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Black holes at all scales

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Outline

1. Introduction

·To make sense of the bewildering variety of information on black holes it is useful to compartmentalize the subject into three cathegories: Mathematical black holes, astrophysical black holes, and quantum black holes. This order also reflects history.

·Mathematical black holes: Given general relativity, one seeks to understand how do they form mathematically. Also wants to know what are their properties in every detail, those coming from a specific theory of gravitation, and those independent of any theory.

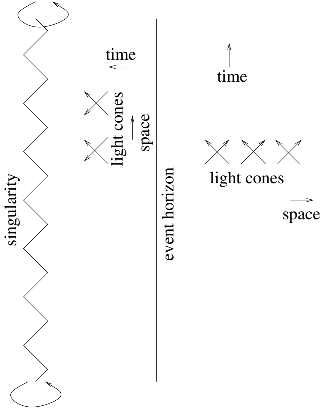
·Astrophysical black holes: Given that black holes are a robust theoretical prediction, and gravitation dominates at astrophysical scales, where can the telescopes find them and what can they tell? In turn, how the phenomena that is seen can be explained.

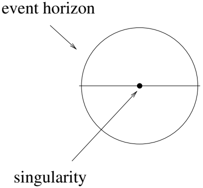
·Quantum black holes: The world has a quantum nature at a fundamental level. Black holes are no exception, thus it is of preeminent interest to know how they interact with quantum matter and at what scales. Being the spacetime objects par excellence, they can lead us to a proper quantum gravity theory.

2. Mathematical black holes

·Mathematical definition: a black hole is a region inside the boundary of the causal past of future null infinity.

·Pictoric realization of a Schwarzschild black hole:





Left: A spacetime diagram of a black hole with its light cones. Right: A space diagram of a black hole, the one that appears in snapshots.

2. Mathematical black holes

- Black holes were invented by Oppenheimer and Snyder (1939). Oppenheimer understood that a stage next to the neutron star stage should exist. It is an amazing discovery. Chandrasekhar was after it since 1929.
- It is an exact solution of total gravitational collapse. The solution is Friedmann inside matched to Schwarzschild outside. Outstanding mix of physics and mathematics.

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On Continued Gravitational Contraction

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(Received July 10, 1939)

When all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse. Unless fission due to rotation, the radiation of mass, or the blowing off of mass by radiation, reduce the star's mass to the order of that of the sun, this contraction will continue indefinitely. In the present paper we study the solutions of the gravitational field equations which describe this process. In I, general and qualitative arguments are given on the behavior of the metrical tensor as the contraction progresses: the radius of the star approaches asymptotically its gravitational radius; light from the surface of the star is progressively reddened, and can escape over a progressively narrower range of angles. In II, an analytic solution of the field equations confirming these general arguments is obtained for the case that the pressure within the star can be neglected. The total time of collapse for an observer comoving with the stellar matter is finite, and for this idealized case and typical stellar masses, of the order of a day; an external observer sees the star asymptotically shrinking to its gravitational radius.

I

RECENTLY it has been shown[1] that the general relativistic field equations do not possess any static solutions for a spherical distribution of cold neutrons if the total mass of the neutrons is greater than $\sim 0.7\odot$. It seems of interest to investigate the behavior of nonstatic solutions of the field equations.

In this work we will be concerned with stars which have large masses, $>0.7\odot$, and which particles in the star, (2) radiation, (3) potential and kinetic energy of the outer layers of the star which could be blown away by the radiation, (4) rotational energy which could divide the star into two or more parts. If the mass of the original star were sufficiently small, or if enough of the star could be blown from the surface by radiation, or lost directly in radiation, or if the angular momentum of the star were great enough to split it into small fragments, then the re-

2. Mathematical black holes

·Doubts of existence of such objects even mathematically. No spherical symmetry could imply no singularities. Lifshitz and Khalatnikov (1963) showed a bounce solution: Singularities would not form when asymmetrical perturbations of general character are taken into account.

·But general relativity is nonlinear. One might think that cases where the collapse is relentless and sufficiently strong, such as in the original Oppenheimer–Snyder model of collapse, the formation of a singularity essentially holds and a singularity is inescapable.

·To solve the problem Penrose hit on the notion of a trapped surface: A closed spacelike 2-surface all of whose null normals converge in future directions. This means that ingoing and outgoing null geodesics, realized by light rays, both converge. Characterizes the notion of a very strong gravitational field.

·By the essence of the trapped surface concept, any small perturbation of some initial data containing trapped surfaces, should also contain trapped surfaces, independently of any symmetry hypotheses.

2. Mathematical black holes

·Penrose theorem (1965): Spacetime (M,g) in general relativity cannot be null geodesically complete if: (i) At every point of M, all future pointing timelike vectors v^a satisfy $T_{ab}v^a v^b \geq 0$, weak energy condition; (ii) there is a noncompact Cauchy surface H^3 in M; (iii) there is a closed trapped surface T^2 in M.

·This is the Nobel Prize theorem. The Royal Swedish Academy of Sciences has decided to award the Nobel Prize in Physics 2020 with one half to Roger Penrose, University of Oxford, UK, "for the discovery that black hole formation is a robust prediction of the general theory of relativity". See Herdeiro, Lemos (2018).

