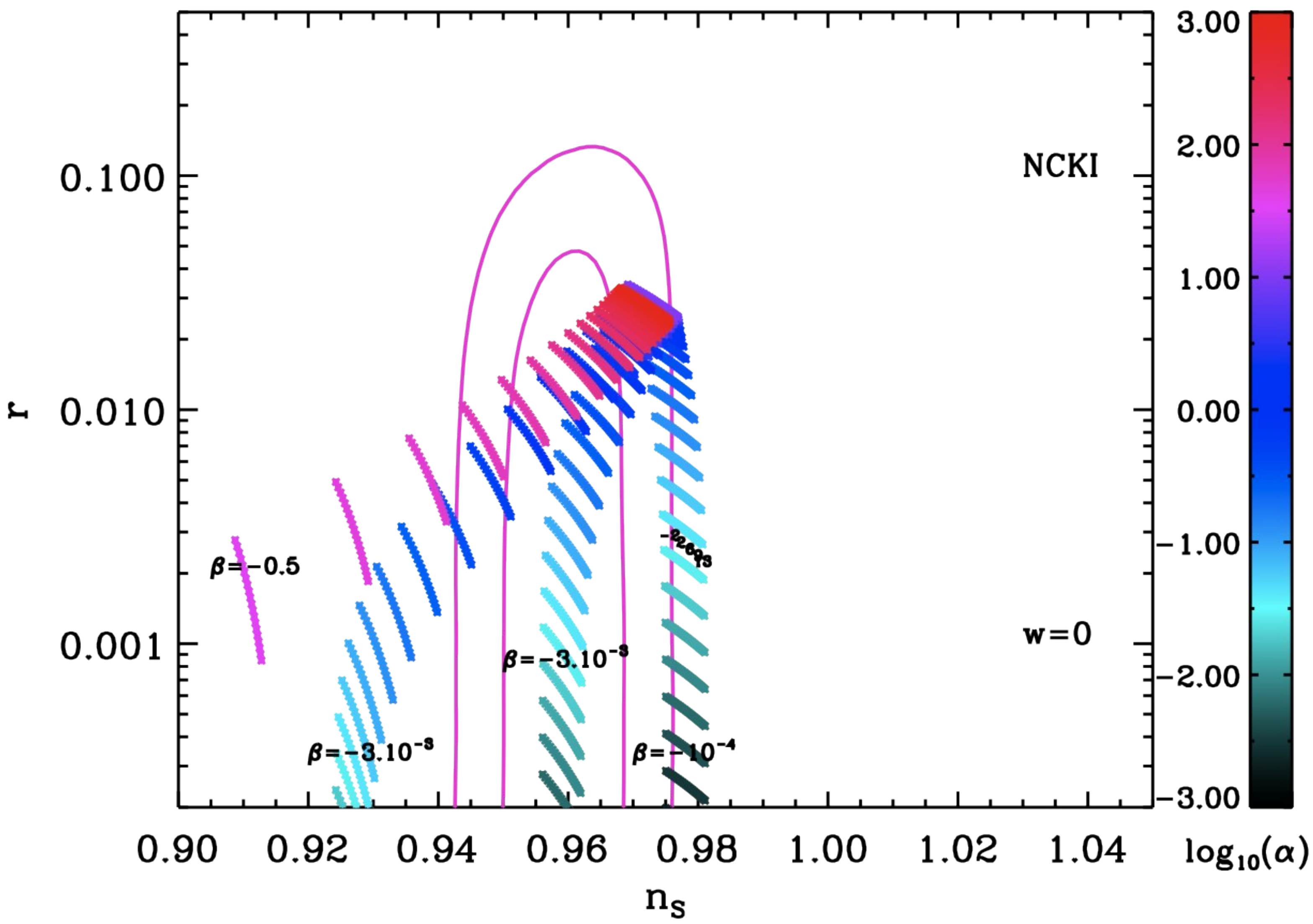
A few examples



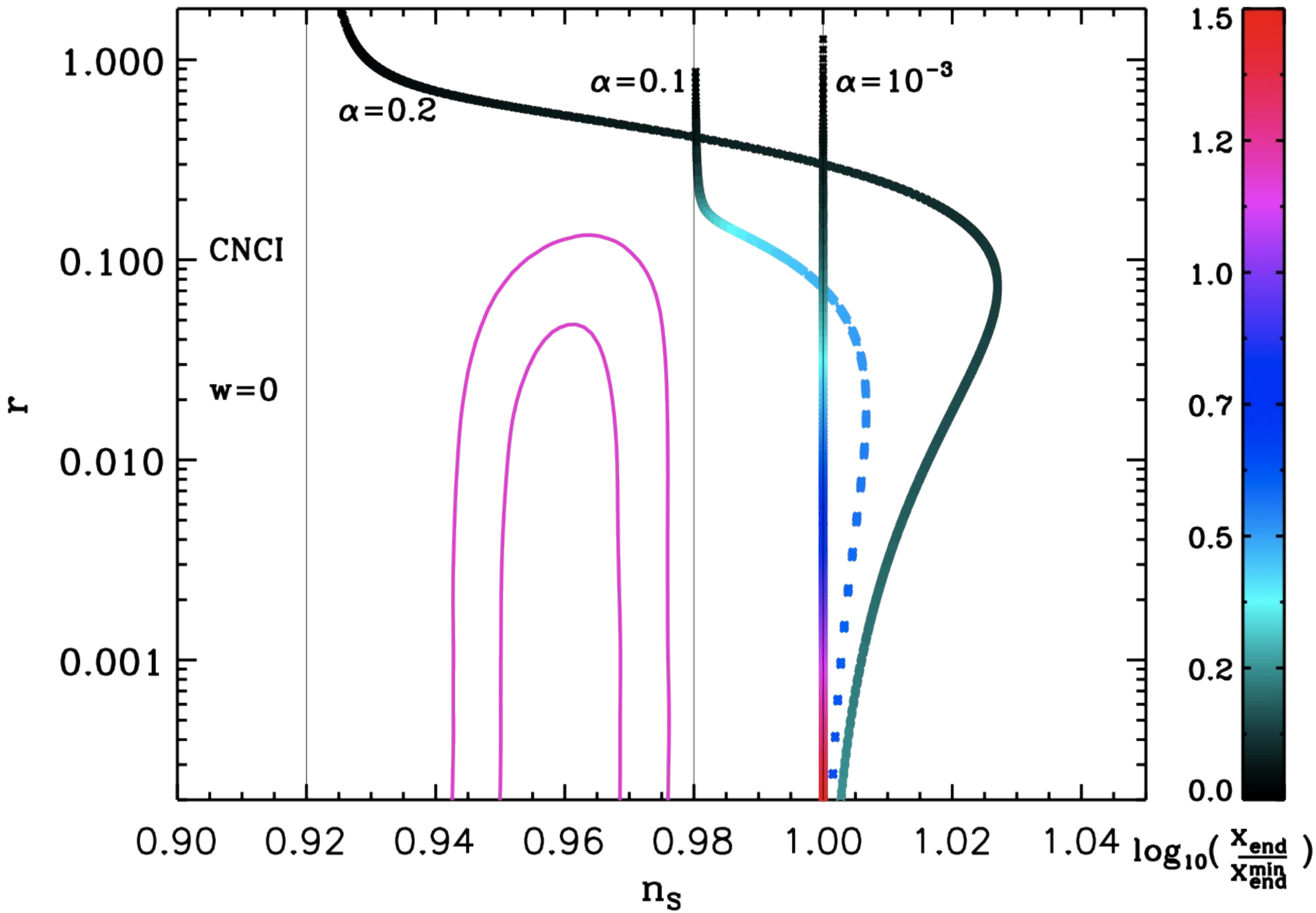
A few examples



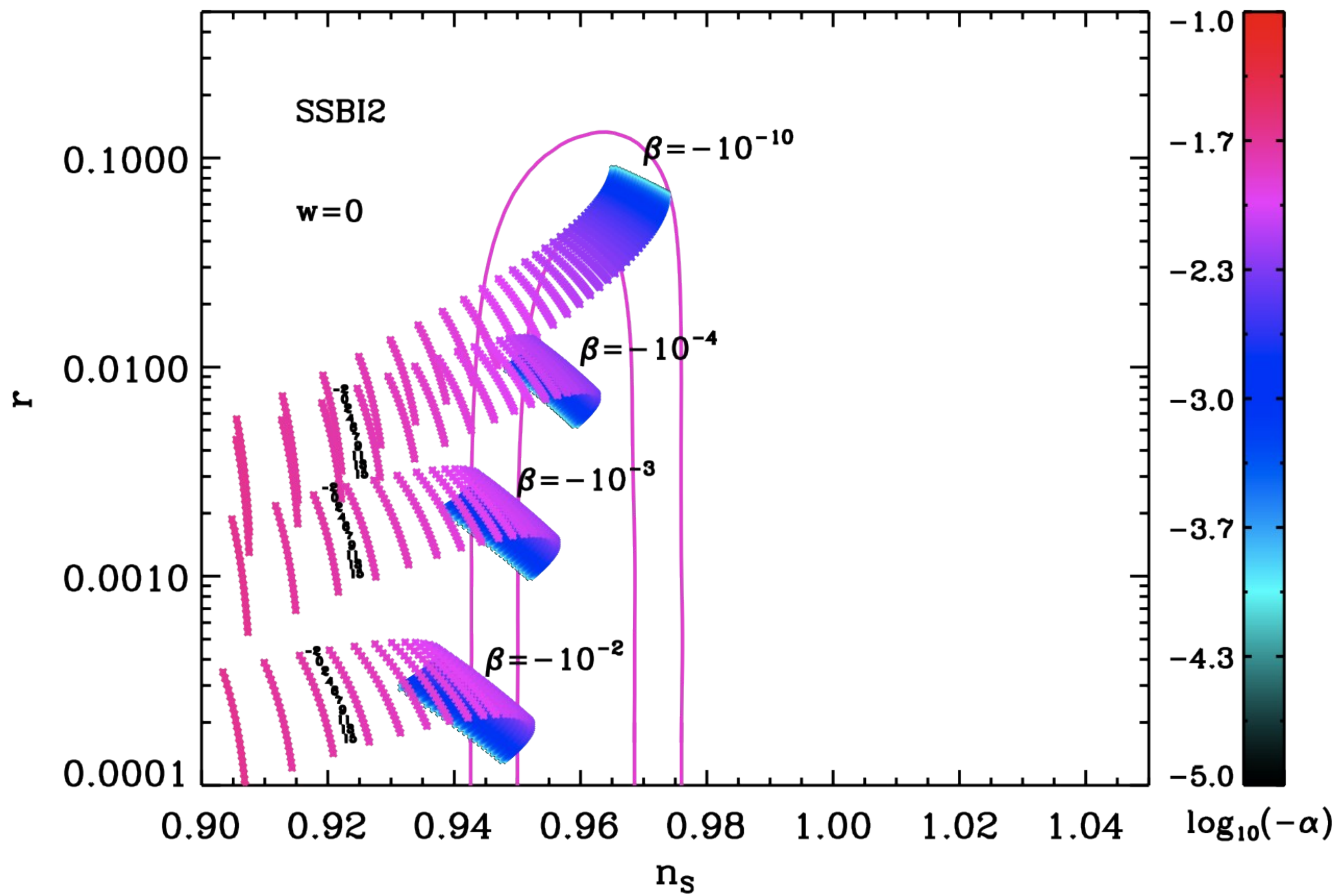
A few examples



