Dynamical invariants and quantization of the one-dimensional time-dependent, damped, and driven harmonic oscillator

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Statement of the problem

- General linear 1D oscillator:

$$\ddot{x} + 2g(t)\,\dot{x} + \omega^2(t)\,x = \frac{1}{m}F(t)$$

- First-oder equations (usual):

$$\dot{x} = y, \qquad \dot{y} = -2gy - \omega^2 x + F/m$$

- x and y are not canonical variables:

$$\frac{d}{dt}[x,y] = -2g[x,y], \quad \Longrightarrow \quad [x,y] = i\hbar\exp(-2gt)$$

- Transformation to canonical variables

$$x = q, \qquad y = e^{-2gt}p/m$$

$$\dot{q} = e^{-2gt}p/m, \qquad \dot{p} = -m\omega^2 e^{2gt}q + e^{2gt}F$$

- Bateman-Caldirola-Kanai (BCK) type Hamiltonian ($G \equiv 2gt$) [2, 3, 4]:

$$H = \frac{1}{2m}e^{-G(t)}p^2 + \frac{1}{2}m\omega^2(t)\,e^{G(t)}q^2 - e^{G(t)}F(t)\,q$$

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Linear invariants of the BCK hamiltonian

- Fundamental commutation relations

$$[q,p] = i\hbar\mathbf{1},\ [q,q] = 0, \quad [p,p] = 0$$

- Two functions $\alpha(t)$, $\beta(t)$: [5]

$$\begin{aligned} \dot{q} &= e^{-G}p/m, \qquad \times\alpha \\ \dot{p} &= e^{G}F - e^{G}m\omega^2 q, \qquad \times\beta \end{aligned}$$

- And a function $\mathcal{F}$:

$$\mathcal{F}(\beta,t) \equiv \int_{t_0}^{t} e^{G(\tau)}\beta(\tau)F(\tau)\,d\tau,$$

$$\beta(t_0)F(t_0) = 0, \quad \Longrightarrow \quad \beta e^{G}F = \frac{d\mathcal{F}}{dt},$$

- Resulting

$$\frac{d}{dt}(\beta p + \alpha m q - \mathcal{F}) = \left(\alpha e^{-G} + \dot{\beta}\right)p + m\left(\dot{\alpha} - e^{G}\omega^2\beta\right)q.$$

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Linear invariants of the BCK hamiltonian

- The operator

$$I = \beta p + \alpha m q - \mathcal{F}(\beta, t)$$

is a first-order invariant iff

$$\alpha + e^{G}\frac{d\beta}{dt} = 0, \qquad \frac{d\alpha}{dt} - e^{G}\omega^2\beta = 0,$$

- We may eliminate α:

$$\ddot{\beta} + 2g\dot{\beta} + \omega^2\beta = 0$$

- So

$$I = \beta p - m e^{G}\dot{\beta} q - \mathcal{F} \qquad \Longrightarrow \qquad \frac{dI}{dt} = 0.$$

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- If $\beta = \rho\left(t\right)e^{i\phi(t)}$ is a solution for ρ and ϕ real function, $\beta^* = \rho\left(t\right)e^{-i\phi(t)}$ is also a solution, so we have two L.I. first-order invariants:

$$I \equiv \beta p - me^G\dot{\beta}q - \mathcal{F}$$
$$I^\dagger \equiv \beta^* p - me^G\dot{\beta}^* q - \mathcal{F}^*.$$

- We have

$$\left[I, I^\dagger\right] = \Omega\mathbf{1},\ \Omega \equiv im\hbar e^G W,\ \text{and}\ W \equiv \dot{\beta}^*\beta - \beta^*\dot{\beta},$$

  the remaining relations are just $[I, I] = \left[I^\dagger, I^\dagger\right] = 0$.

- Besides, Ω is a constant function.