·So, if a spacetime subject to certain physically reasonable restrictions contains a trapped surface, then it cannot be extended. This signals the presence of a singularity. It is usually taken to mean that infinite curvatures characterize the singularity which prevents the further evolution of the spacetime. Belinski, Khalatnikov, and Lifshitz (1970) corrected an own error and found that there were many more general solutions and in effect singularities arise in collapse. It gave rise to the BKL proposal for singularities.

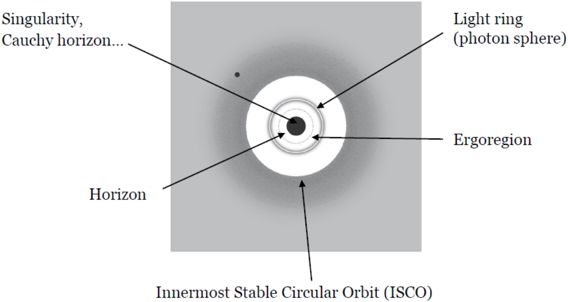
2. Mathematical black holes

- Around that time appeared the Kerr black hole (1963) with mass M and angular momentum $J = aM$,

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 + \frac{2a(\Delta - r^2 - a^2)\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2,$$

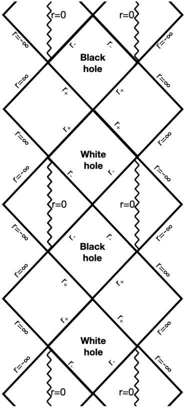
$\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 + a^2 - 2Mr$, a new impetus. The Kerr-Newman black hole (1965) includes electric charge.

- A snapshot of a Kerr black hole.



2. Mathematical black holes

- The maximal extension of a black hole spacetime (Kruskal, Szekeres 1960, Fueller, Wheeler 1962) was a revelation, and the importance of a conformal analysis on a spacetime was manifest in bringing infinities to finite regions, and the consequent understanding of the full spacetime causal structure and its null rays, as displayed in Carter-Penrose diagrams (1963,1967).



The conformal diagram for a Kerr black hole.

2. Mathematical black holes

- Uniqueness (Israel 1967, Carter 1971, Robinson 1971), cosmic censorship (Penrose 1969), black holes have no hair (Wheeler 1971), hoop conjecture (Thorne 1972). Very hard problems.
- Uniqueness: If the conditions of the theorems do not hold then uniqueness falls. Hawking (1973) showed that the event horizon has spherical topology. But cylindrical black holes with angular momentum exist in general relativity (Lemos 1994).
- Cosmic censorship: There are no naked singularities. However, negative mass Schwarzchild, overcharged Reissner-Nordström, overspinning Kerr are naked singularities. Formation of naked singularities from gravitational collapse have been found (Christodoulou 1984, Lemos 1992, James 2022), perhaps nongeneric.
- Reversing the question, are there nonsingular black holes? Have to be null geodesically complete in spite of trapped surfaces. Spacetime exhibits smooth changes of topology from open to closed and compact. So, there is violation of weak energy condition according to the Penrose theorem (Bardeen 1968, Lemos, Zanchin 2011, Lemos, Luz 2021, Masa, Lemos, Zanchin 2023).

2. Mathematical black holes

·What about objects on the verge of becoming a black hole? These are the quasiblack holes (Lemos, Weinberg 2003). Let, $ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2\,d\Omega^2$. Let an inner matter configuration, with an asymptotic flat exterior region, exist with the properties (a) the function $\frac{1}{A(r)}$ attains a minimum at some $r^* \neq 0$, such that $\frac{1}{A(r^*)} = \varepsilon$, with $\varepsilon \ll 1$, this minimum being achieved from both sides of r^*, (b) for such a small but nonzero ε the configuration is regular everywhere with a nonvanishing metric function B, and (c) in the limit $\varepsilon \to 0$ the metric coefficient $B \to 0$ for all $r \leq r^*$. These three features define a QBH (Lemos, Zaslavskii 2007, Lemos, Fernandes 2023). See also gravastars and extremely compact objects, related to quasiblack holes with different aims.

·No hair: Well, no hair generically is a failure. Any hair is a hair.

·Hoop conjecture: a collapsing mass forms a black hole when and only when a circular hoop with a specific critical circumference can be placed around the object and rotated about its diameter. Hard to prove (Schoen, Yau 1983, Hirsch, Kazaras, Khuri, Zhang 2024).

·Are there Newtonian black holes? If there are (Lemos 2025), they have nothing to do with the dark stars of Mitchell (1784) and Laplace (1798).

3. Astrophysical black holes

- It is relevant to go back to 1939. With Chandrasekhar's maximum mass for white dwarfs (1929), Landau's postulation of neutron stars (1932), plus Zwicky's ideas for neutron stars at the center of supernovas (1934), Oppenheimer and Volkoff (1939) set to make a proper calculation of the maximum mass for neutron stars. Then, came the Oppenheimer-Snyder 1939 paper. After neutron stars ideas and gravitational collapse all seemed a dead end.
- The discovery of quasars by Schmidt (1963) with the 5m telescope at Palomar, using the accurate position for one of powerful radio sources identified as 3C273, was decisive. Its starlike appearance suggested it was nearby, but its spectrum had a high redshift of 0.16, so it was far in the cosmos, and should have an extraordinarily high luminosity. Moreover, did not seem to be associated with any galaxy. Termed quasi stellar objects, then quasars.
- With it, the idea of total gravitational collapse of millions of stars appeared in the Relativistic Astrophysics Symposium in Dallas Texas (1963). The remarkable presentation by Wheeler brought out an extraordinary book: Gravitational Theory and Gravitational Collapse, signed by Harrison, Thorne, Wakano, and Wheeler (1965).