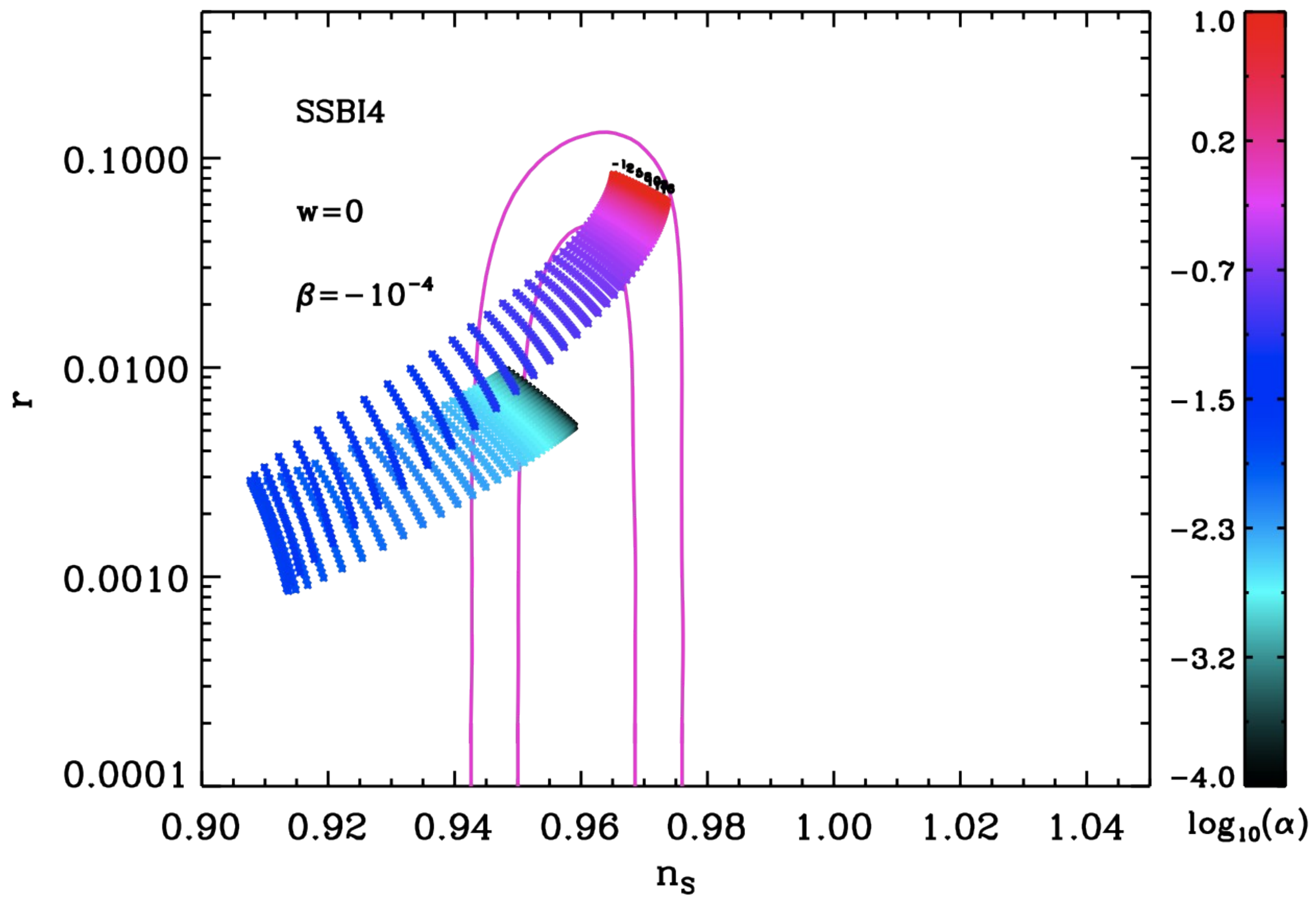
A few examples



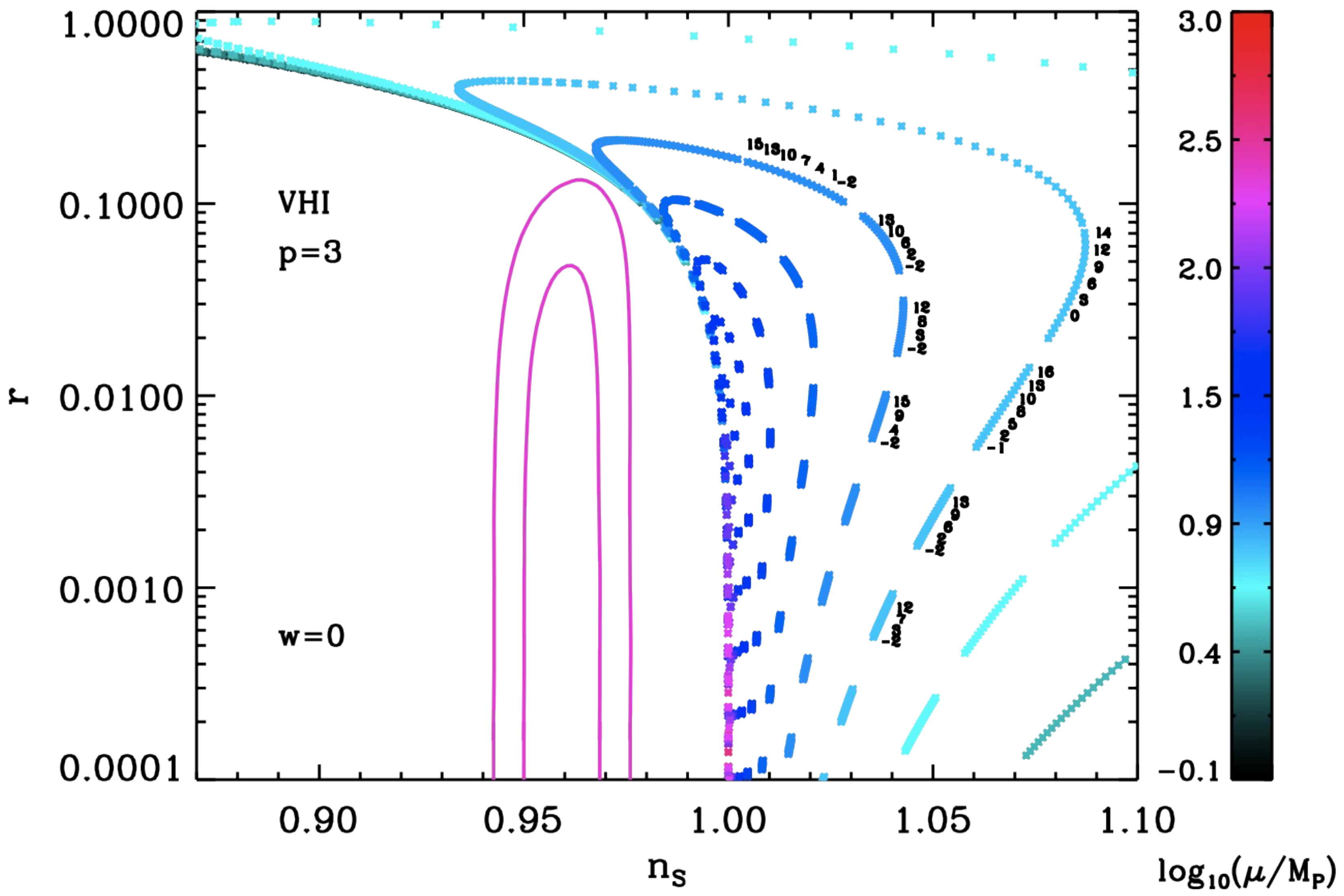
A few examples



A few examples



A few examples



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