Duality between k-essence and Rastall gravity

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- *On Horizons and Wormholes in k-Essence Theories* (Bronnikov et al., 2016; arXiv:1511.08036)
- *Static, spherically symmetric solutions with a scalar field in Rastall gravity* (Bronnikov et al., 2016; arXiv:1606.06242)

K-essence

The k-essence theories can be defined by the following most general Lagrangian:

$$\mathcal{L} = \sqrt{-g}[R + F(X, \phi) + L_m]$$

with

$$X = \eta \phi_\mu \phi^\mu,$$

where $\eta = \pm 1$ is set to avoid ill-defined power-law.

The field equations are

$$G^\nu_\mu \equiv R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R = -T^\nu_\mu[\phi] - T^\nu_\mu[m],$$

$$T^\nu_\mu[\phi] \equiv \eta F_X \phi_\mu \phi^\nu - \frac{1}{2}\delta^\nu_\mu F,$$

$$\eta \nabla_\alpha (F_X \phi^\alpha) - \frac{1}{2} F_\phi = 0,$$

Rastall gravity

Rastall gravity field equations

$$R^{\nu}_{\mu} - \frac{\lambda}{2}\delta^{\nu}_{\mu}R = -T^{\nu}_{\mu},$$

$$\nabla_{\nu}T^{\nu}_{\mu} = \frac{\lambda - 1}{2}\partial_{\mu}R,$$

where GR is recovered with $\lambda = 1$.

$$G^{\nu}_{\mu} = -\left\{T^{\nu}_{\mu} - \frac{b-1}{2}\delta^{\nu}_{\mu}T\right\} \equiv -\widetilde{T}^{\nu}_{\mu},$$

$$\nabla_{\nu}T^{\nu}_{\mu} = \frac{b-1}{2}\partial_{\mu}T, \quad b := \frac{3\lambda - 2}{2\lambda - 1}, \quad T = T^{\alpha}_{\alpha}.$$

In this parameterization, GR is recovered with $b = 1$.

Duality k-essence-Rastall ($k - R$)

- Let us assume the k-essence and Rastall theories must coincide, in that the important quantities depend on the temporal or spatial coordinate.
- The metric has the form

$$ds^2 = \eta e^{2\alpha(u)} du^2 + h_{ik} dx^i dx^k,$$

where i, k are the numbers of coordinates other than u.

- The scalar field ϕ is a function of the coordinate u, where u can be spatial or temporal.

Adopting the model

$$F(X) = F_0 X^n - 2\Lambda, \qquad F_0, \Lambda = const,$$

SET of the ϕ field has the following nonzero components:

$$T^u_u[\phi] = (n - \frac{1}{2})F_0 X^n + V,$$

$$T^i_i[\phi] = -\frac{1}{2}F_0 X^n + V,$$

with $X = \mathrm{e}^{-2\alpha(u)}\phi_u^2$.

The scalar field equation has the form

$$\mathrm{e}^{-2n\alpha}\phi_u^{2n-2}[(2n-1)\phi_{uu} + \sigma_u\phi_u - (2n-1)\alpha_u\phi_u] = -\frac{1}{nF_0}\frac{dV}{d\phi}.$$

The nonzero components of the modified SET, $\widetilde{T}^{\nu}_{\mu}$ are:

$$\widetilde{T}^{u}_{u}[\psi] = \frac{1}{2}\epsilon b\eta \mathrm{e}^{-2\alpha}\psi_u^2 + (3-2b)W(\psi),$$

$$\widetilde{T}^{i}_{i}[\psi] = \frac{1}{2}\epsilon (b-2)\eta \mathrm{e}^{-2\alpha}\psi_u^2 + (3-2b)W(\psi),$$

The equation of the scalar field takes the form

$$\mathrm{e}^{-2\alpha}[b\psi_{uu} + \psi_u(\sigma_u - b\alpha_u)] = -\epsilon\eta(3-2b)W_{\psi}.$$