3. Astrophysical black holes

- Neutron stars were confirmed in 1967 in the form of pulsars by Hewish and Bell, and the black hole puzzle in our galaxy was closing. Indeed, Cygnus X1, was identified as a $21M_\odot$ black hole in 1971 (Webster, Murdin 1971). From then on extraordinary evidence for black holes in the Galaxy. There are around 10^8.
- Turning against the common view that quasars were not associated with galaxies, Salpeter (1967) treated them as an accretion disk around a large black hole. Derived the power emitted from the gravitational binding energy of the last stable circular orbit and deduced consequences for such black holes accreting as they traveled through the interstellar gas of a galactic disc.
- For quasars, the black hole received little attention until Lynden-Bell (1969) argued that dead quasars in the form of black holes were common in galactic nuclei. Quiescent ones might be detectable through their effect on the mass-to-light ratio of nearby galaxies. Different values of the black hole mass and accretion rate give an explanation for all of the phenomena of high-energy astrophysics: Galactic nuclei, Seyfert galaxies, and quasars. Estimated the black hole masses of the Galaxy, the Magellanic clouds, M31, M32, M81, M82, NGC4151, and M87.

3. Astrophysical black holes

· Then, Bardeen (1970) showed how accretion spins up a black hole leaving a near-extremal Kerr one of greater efficiency. Reversal of torque in accreting disks around Kerr black holes was found (Anderson, Lemos 1988).

·The prediction that many large galaxies, including our own, contain giant black hole remnants of dead quasars in their nuclei, took over 26 years before the first definitive case was found (Miyoshi 1995), a work on megamasers definitely confirming that such inner discs orbit around black holes.

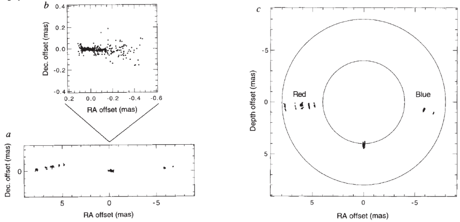
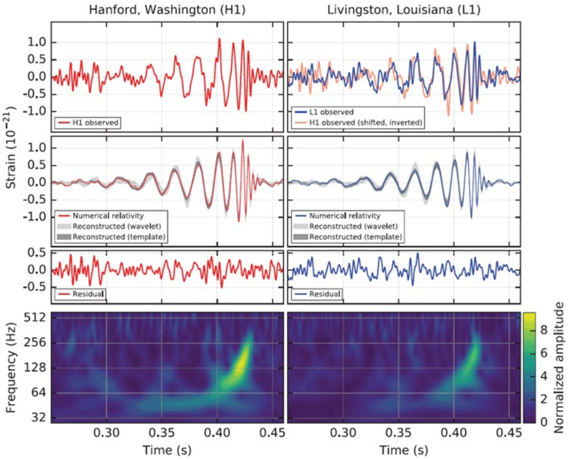


FIG. 2 *a*, Map of the distribution of the water masers in the NGC4258 (RA, right ascension; dec., declination). All features greater than 5σ are plotted (20–40 mJy). The position errors are $\sim 5 \times 10$ microarcsec for the systemic features, and $\sim 20 \times 40$ microarcsec for the high-velocity features. The origin of the coordinates is referred to the feature at 512.5 km s^{-1}. *b*, Expanded view of the emission near the systemic velocity of the galaxy. The features at RA < -0.2 mas are weak and the scatter in positions is due to measurement noise. *c*, Top view of the molecular disk. The positions along the line-of-sight (depth) direction were calculated from the line-of-sight velocities and the calculated velocity field of a simple edge-on disk based on the binding mass required by the high-velocity features. Only those features that were measured accurately enough to provide errors in the depth direction of <0.15 mas are plotted. The circles mark the inner and outer edges of the observed molecular disk.

·Now taken for granted. Work by Kormendy and Richstone (1995) on external galaxies, and the beautiful results of Genzel (2003) and Ghez (2004) on our galaxy, for which they shared the Nobel prize 2020, transformed the situation.

3. Astrophysical black holes

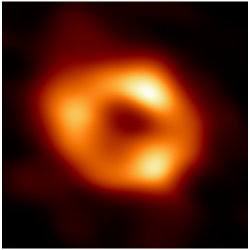
·Game changer year: 2015. The Physical Review Letters of 12 February 2016, published a paper by Ligo/Virgo on the gravitational wave detection on 14 September 2015, 08:50:45 (Rio time).



·These data require a binary black hole merger of $35M_\odot$ each.

3. Astrophysical black holes

·Event Horizon Telescope in 2019 photographed at microwave band the black hole in M87. In 2022 photographed the central black hole of $4.3 \times 10^6 M_\odot$ of our galaxy:



·Pulsar Timing Array in 2023 has reported evidence of a background of gravitational waves indicating mergers of supermassive black holes.

·More will come. Soon we will have black holes from $3M_\odot$ to $10^{11}M_\odot$.

·Primordial black holes of masses of $10^{-18}M_\odot$ are speculated to exist and could be detected in its final explosion by now due to Hawking radiation.

4. Quantum black holes

·We can say that the embryonic ideas started with the Penrose process (1969), where energy could be extracted from black holes. Showed nontrivial dynamics from Kerr black holes.