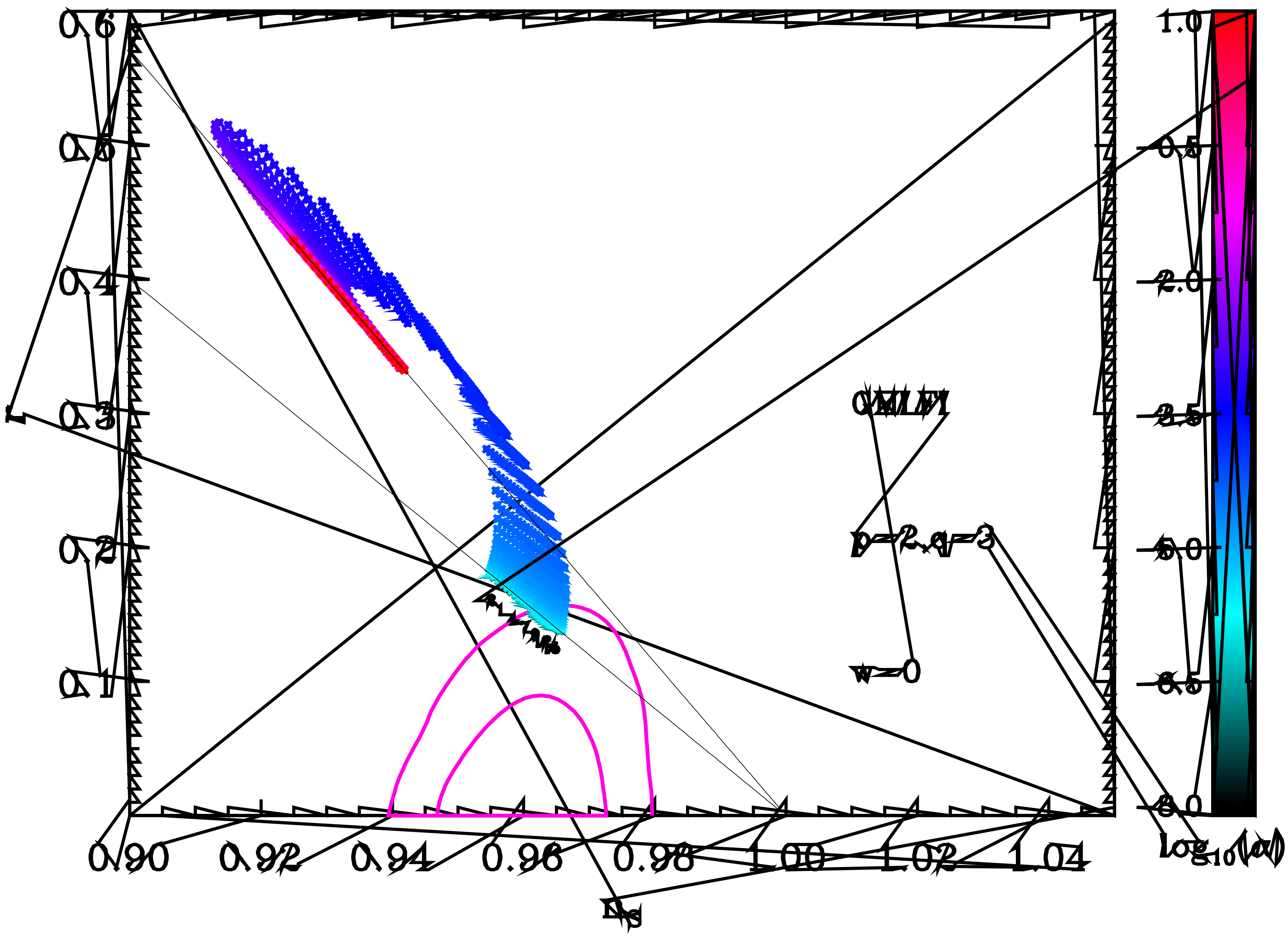


How to do model comparison?

- For model comparison, we compute the Bayesian evidence (integral of the likelihood over all parameter priors~probability of a model), ie the probability of a model, for each inflationary scenario

Bayesian evidence of the model "i"

$$\frac{p\left(\mathcal{M}_i|D\right)}{p\left(\mathrm{HI}|D\right)} = B_{i-\mathrm{HI}}$$