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Ladder and number operators

- We introduce

$$a \equiv \frac{I}{\sqrt{\Omega}}, \qquad a^\dagger \equiv \frac{I^\dagger}{\sqrt{\Omega}},$$

- Which obey the commutation relations

$$\left[a, a^\dagger\right] = \mathbf{1}, \quad [a, a] = \left[a^\dagger, a^\dagger\right] = 0.$$

- Since a and $a^\dagger$ are invariants, the number operator

$$\hat{\boldsymbol{n}} \equiv a^\dagger a,$$

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Quantum states

- In this case, we suppose a basis $\{|n\rangle\}$:

$$\hat{\boldsymbol{n}}\,|n\rangle = n\,|n\rangle\,, \qquad n \in \mathbb{R}^+.$$

- The complete algebra of the oscillator is shown to be given by

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from where we derive

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \qquad a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,.$$

- We suppose the existence of a fundamental state, defined by $a\,|0\rangle = 0$, and therefore n must be a natural number.
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The second-order invariant

- $\hat{\boldsymbol{n}}$ may be written as

$$\hat{\boldsymbol{n}} = \frac{1}{2}\left(\frac{1}{\Omega}\left\{I^{\dagger}, I\right\} - \mathbf{1}\right).$$

- The quantity $I_Q \equiv \frac{1}{2}\left\{I^{\dagger}, I\right\}$ is a quadratic self-adjoint dynamical invariant:

$$I_Q = \frac{1}{2}\left(me^G\right)^2\left(\frac{1}{2}\frac{d^2\gamma}{dt^2} + g\frac{d\gamma}{dt} + \omega^2\gamma\right)q^2 - \frac{1}{4}me^G\frac{d\gamma}{dt}\left\{q, p\right\} + \frac{1}{2}\gamma p^2 + \\ - me^G\left(\frac{d\sigma}{dt} + \gamma e^G F\right)q + \sigma p - \mathcal{F}\left(\sigma, t\right),$$

with $\gamma \equiv 2\beta^*\beta$, and $\sigma \equiv -\beta^*\mathcal{F} - \mathcal{F}^*\beta$.

- Therefore,

$$I_Q = \Omega\left(\hat{\boldsymbol{n}} + \frac{1}{2}\right).$$

- Physical states are eigenstates of I_Q but still obey the time-dependent Schrödinger equation.

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$$\hat{\boldsymbol{n}} = \frac{1}{2}\left(\frac{1}{\Omega}\left\{I^{\dagger}, I\right\} - \mathbf{1}\right).$$

- The quantity $I_Q \equiv \frac{1}{2}\left\{I^{\dagger}, I\right\}$ is a quadratic self-adjoint dynamical invariant:

$$\begin{aligned} I_Q = & \frac{1}{2}\left(me^{G}\right)^{2}\left(\frac{1}{2}\frac{d^{2}\gamma}{dt^{2}} + g\frac{d\gamma}{dt} + \omega^{2}\gamma\right)q^{2} - \frac{1}{4}me^{G}\frac{d\gamma}{dt}\left\{q, p\right\} + \frac{1}{2}\gamma p^{2} + \\ & - me^{G}\left(\frac{d\sigma}{dt} + \gamma e^{G}F\right)q + \sigma p - \mathcal{F}\left(\sigma, t\right), \end{aligned}$$

with $\gamma \equiv 2\beta^{*}\beta$, and $\sigma \equiv -\beta^{*}\mathcal{F} - \mathcal{F}^{*}\beta$.

- Therefore,

$$I_Q = \Omega\left(\hat{\boldsymbol{n}} + \frac{1}{2}\right).$$

- Physical states are eigenstates of I_Q but still obey the time-dependent Schrödinger equation.