·Then appeared Christodolou's irreducible mass (1970), the area theorem of Hawking (1971), Bekenstein black hole entropy (1972), Smarr formula (1973), and the four laws of black hole mechanics of Bardeen, Carter, Hawking (1973).

·Hawking black hole radiation (1974) revolutionized the field, black holes were quantum and thermodynamic objects.

·Definition of vacua, statistical mechanics, and thermodynamics of black holes. In particular the Euclidean path integral formulation of quantum gravity of Gibbons and Hawking (1977) put on firm ground by York (1986).

·My works developed it in several directions: Lemos (1996), Peça, Lemos (1999), André, Lemos (2020, 2021). Lemos, Zaslavskii (2023, 2024), Fernandes, Lemos (2024), Fernandes, Gandum, Lemos (2025), Fernandes, Lemos, Zaslavskii (2025).

4. Quantum black holes

- It is granted that there are three fundamental constants: everything gravitates G, everything moves c, and all is quantum $\hbar$. All the other constants are not fundamental.
- With these three fundamental constants one can make a mass, a length, and a time,

$$m_{\rm pl} = \sqrt{\frac{\hbar c}{G}},$$

$$l_{\rm pl} = \sqrt{\frac{\hbar G}{c^3}},$$

$$t_{\rm pl} = \sqrt{\frac{\hbar G}{c^5}},$$

put forward by Planck in 1900, and devised earlier by Stoney in 1881 with e instead of $\hbar$. Since $\frac{e^2}{\hbar c} = \frac{1}{137}$, either e or $\hbar$ will do.
- To have an idea, the values are $m_{\rm pl} = 10^{-5}\,$gm, $l_{\rm pl} = 10^{-33}\,$cm, $t_{\rm pl} = 10^{-44}\,$s. One can put all three equal to one and everything is measured in terms of these scales.
- The Planck scale is where quantum gravity is supposed to take hold.

4. Quantum black holes

·Let us start with the Hawking thermal temperature for a black of hole of mass M, $T_{\rm th} = \frac{1}{8\pi k_{\rm B}}\left(\frac{\hbar c^3}{G}\right)\frac{1}{M}$. It involves the three fundamental constants and $k_{\rm B}$. Is it fundamental? Perhaps yes. Recall, $m_{\rm pl} = \sqrt{\frac{\hbar c}{G}}$. So $\frac{k_{\rm B}T_{\rm th}}{c^2} = \frac{1}{8\pi}\frac{m_{\rm pl}^2}{M}$. Define $m_{\rm th} = \frac{k_{\rm B}T_{\rm th}}{c^2}$ and turn things around to define a Hawking cloud with Hawking mass $M_{\rm H}$ once $m_{\rm th}$ is given,

$$M_{\rm H} = \frac{1}{8\pi}\left(\frac{m_{\rm pl}}{m_{\rm th}}\right) m_{\rm pl}\,.$$

·Compare with the Chandrasekhar mass $M_{\rm C} = \frac{\sqrt{3\pi}}{4}\left(\frac{\hbar c}{G}\right)^{\frac{3}{2}}\frac{1}{m_{\rm n}^2}$, $m_{\rm n} = m_{\rm p}$ the neutron or the proton mass. So,

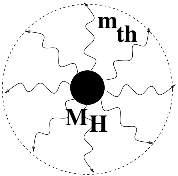
$$M_{\rm C} = \frac{\sqrt{3\pi}}{4}\left(\frac{m_{\rm pl}}{m_{\rm n}}\right)^2 m_{\rm pl}\,.$$

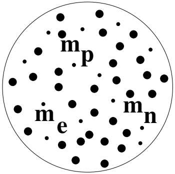
This is the mass of a Chandrasekhar star, i.e., the maximum mass of a white dwarf. It is also the mass of a Landau-Oppenheimer-Volkoff star, i.e., the maximum mass of a neutron star.

·So $m_{\rm th}$ and $m_{\rm n}$ play a similar role in defining the masses $M_{\rm H}$ and $M_{\rm C}$. The tendency is to say that the neutron mass $m_{\rm n}$, or $m_{\rm p}$, is not fundamental.

4. Quantum black holes

·Let us draw the two corresponding figures.





$$M_{\rm H} = \frac{1}{8\pi}\left(\frac{m_{\rm pl}}{m_{\rm th}}\right) m_{\rm pl}\,, \qquad M_{\rm C} = \frac{\sqrt{3\pi}}{4}\left(\frac{m_{\rm pl}}{m_{\rm n}}\right)^2 m_{\rm pl}\,,$$

$$R_{\rm H} = \lambda_{\rm th}\,, \qquad R_{\rm C} = \frac{\lambda_{\rm e}\lambda_{\rm n}}{l_{\rm pl}^2}\,, \quad R_{\rm LOV} = \frac{\lambda_{\rm n}\lambda_{\rm n}}{l_{\rm pl}^2}\,,$$

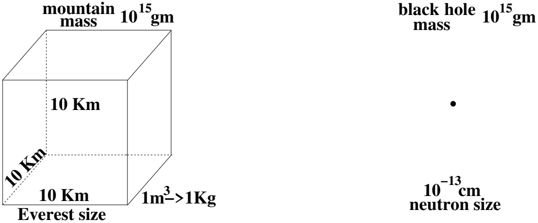
where λs are the associated Compton wavelengths, $\lambda = \frac{\hbar}{mc}$. The expressions are somehow similar. Why consider them of different character? Possible answer: $m_{\rm th}$ and $\lambda_{\rm th}$ are statistical and so involve only quantities that enter into the system itself, i.e., Planck quantities.