The SET of the scalar field ψ is given by

$$T^{\nu}_{\mu}[\psi] = \epsilon(\psi_\mu\psi^\nu - \frac{1}{2}\delta^{\nu}_{\mu}\psi_\alpha\psi^\alpha) + \delta^{\nu}_{\mu}W(\psi),$$

Comparison

To begin with, we identify the potentials:

$$V(\phi) = (3 - 2b)W(\psi).$$

Next, we equate the ratios of the kinetic parts of $T^{\nu}_{\mu}[\phi]$ and $\widetilde{T}^{\nu}_{\mu}[\psi]$, to obtain

$$\frac{2n-1}{-1} = \frac{b}{b-2} \quad \Rightarrow \quad (2-b)n = 1.$$

Then, equating the kinetic parts themselves, we find that

$$\psi_u^2 = \epsilon\eta n F_0 \phi_u^{2n} \mathrm{e}^{2(1-n)\alpha}.$$

Cosmology with matter

We will consider cosmological FLRW spatially flat metrics

$$ds^2 = \eta e^{2\alpha(u)} du^2 + h_{ik} dx^i dx^k,$$

becomes

$$ds^2 = dt^2 - a(t)^2[dx^2 + dy^2 + dz^2],$$

assuming $\eta = 1$, $e^{\alpha} = 1$, and $e^{\sigma} = a(t)^3$.

Matter will be taken in the form of a perfect fluid, so that

$$T^{\nu}_{\mu}[m] = \mathrm{diag}(\rho, -p, -p, -p),$$

where ρ is the density and p is the pressure.

In the FLRW metric and with $\phi = \phi(t)$, the field equations with matter take the form

$$3H^2 = \frac{1}{2}(2n-1)F_0\dot{\phi}^{2n} + V(\phi) + \rho_k,$$

$$2\dot{H} + 3H^2 = -\frac{1}{2}F_0\dot{\phi}^{2n} + V(\phi) - p_k,$$

$$\dot{\phi}^{2n-2}[(2n-1)\ddot{\phi} + 3H\dot{\phi}] = -\frac{1}{nF_0}V_\phi,$$

where $H = \dot{a}/a$ is the Hubble parameter and $V_\phi \equiv dV/d\phi$. The SET of matter satisfies the conservation law $\nabla_\nu T^\nu_\mu[m] = 0$, whence

$$\dot{\rho}_k + 3H(\rho_k + p_k) = 0.$$

Rastall cosmology with a scalar field and matter

The total SET is given by

$$T^{\nu}_{\mu} = T^{\nu}_{\mu}[\psi] + T^{\nu}_{\mu}[m].$$

The modified SET $\widetilde{T}^{\nu}_{\mu}$ is a sum of $\widetilde{T}^{\nu}_{\mu}[\psi]$

$$\widetilde{T}^{u}_{u}[\psi] = \frac{1}{2}\epsilon b\eta \mathrm{e}^{-2\alpha}\psi_u^2 + (3-2b)W(\psi),$$

$$\widetilde{T}^{i}_{i}[\psi] = \frac{1}{2}\epsilon (b-2)\eta \mathrm{e}^{-2\alpha}\psi_u^2 + (3-2b)W(\psi),$$

and $\widetilde{T}^{\nu}_{\mu}[m]$, with components

$$\widetilde{T}^{t}_{t}[m] = \frac{1}{2}[(3-b)\rho + 3(b-1)p] \equiv \widetilde{\rho},$$

$$\widetilde{T}^{i}_{i}[m] = \frac{1}{2}[(1-b)\rho + (3b-5)p] \equiv -\widetilde{p}$$

Therefore, the equations in Rastall lead us to

$$3H^2 = \frac{1}{2}\epsilon b\dot{\psi}^2 + (3-2b)W + \widetilde{\rho},$$

$$2\dot{H} + 3H^2 = \frac{1}{2}\epsilon(b-2)\dot{\psi}^2 + (3-2b)W - \widetilde{p},$$