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Ermakov's type equations for ρ and σ

- It can be shown that γ and σ obey the generalized Ermakov equation [1],

$$\frac{1}{2}\frac{d^3\gamma}{dt^3} + 3g\frac{d^2\gamma}{dt^2} + \left(\dot{g} + 4g^2 + 2\omega^2\right)\frac{d\gamma}{dt} + \left(\frac{d\omega^2}{dt} + 4\omega^2 g\right)\gamma = 0,$$

- Together with

$$\frac{d^2\sigma}{dt^2} + 2g\frac{d\sigma}{dt} + \omega^2\sigma = -\frac{3}{2}e^G F\frac{d\gamma}{dt} - e^G\left(\frac{dF}{dt} + 4gF\right)\gamma.$$

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Wave functions

- Considering $\langle q \left| a \right| 0 \rangle = 0$,

$$\left(\mathcal{F} + me^{G}\dot{\beta}q + i\hbar\beta\frac{d}{dq} \right) \psi_0\left(q\right) = 0,$$

- Solution

$$\psi_0 = A\exp\left[-\frac{1}{2}\frac{1}{i\hbar\beta}\left(me^{G}\dot{\beta}q^2 + 2\mathcal{F}q \right) \right],$$

$$A = \left(\frac{1}{2\pi\hbar^2}\frac{\Omega}{\beta^*\beta} \right)^{1/4} \exp\left[-\left(\frac{1}{\beta^*\beta} \right)^2 \frac{1}{\Omega}\left(\mathrm{Im}\left(\beta^*\mathcal{F} \right) \right)^2 \right].$$

- The complete set of normalized eigenfunctions are found to be

$$\psi_n = \frac{1}{\sqrt{2^n \cdot n!}}\left(i\sqrt{\frac{\beta^*}{\beta}} \right)^n \psi_0 H_n\left[\sqrt{\frac{\Omega}{2\beta^*\beta}}\left(\frac{q}{\hbar} + \frac{2}{\Omega}\mathrm{Im}\left(\beta^*\mathcal{F} \right) \right) \right],$$

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Uncertainty relations

- Writing the canonical variables in the form

$$q = \frac{i\hbar}{\sqrt{\Omega}}\left(\beta^* a - \beta a^\dagger\right) - 2\frac{\hbar}{\Omega}\mathrm{Im}\left(\beta^* \mathcal{F}\right),$$

$$p = \frac{im\hbar e^G}{\sqrt{\Omega}}\left(\dot{\beta}^* a - \dot{\beta} a^\dagger\right) - \frac{2m\hbar e^G}{\Omega}\mathrm{Im}\left(\dot{\beta}^* \mathcal{F}\right),$$

- allows us to calculate the uncertainty relations

$$(\Delta q)_n^2 (\Delta p)_n^2 = \frac{2m^2\hbar^4 e^{2G}}{\Omega^2}\gamma\dot{\beta}^*\dot{\beta}\left(n + \frac{1}{2}\right)^2.$$

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The underdamping oscillator

- $g^2 \leq \omega^2$, ω and g constant parameters, $F = F(t)$, $G = 2gt$:

$$\beta = \exp(-gt)\exp(i\bar{\omega}t), \qquad \bar{\omega}^2 \equiv \omega^2 - g^2,$$

  β^* is just the complex conjugate.

- The linear dynamical invariants become ($\bar{g} \equiv g - i\bar{\omega}$)

$$I = e^{i\bar{\omega}t}\left[e^{-gt}p + m\bar{g}e^{gt}q - e^{-i\bar{\omega}t}\mathcal{F}\right],$$

  together with the adjoint operator $I^\dagger$.

- We also have $\Omega = 2m\bar{\omega}\hbar$, which gives the ladder operators

$$a = \frac{e^{i\bar{\omega}t}}{\sqrt{2m\bar{\omega}\hbar}}\left[e^{-gt}p + m\left(g - i\bar{\omega}\right)e^{gt}q - e^{-i\bar{\omega}t}\mathcal{F}\right],$$

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- The quadratic invariant results in

$$I_Q = 2mH - me^{2gt}\frac{d\sigma}{dt} + \sigma p - \mathcal{F}(\sigma, t).$$

$$H = \frac{1}{2m}e^{-2gt}p^2 + \frac{1}{2}m\omega^2 e^{2gt}q^2 - e^{2gt}Fq.$$

- Note that I_Q is an invariant observable, so the invariant eigenvalues

$$I_n = 2m\overline{\omega}\hbar\left(n + \frac{1}{2}\right)$$

represent invariant characteristic values of the oscillator.