4. Quantum black holes

·Let us put numbers.

$$T_{\rm th} = 6\times 10^{-8}\left(\frac{M_\odot}{M_{\rm H}}\right)\ {\rm K}\,.$$

$M_{\rm H} = M_\odot$, $r_{\rm H} = 3\,{\rm Km}$, $T_{\rm th} = 6\times 10^{-8}$ K.
$M_{\rm H} = M_{\rm earth} = 3\times 10^{-6} M_\odot$, $r_{\rm H} = 2\,{\rm cm}$, $T_{\rm th} = 2\times 10^{-2}$ K.
$M_{\rm H} = M_{\rm mercury} = 1.15\times 10^{-7} M_\odot$, $r_{\rm H} = 0.03\,{\rm cm}$, $T_{\rm th} = 0.5$ K.
$M_{\rm H} = M_{\rm asteroid} = 2\times 10^{-10} M_\odot$, $r_{\rm H} = 6\times 10^{-5}\,{\rm cm}$, $T_{\rm th} = 300$ K.
$M_{\rm H} = M_{\rm mountain} = 5\times 10^{-19} M_\odot$, $r_{\rm H} = 10^{-13}\,{\rm cm}$, $T_{\rm th} = 1.2\times 10^{11}$ K.
For mountain mass black holes $M_{\rm H} = M_{\rm mountain} = 5\times 10^{-19} M_\odot = 10^{15}$ gm, Hawking radiation makes an impact. Such a black hole has the mass of an Everest 10^{15} gm and size of a neutron 10^{-13} cm, a really interesting black hole.



Comparing the Everest as a cube and a black hole of neutron size.

4. Quantum black holes

·This is the kind of system we are interested: Microscopic 10^{-13} cm and high T 10^{11} K. Quantum effects are important, but not full quantum gravity.

·If the system is a pure black hole, left by itself it evanesces. Due to the breaking of the vacuum from the strong gravitational field near the event horizon of a black hole, particles, such as gravitons or other matter fields, are emitted to infinity with a definite temperature, the Hawking temperature, the black hole would radiate on and on and eventually disappears.

·One way to understand more deeply the conversion between black holes and hot matter fields, one can enclose the black hole and the hot matter in a cavity inside a heat reservoir maintained at constant temperature and constant radius, characterizing the statistical mechanics canonical ensemble. A thermodynamic treatment for the system black hole plus hot matter makes sense and is then possible (Lemos, Zaslavskii 2024).

·But here let us simplify. Put matter in anti-de Sitter (AdS) space in the form of a thin shell and then compare with Hawking-Page (1983) black hole results. AdS is a space that acts as a cavity, have to worry with asympotic boundary conditions. Let us start, see also (Fernandes, Gandum, Lemos 2025).

4. Quantum black holes

·Hawking and Page (1983) studied black holes in AdS using Gibbons-Hawking (1977) path integral approach to quantum gravity. Here we study matter in AdS using York's (1986) path integral formalism which is much more powerful.

·Feynman's path integral expresses the amplitude of quantum states at different times as given by

$$\langle g(0),\phi(0)|g(t_1),\phi(t_1)\rangle = \langle g(0),\phi(0)|\mathrm{e}^{-i\int_0^{t_1} H dt}|g(0),\phi(0)\rangle = \int_{g(0),\phi(0)} DgD\phi\, \mathrm{e}^{iI[g,\phi]}.$$

·We want to use the Euclidean path integral approach to quantum gravity. Perform a complex analytic extension of metrics with the identification $t=-i\tau$, $\int_0^{2\pi}\sqrt{g_{\tau\tau}}d\tau=\beta$, and sum over a basis to get $Z[\beta]=\mathrm{Tr}\,\mathrm{e}^{-\beta H}=\sum_i \mathrm{e}^{-\beta E_i}$.

·Then,

$$Z[\beta]=\int_{g(0)=g(2\pi),\phi(0)=\phi(2\pi)} DgD\phi\, \mathrm{e}^{-I[g,\phi]} \quad ,$$

I is now the Euclidean action. Thus, identify $T=\frac{1}{\beta}$, T the reservoir temperature. Clearly, the Euclidean path integral yields the canonical statistical ensemble partition function $Z[\beta]$ or $Z[T]$.

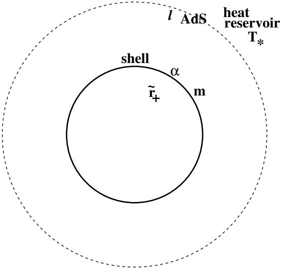
4. Quantum black holes

·Since there is gravity and matter, we split the path integral as

$$Z[\beta] = \int_{g(0)=g(2\pi)} Dg\, e^{-I_g[g]} \int_{\phi(0)=\phi(2\pi)} D\phi\, e^{-I_\phi[g,\phi]}.$$