posterior odds

Bayesian evidence of the reference model=Starobinsky model

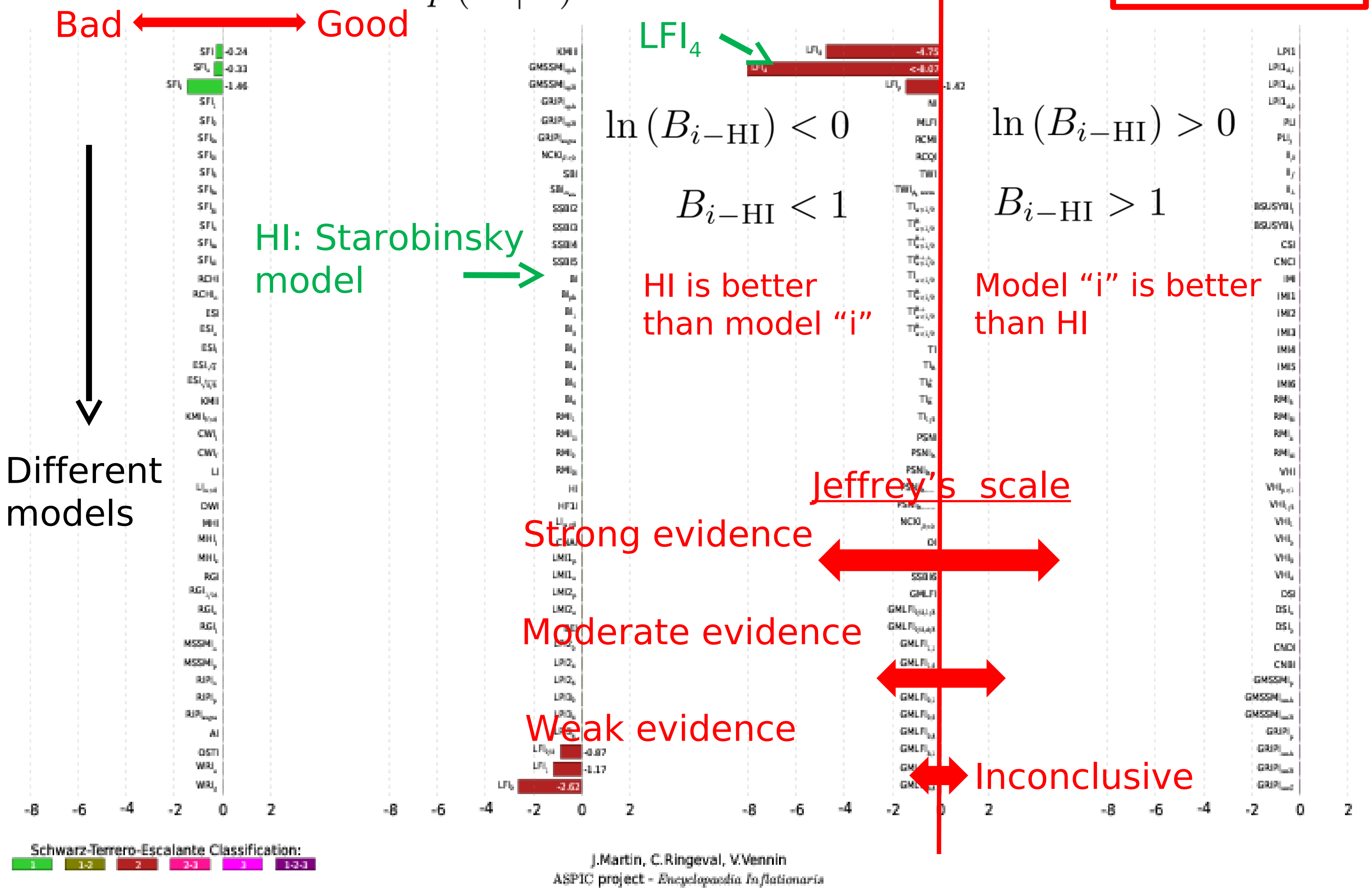
$B_{i-\mathrm{HI}} > 1$ Model "i" is better than HI

$B_{i-\mathrm{HI}} < 1$ HI is better than model "i"

First calculation of inflationary Bayesian evidence in 2010

$$\frac{p(\mathcal{M}_i|D)}{p(\mathrm{HI}|D)} = B_{i-\mathrm{HI}} \qquad \ln(B_{i-\mathrm{HI}}) = 0$$

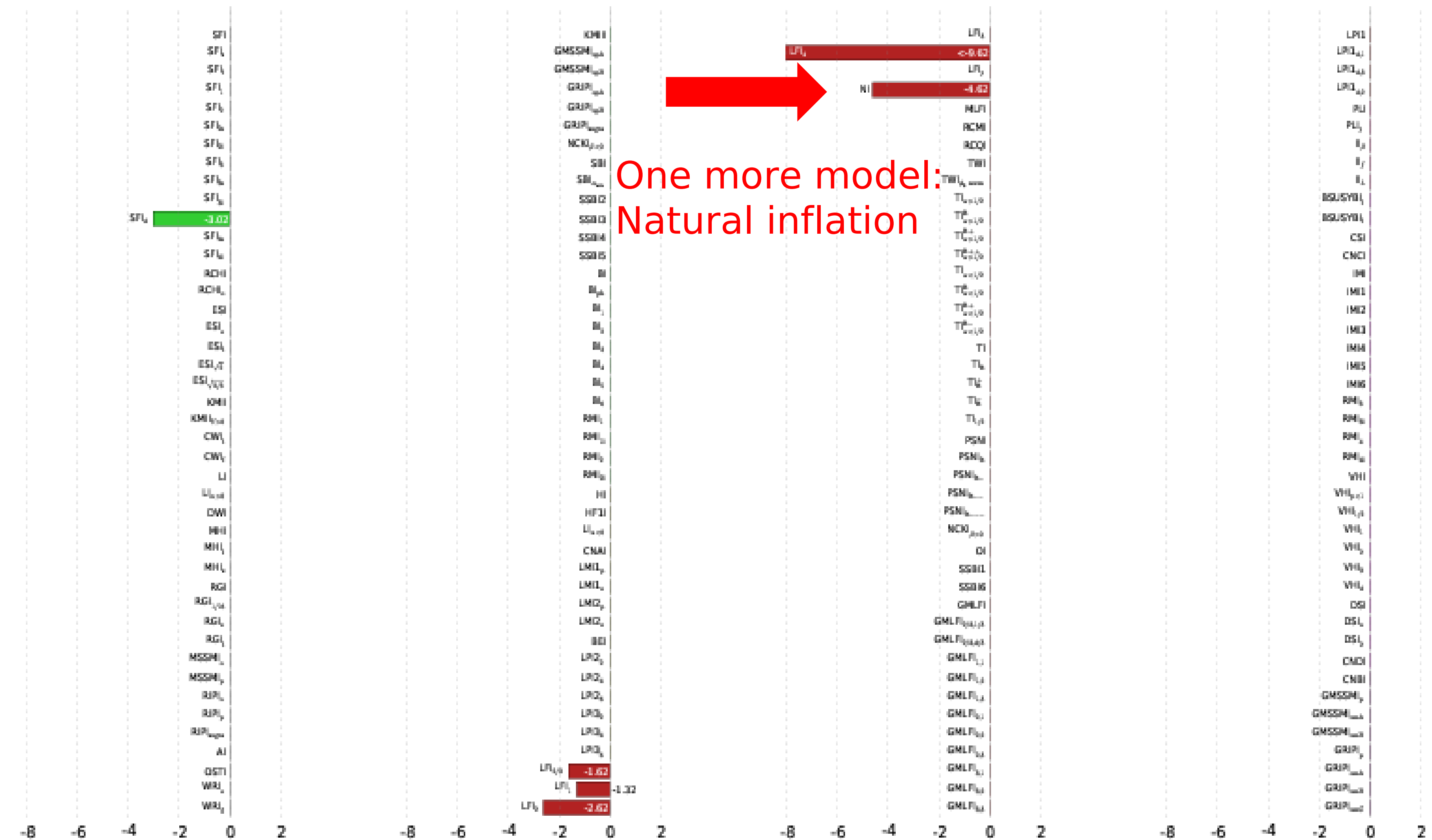
WMAP7: Martin, Ringeval & Trotta
arXiv:1009.4157