- The normalized wave function of the fundamental state is given by

$$\psi_0 = e^{gt/2}\left(\frac{m\bar{\omega}}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{e^{2gt}}{2m\hbar\bar{\omega}}\left(\mathrm{Im}\left(e^{-i\bar{\omega}t}\mathcal{F}\right)\right)^2\right]\times$$
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- A straightforward calculation shows that

$$\psi_n = \frac{(i)^n}{\sqrt{2^n \cdot n!}} e^{-in\bar{\omega}t} \psi_0 H_n(x)$$

are the normalized eigenfunctions, where

$$x = \sqrt{\frac{m\bar{\omega}}{\hbar}} e^{gt} q - \sqrt{\frac{1}{m\hbar\bar{\omega}}} \mathrm{Im}\left(e^{-i\bar{\omega}t}\mathcal{F}\right).$$

- Moreover, we have the uncertainty relations

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The sine force term

- Now, let us display some results for the case $F = F_0 \sin(\alpha t)$:

$$\mathcal{F} = F_0 e^{\bar{g}^* t} \frac{\bar{g}^* \sin(\alpha t) - \alpha \cos(\alpha t) + \alpha e^{-\bar{g}^* t}}{(\bar{g}^*)^2 + \alpha^2}.$$

- The parameter σ is given by

$$\sigma = \frac{-2F_0}{\left[\left(g^2 - \overline{\omega}^2 + \alpha^2\right)^2 + 4\overline{\omega}^2 g^2\right]} \{2\overline{\omega} g \left[\overline{\omega} \sin(\alpha t) + \right.$$
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$$\mathcal{F}(\sigma, t) = \frac{-2F_0^2 e^{2gt}}{\left[\left(g^2 - \overline{\omega}^2 + \alpha^2\right)^2 + 4\overline{\omega}^2 g^2\right]} \{\left[\overline{\omega} \sin(\alpha t) + \right.$$
$$\left. -\alpha e^{-gt} \sin(\overline{\omega} t)\right]^2 + \left[g \sin(\alpha t) - \alpha \cos(\alpha t) + \alpha e^{-gt} \cos(\overline{\omega} t)\right]^2\}.$$

- The simple harmonic oscillator is trivially recovered in the case $G = 0$, $\overline{\omega} = \omega$, and $\mathcal{F} = 0$, which also gives the condition $\sigma = 0$.

The sine force term

- Now, let us display some results for the case $F = F_0 \sin(\alpha t)$:

$$\mathcal{F} = F_0 e^{\bar{g}^* t} \frac{\bar{g}^* \sin(\alpha t) - \alpha \cos(\alpha t) + \alpha e^{-\bar{g}^* t}}{(\bar{g}^*)^2 + \alpha^2}.$$

- The parameter σ is given by

$$\sigma = \frac{-2F_0}{\left[\left(g^2 - \overline{\omega}^2 + \alpha^2\right)^2 + 4\overline{\omega}^2 g^2\right]} \left\{2\overline{\omega} g \left[\overline{\omega} \sin(\alpha t) + \right.\right.$$
$$\left. - \alpha e^{-gt} \sin(\overline{\omega} t)\right] + \left(g^2 - \overline{\omega}^2 + \alpha^2\right) \left[g \sin(\alpha t) + \right.$$
$$\left.\left. - \alpha \cos(\alpha t) + \alpha e^{-gt} \cos(\overline{\omega} t)\right]\right\},$$

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Conclusions

- We showed a procedure for the quantization of the general one dimensional linear oscillator.
- The procedure is based on the construction of the linear invariants of the BCK Hamiltonian, which turns out to be ladder operators.
- We also construct the Hilbert space of the system and calculate the wave eigenfunctions.
- The wave functions ψ_n are time-dependent and lead, in the general case, to time-dependent expectation values and uncertainty relations for the canonical operators.
- In the special case of constant parameters, however, the uncertainty relations between q and p are time-independent.

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Thank you!