One cannot perform the path integral over matter in a closed form. Yet, we consider spherically symmetric metrics and matter described by spherically symmetric shell. Then, $\int_{\phi(0)=\phi(2\pi)} D\phi\, e^{-I_\phi[g,\phi]} = e^{-I_m[g]}$. Now, put $I_m[g] = \int_M \mathscr{F}[g]\sqrt{g}\, d^4x$, where $\mathscr{F}[g]$ is the free energy density of the matter in the shell, with $d\mathscr{F} = -s_m dT_m + \frac{\mathscr{F}+p_m}{n} dn$, and $n = \frac{1}{\sqrt{hh^{\tau\tau}}}$, with h_{ab} the induced metric. The partition function is

$$Z[\beta] = \int_{g(0)=g(2\pi)} Dg\, e^{-I_{\text{eff}}[g]}, \qquad I_{\text{eff}}[g] = I_g[g] + \int_M \mathscr{F}[g]\sqrt{g}\, d^4x.$$



4. Quantum black holes

·The action is

$$I_{\rm eff} = -\int_{M-\{A\}} \frac{R+\frac{6}{l^2}}{16\pi}\sqrt{g}\,d^4x - \int_{\partial M}\frac{K}{8\pi}\sqrt{\gamma}\,d^3x + \int_A\left(\frac{[K]}{8\pi}+\mathscr{F}[h]\right)\sqrt{h}\,d^3x - I_0,$$

M is the spacetime, A the shell, R the Ricci scalar of g_{ab}, l the AdS length, K the extrinsic curvature at infinity, $[K]=K_2-K_1$, $\mathscr{F}[h]$ the free energy density at the shell, h_{ab} the induced metric on the shell, and I_0 a reference value.

·The metric is

$$\cdot\, ds_1^2 = \frac{b_1^2(y)b_2^2(y_m)}{b_1^2(y_m)}d\tau^2 + v_1^2(y)dy^2 + r^2(y)d\Omega^2\,, \qquad y\in[0,y_m[.$$

$$\cdot \text{ Shell at } r(y_m)=\alpha, \qquad y=y_m$$

$$\cdot\, ds_2^2 = b_2^2(y)d\tau^2 + v_2^2(y)dy^2 + r^2(y)d\Omega^2\,, \qquad y\in]y_m,\infty[.$$

·The boundary conditions are

$$\cdot y=0\text{: } b_1(0) \text{ is finite}, \frac{b_1'}{v_1}(0)=0, r(0)=0, \qquad \text{i.e., nonsingular conditions.}$$

$$\cdot y=\infty\text{: } \frac{b_2(y)}{r(y)} = \frac{\beta_*}{2\pi l}, \qquad \text{i.e., asymptotic AdS conditions.}$$

4. Quantum black holes

·To find the zero loop approximation, minimize the action by varying it in relation to the metric function b. One obtains the Hamiltonian constraint $G^\tau_\tau = 0$. It gives for the three regions,

- $\left(\frac{r'}{v_1}\right)^2 = 1 + \frac{r^2}{l^2} \equiv f_1(r),$ pure AdS,
- $m(\tilde{r}_+, \alpha) = \alpha\left(\sqrt{f_1(\alpha)} - \sqrt{f_2(\alpha)}\right),$ junction condition,
- $\left(\frac{r'}{v_2}\right)^2 = 1 + \frac{r^2}{l^2} - \frac{\tilde{r}_+}{r}\left(1 + \frac{\tilde{r}_+^2}{l^2}\right) \equiv f_2(r),$ Schwarzschild AdS,

  where $r' = \frac{dr}{dy}$.

·The Hamiltonian constraint yields no dependence on $b_1(y)$ and $b_2(y)$. One can choose $y(r)$, a coordinate change. It only changes $\alpha = r(y_m)$. The function v_1 is fully determined, while v_2 is a functional of $\tilde{r}_+$. Thus, $Dv_1 Dv_2 Dr = D\tilde{r}_+ D\alpha$. The partition function is

$$Z[\beta_*] = \int D\tilde{r}_+ D\alpha \, e^{-I[\beta_*;\tilde{r}_+,\alpha]}, \quad I[\beta_*;\tilde{r}_+,\alpha] = \frac{\beta_*}{2}\left(\tilde{r}_+ + \frac{\tilde{r}_+^3}{l^2}\right) - S_m(m(\tilde{r}_+,\alpha),\alpha),$$

with I being the reduced action.

4. Quantum black holes

·In performing the path integral based on the action, we enforced the constraints, only those histories that stay on the constraint hypersurface of the gravitational phase space are counted. I.e., the constraints were solved and eliminated from the Einstein-Hilbert action, producing a reduced action that depends upon two remaining degrees of freedom, the shell's gravitational radius $\tilde{r}_+$ and the shell's radius α, as well as on the thermodynamic data β_*.

·One could then perform the integral $Z[\beta_*] = \int D\tilde{r}_+ D\alpha \, \mathrm{e}^{-I[\beta_*;\tilde{r}_+,\alpha]}$ with $I[\beta_*;\tilde{r}_+,\alpha] = \frac{\beta_*}{2}(\tilde{r}_+ + \frac{\tilde{r}_+^3}{l^2}) - S_m(m(\tilde{r}_+,\alpha),\alpha)$. More practical, to find where the action is a minimum, I_0, with $\frac{\partial I}{\partial \alpha} = 0$ and $\frac{\partial I}{\partial \tilde{r}_+} = 0$. Then,

$$Z[\beta_*] = \mathrm{e}^{-I_0[\beta_*]} \quad I_0[\beta_*] = I[\beta_*;\tilde{r}_+(\beta_*),\alpha(\beta_*)].$$

The solution depends only on the boundary data β_*, i.e., on the reservoir temperature $T_* = \frac{1}{\beta_*}$.