Planck team calculation of Bayesian evidence

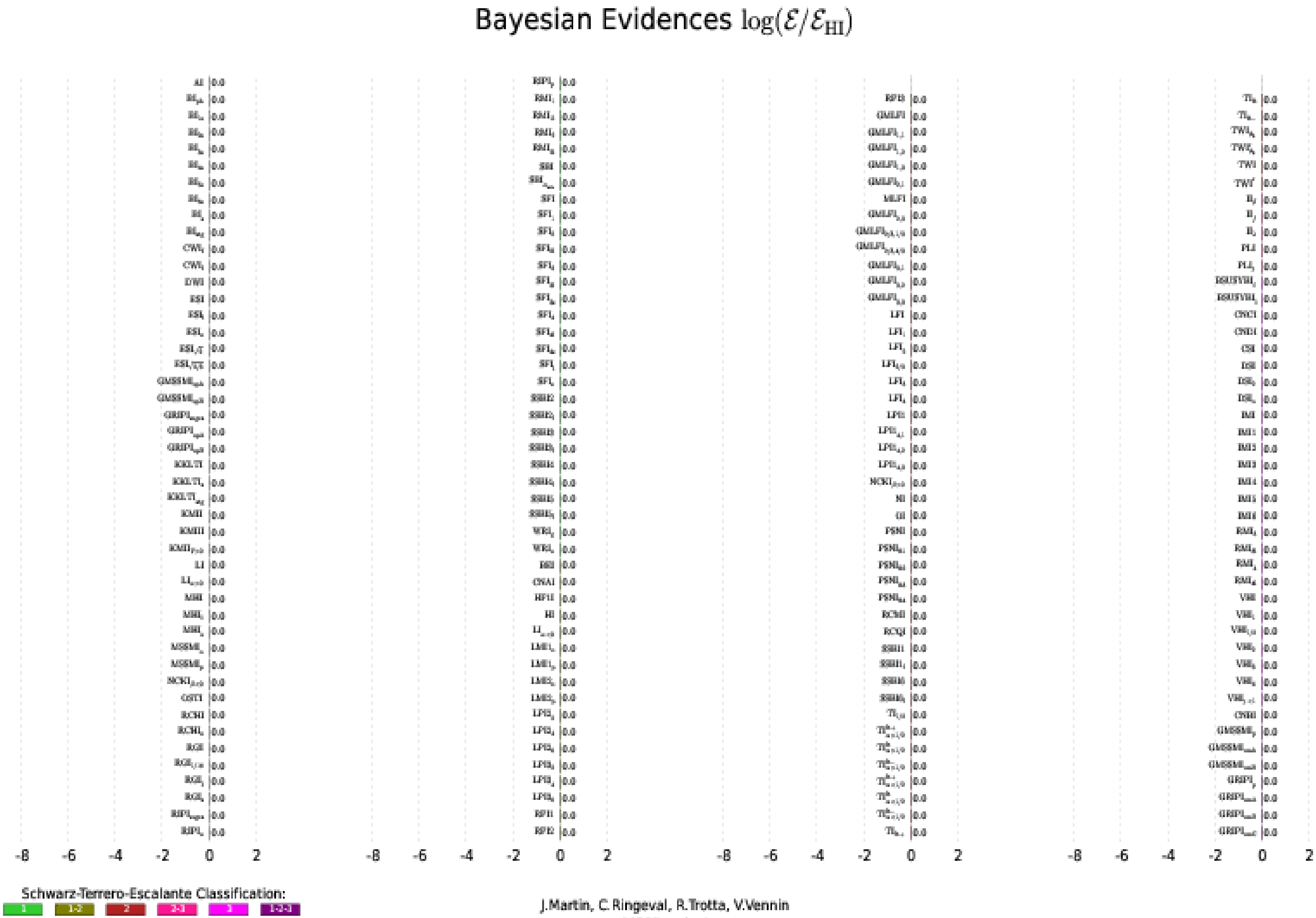
Planck collaboration arXiv:1303.5082

Bayesian Evidences $\log(\mathcal{E}/\mathcal{E}_{\rm HI})$ for the inflationary models