One can notice that

$$\widetilde{\rho} + \widetilde{p} = \rho + p,$$

and

$$\widetilde{\rho} - \rho = p - \widetilde{p} = \frac{1-b}{2}(\rho - 3p).$$

Case R1: No mixing of scalar field and matter

In this case, we have

$$\nabla_\nu T^\nu_\mu[\psi] = \frac{b-1}{2}\partial_\mu T[\psi],$$

$$\nabla_\nu T^\nu_\mu[m] = \frac{b-1}{2}\partial_\mu T[m],$$

The first equation leads us to

$$b\ddot{\psi} + 3H\dot{\psi} = -\epsilon(3-2b)W_\psi,$$

while the second

$$\dot{\widetilde{\rho}} + 3H(\widetilde{\rho} + \widetilde{p}) = 0.$$

Therefore, we can identify

$$\rho_k = \widetilde{\rho}, \qquad p_k = \widetilde{p},$$

and

$$\dot{\psi}^2 = \epsilon n F_0 \dot{\phi}^{2n}.$$

Case R2: Conservative matter

We now have

$$\nabla_\nu T^\nu_\mu[m] = 0,$$

which results in

$$\dot{\rho} + 3H(\rho + p) = 0.$$

In this case, for the scalar field SET we have

$$\nabla_\nu T^\nu_\mu[\psi] = \frac{1}{2}(b-1)(\partial_\mu T[\psi] + \partial_\mu T[m]),$$

which leads to the scalar field equation

$$\dot{\psi}(b\ddot{\psi} + 3H\dot{\psi}) = -\epsilon(3-2b)W_\psi\dot{\psi} + \frac{1}{2}\epsilon(b-1)(\dot{\rho} - 3\dot{p}).$$

Case R2: Conservative matter

We can identify the SET of matter in the k-essence and Rastall theories:

$$\rho = \rho_k, \qquad p = p_k.$$

Identifying, as before, the potentials

$$V = (3 - 2b)W.$$

From Friedmann-like equations, we have

$$\epsilon\dot{\psi}^2 = nF_0\dot{\phi}^{2n},$$

and from scalar field equation

$$\epsilon\dot{\psi}^2\left[b - \frac{2n-1}{n}\right] = (b-1)(\rho - 3p),$$

from the two equations above

$$\dot{\phi}^{2n-2}[(2n-1)\ddot{\phi} + 3H\dot{\phi}] = -\frac{1}{nF_0}V_\phi,$$

Perturbations and the speed of sound

Equivalence between k-essence (power law) and a perfect fluid

- EoS:$p = w\rho$
- the constant w is related to the power n:

$$w = \frac{1}{2n-1}.$$

In a fluid, adiabatic perturbations propagate as a sound with the speed v_s such that

$$v_s^2 = dp/d\rho = w.$$

In k-essence theories with Lagrangians in the form $F(X, \phi)$, the speed of sound is given

$$v_s^2 = \frac{F_X}{F_X + 2XF_{XX}},$$

Perturbations and the speed of sound

which results in

$$v_s^2 = \frac{1}{2n-1},$$

In particular, if $w < 0$, corresponding to $n < 1/2$, the model is perturbatively unstable since it implies $v_s^2 < 0$.

In Rastall's theory, things may be different. The speed of sound for a scalar field is given by

$$v_s^2 = \frac{2-b}{b}.$$

- Caso R1:
  - $(2-b)n = 1$: we found the same expression for k-essence
  - $d\widetilde{p}/d\widetilde{\rho} = dp_k/d\rho_k$.
- Caso R2:

$$\epsilon\dot{\psi}^2\left[b - \frac{2n-1}{n}\right] = (b-1)(\rho - 3p),$$

Conclusion

- Case R1:

  We can conclude that the models belonging to the two theories coincide not only at the background level but also at the level of adiabatic perturbations

- Case R2:

  We have another relation between the parameters b and n, without such a simple connection. As a consequence, in principle it is possible that an unstable model in a k-essence model may correspond to a stable model in Rastall's theory, or vice versa.

- *Duality between k-essence and Rastall gravity* (Bronnikov et al., 2017; arXiv:1701.06662)

Obrigado!