4. Quantum black holes

- To determine if the solutions are minima of the action and thus stable, we must go beyond the zero loop approximation and expand the action and the path integral around the stationary point. Thus, we write $I_* = I_0 + \left(\frac{\partial^2 I_*}{\partial \tilde{r}_+^2}\right)_0 \delta\tilde{r}_+^2 + \left(\frac{\partial^2 I_*}{\partial \alpha^2}\right)_0 \delta\alpha^2$, where the subscript 0 means that the quantity inside parenthesis is evaluated at the stationary point, $I_0 = I_*(\beta_*, R; (\tilde{r}_+)_0, (\alpha)_0)$, and $\delta\tilde{r}_+ = \tilde{r}_+ - (\tilde{r}_+)_0$, $\delta\alpha = \alpha - (\alpha)_0$. Then, the partition function can be expanded as
$$Z = \mathrm{e}^{-I_0} \int D\delta\tilde{r}_+ \mathrm{e}^{-\left(\frac{\partial^2 I_*}{\partial \tilde{r}_+^2}\right)_0 \delta\tilde{r}_+^2} \int D\delta\alpha \mathrm{e}^{-\left(\frac{\partial^2 I_*}{\partial \alpha^2}\right)_0 \delta\alpha^2}.$$
The partition function contains one loop contributions that obey the spherical symmetry of the geometry, the boundary conditions, and the Hamiltonian and Gauss constraints. For the path integral to be well-defined, we must have
$$\left(\frac{\partial^2 I_*}{\partial \tilde{r}_+^2}\right)_0 > 0, \quad \left(\frac{\partial^2 I_*}{\partial \alpha^2}\right)_0 > 0.$$
so that the stationary point is a minimum and stable.

- Before we calculate the differentials, we have to state the properties of the matter in the shell.

4. Quantum black holes

·Matter in the shell has to obey the first law of termodynamics

$$T_m(m,\alpha)dS_m = dm + p_m(m,\alpha)dA,$$

$A = 4\pi\alpha^2$.

·Choose an equation of state of the form

$$S_m(m,\alpha) = c_0\, l_c^{\frac{1}{4}}\, m^{\frac{3}{4}}\left(4\pi\alpha^2\right)^{\frac{1}{4}},$$

l_c the Compton wavelength of the matter. From the differentials get

$$p_m = \frac{m}{12\pi\alpha^2}.$$

And get

$$\frac{1}{T_m} = \frac{3}{4}\, c_0\, l_c^{\frac{1}{4}}\left(\frac{4\pi\alpha^2}{m}\right)^{\frac{1}{4}}.$$

Recall that $m = m(\tilde{r}_+,\alpha)$ from the gravitation side so, $p_m = p_m(m(\tilde{r}_+,\alpha),\alpha)$, $T_m = T_m(m(\tilde{r}_+,\alpha),\alpha)$, and $S_m = S_m(m(\tilde{r}_+,\alpha),\alpha)$.

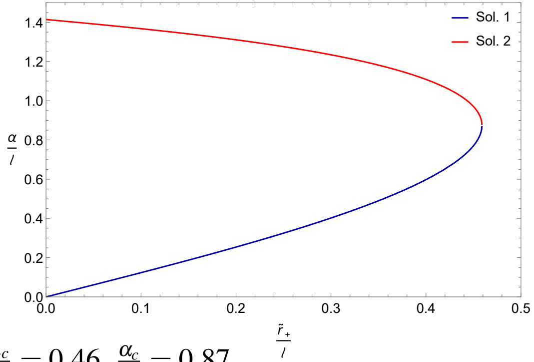
4. Quantum black holes

·One stationary point of the reduced action I is given by $\frac{\partial I}{\partial \alpha} = 0$, i.e.,

$$p_m(m(\tilde{r}_+, \alpha), \alpha) = p_{\text{GR}}(\tilde{r}_+, \alpha).$$

with $p_m = \frac{m(\tilde{r}_+,\alpha)}{12\pi\alpha^2}$ and $p_{\text{GR}}(\tilde{r}_+, \alpha) = \frac{1}{8\pi\alpha}\left(\frac{2f_1(\alpha)+f_2(\alpha)-2}{2\sqrt{f_2(\alpha)}} - \frac{2f_1(\alpha)-1}{2\sqrt{f_1(\alpha)}}\right)$.

·Two solutions.



Critical value $\frac{\tilde{r}_{+c}}{l} = 0.46$, $\frac{\alpha_c}{l} = 0.87$.

·Stability $\frac{\partial^2 I}{\partial \alpha^2} \leq 0$, i.e., $\partial_\alpha(p_{\text{GR}} - p_m) \geq 0$, happens when $\partial_{\tilde{r}_+}\alpha \leq 0$. This is precisely mechanical stability condition.