Schwarz-Terrero-Escalante Classification: 1, 1-2, 2, 2-3, 3, 1-2-3

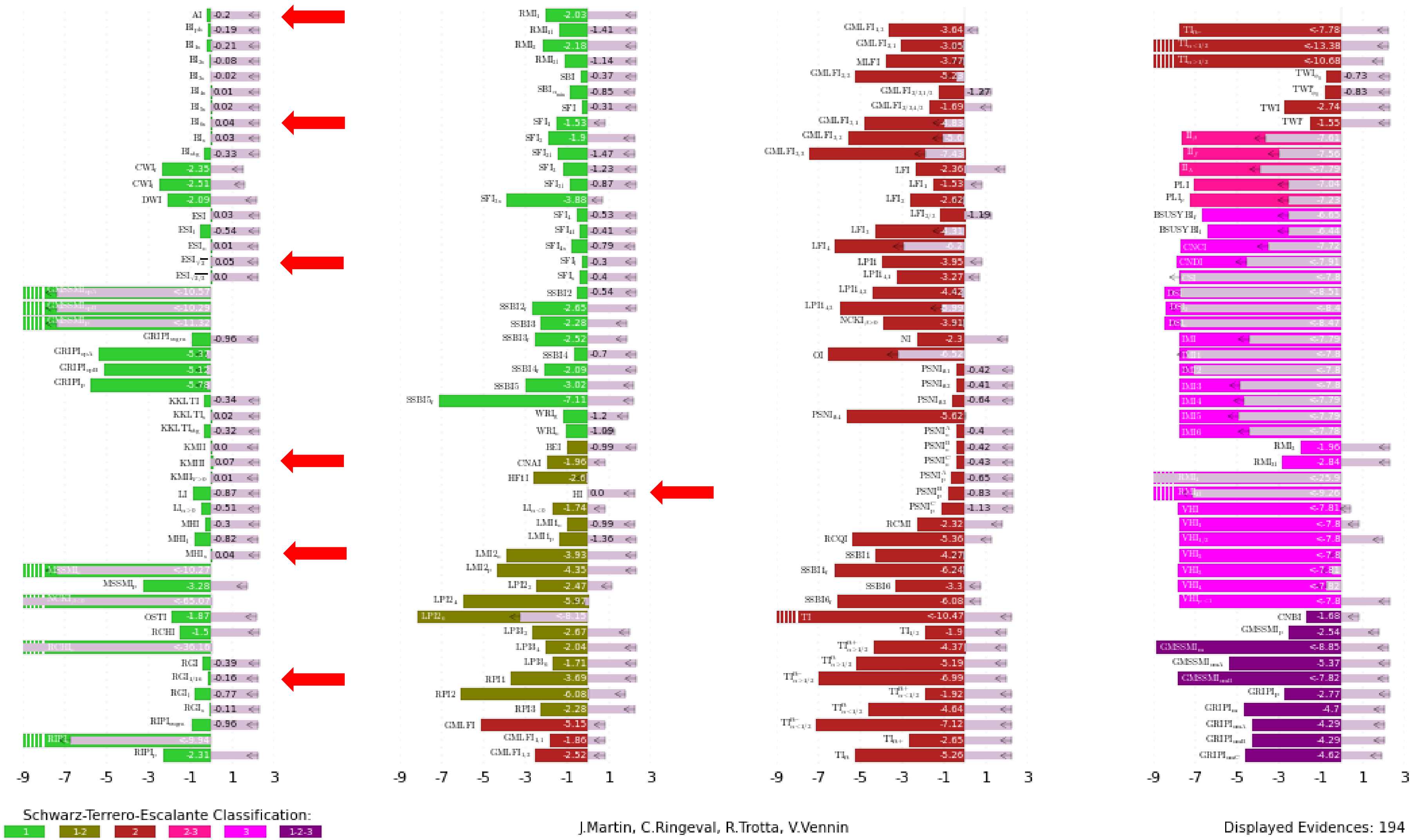
J.Martin, C.Ringeval, V.Vennin
ASPIC project - *Encyclopædia Inflationaris*



J.Martin, C. Ringeval, R. Trotta, V.Vennin
ASPIC project

ASPIC evidence: good models

Bayesian Evidences $\ln(\mathcal{E}/\mathcal{E}_{\mathrm{HI}})$ and $\ln(\mathcal{L}_{\max}/\mathcal{E}_{\mathrm{HI}})$



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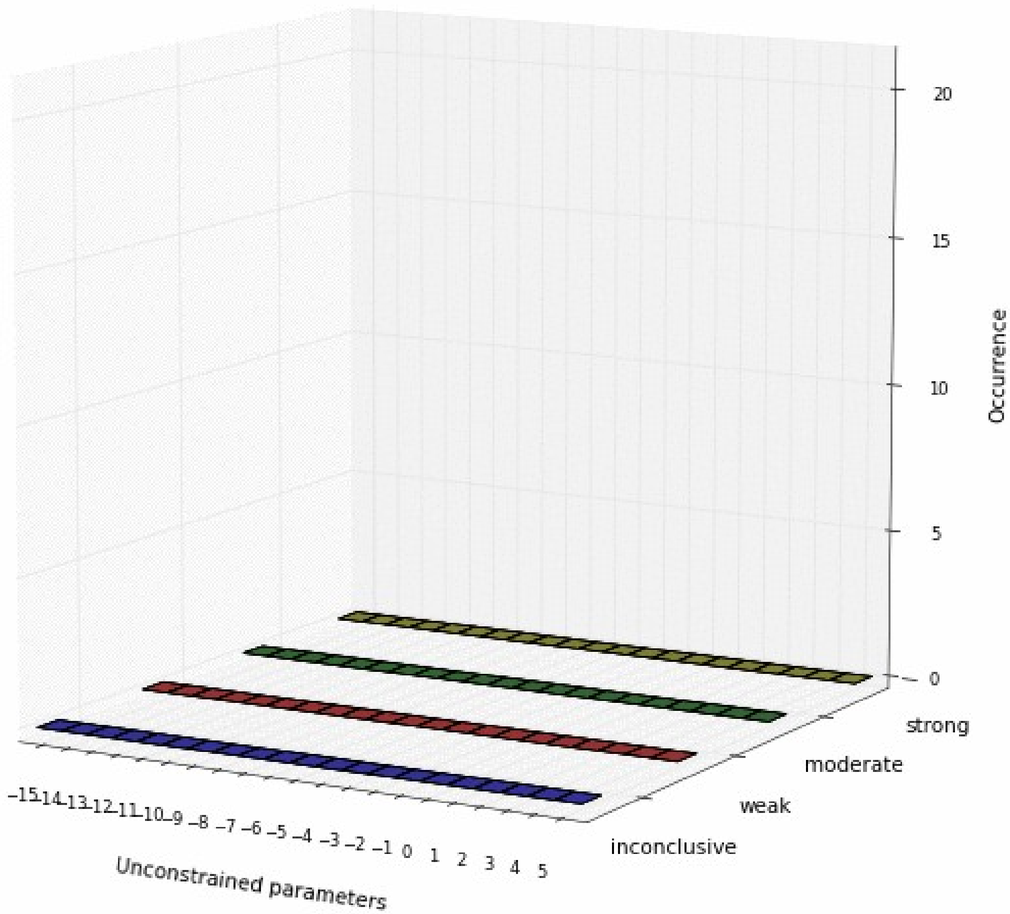
Displayed Evidences: 194