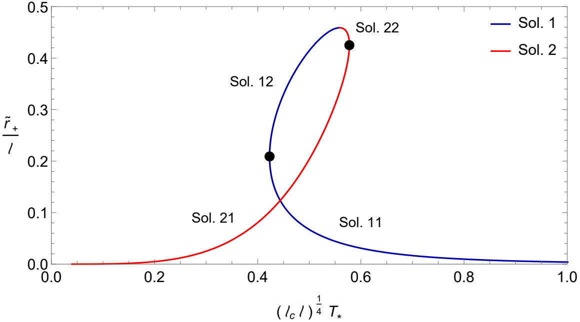
4. Quantum black holes

- The other stationary point of the reduced action I, $\frac{\partial I}{\partial \tilde{r}_+} = 0$, gives $T_* = T_m(m(\tilde{r}_+, \alpha(\tilde{r}_+)), \alpha(\tilde{r}_+)) \sqrt{f_2(\tilde{r}_+, \alpha(\tilde{r}_+))}$. Inverting gives

$$\tilde{r}_+ = \tilde{r}_+(T_*),$$

  or $\alpha = \alpha(T_*)$. The shell solution is given in terms of the ensemble quantities. There are solutions, two for each $\alpha(\tilde{r}_+)$.

- Stability means $\frac{\partial^2 I}{\partial \tilde{r}_+^2} \leq 0$, i.e., $\partial_{T_*} \tilde{r}_+ \geq 0$. Sol_{11} and Sol_{12} are mechanically unstable. Sol_{22} is thermodynamically unstable. Sol_{21} is fully stable.

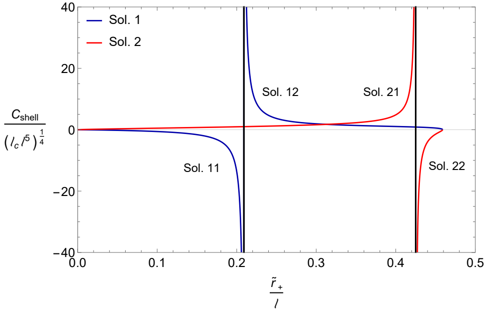


4. Quantum black holes

·To obtain thermodynamics, the zero loop approximation relates free energy F to the reduced action, $F = \frac{I}{T_*}$, $F = \frac{1}{2T_*}(\tilde{r}_+ + \frac{\tilde{r}_+^3}{l^2}) - S_m(m(\tilde{r}_+, \alpha), \alpha)$, with $\tilde{r}_+[T_*]$ and $\alpha[\tilde{r}_+[T_*]]$.

·The energy is given by $E = \frac{1}{2}(\tilde{r}_+ + \frac{\tilde{r}_+^3}{l^2})$ and entropy $S = E - \frac{F}{T_*} = S_m(T_*)$, also $S = S(\tilde{r}_+) = S(\tilde{A}_+) = \frac{1}{3}\tilde{A}_+^{\frac{3}{5}}$, approximately. Compare $S_{\rm bh} = \frac{1}{4}\tilde{A}_+$.

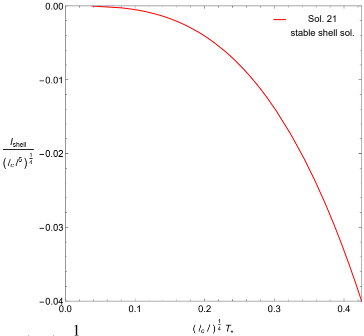
·The thermodynamic stability is given by the heat capacity $C_{\rm shell} \geq 0$, $C_{\rm shell} = \left(\frac{\partial E}{\partial T}\right)_\alpha$.

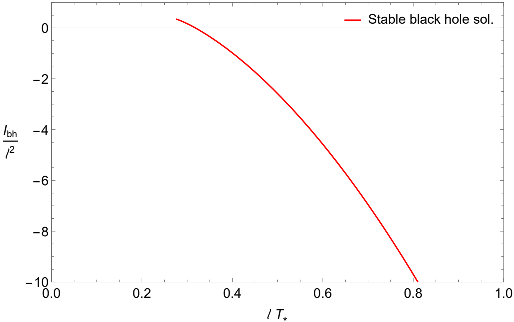


There is a second order phase transition from thermodynamically unstable to stable solutions.

4. Quantum black holes

- Consider now two thermodynamic states, matter shell in AdS and the Hawking-Page black hole in AdS. The one with lower free energy dominates.





If $\left(\frac{l^3}{l_c}\right)^{\frac{1}{4}} > 0.48$, then there is a range of temperatures that both solutions exist. There is a temperature T_{int} at which the free energy of both is the same. From $0 < T < T_{\text{int}}$ the shell is favorable, from $T_{\text{int}} < T$ the black hole is favorable.

- The shell solution acts as version of hot AdS. The shells have temperatures that mimick black holes, so we are considering microscopic objects, where the quantum nature of matter, radiation, and black hole can be revealed.

5. Conclusions

- It has been a long way since 1916 the year of the discovery of the Schwarzschild solution, the solution that would be later transformed in a black hole solution. Indeed, it has passed 108 years, 3.4 billion seconds, with many people working at it relentlessly.
- It was an extremely difficult journey. But it has paid off. We now know much more about black holes than in the beginning of last century.
- What does remain to be understood? A lot.
  - On the mathematical side one has still to prove at various levels uniqueness, cosmic censorship, no hair, and the hoop conjecture.
  - On the astrophysical side it seems we have seen nothing. A plethora of new events in high energy astrophysics involving black holes at all scales are there to be unveiled sooner or later.
  - On the quantum side there is: 1. The information paradox along with its fires to be solved. 2. The interior of the black hole is yet to be explained. What is the singularity made of? Surely it is a quantum Planck scale object, but cannot be that drastic. After all Hawking radiation, a tamed radiation, slowly peels it away.