Summary

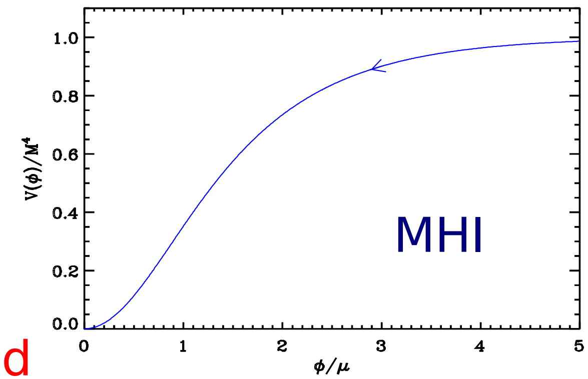
26 % inconclusive zone

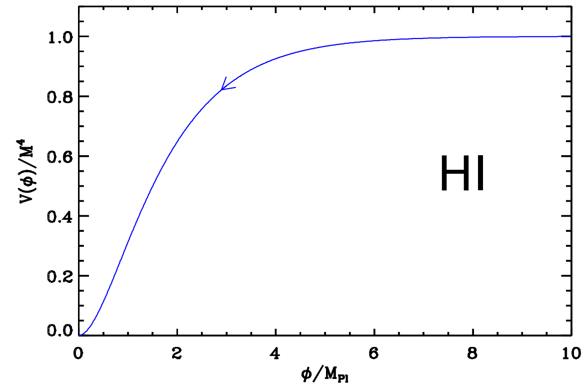
21 % weak zone

18 % moderate zone

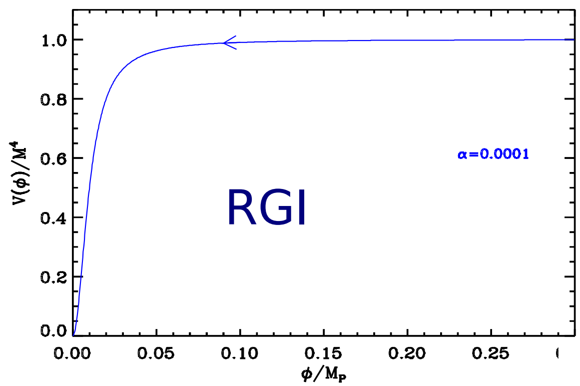
34 % strong zone

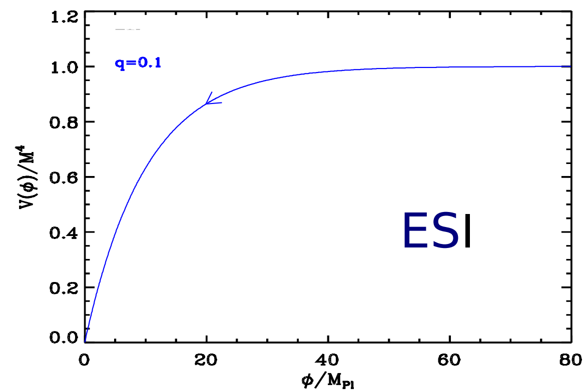
15 different potentials in the inconclusive zone





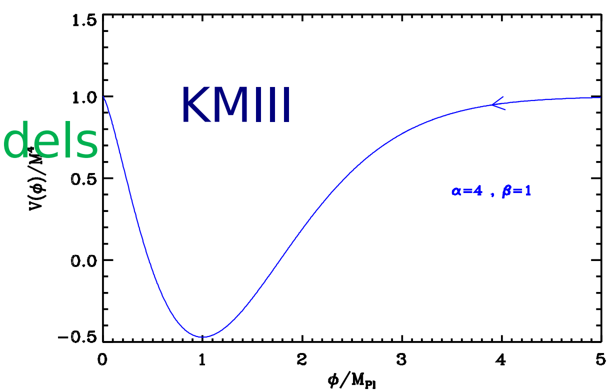
Planck has identified the shape of the inflaton potential:

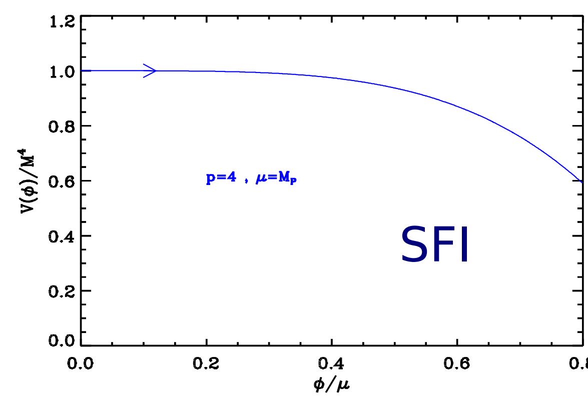




Plateau inflation

NB: the difference between these models is “inconclusive”.





Conclusions

nflation is in good shape after Planck 2013

he data indicate that we deal with the simplest, ie non exotic, version
flation: single field slow roll model with minimal kinetic term

he shape of the potential is constrained: plateau inflation

here are ~9 models that have a very good Bayesian evidence and a number
nconstrained parameters between zero and one. We have come a long road …
 hundreds of models, Planck has identified ~9 favored scenarios!

nteresting enough, most of these models are well justified from high
gy physics & string theory

And the winners are …