Hybrid Theories of Gravitation: Testing deSitter Solution Stability.

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Summary

1 - Introduction/Motivation.

- General Relativity Theory (GRT) (Einstein, A.-1915)
    - From Sobral eclipse 1919 (Eddington, A., Dyson, F.-1920) to gravitational waves (Abbott, B. P., et al. (LIGO/Virgo Collaboration)-2016).

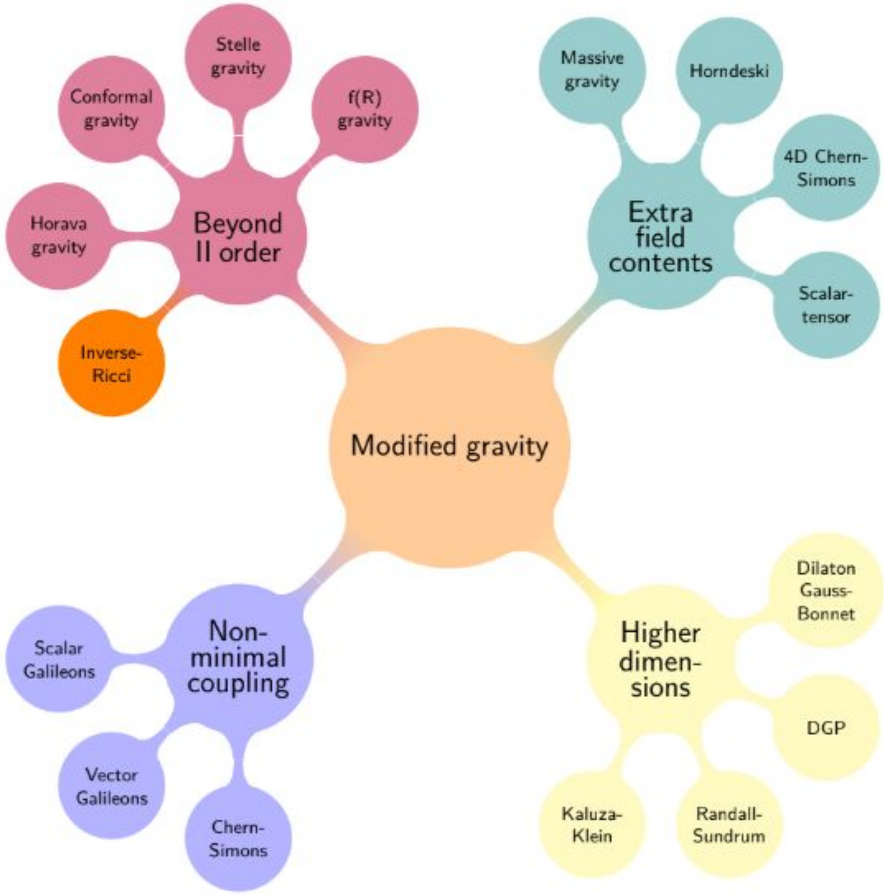
- Accelerating expanding universe, SN Ia (Riess, A. G., et al.-1998).
    - ΛCMD model.
    - Cosmological Constant tensions.

- Recently released data from DESI (DESI et al - 2025) experiment suggest that the $\omega_{de} = \omega_{de}(t)$.

1 - Introduction/Motivation.

The Lovelock Theorem: 'The Einstein field equations are the only second-order local equations of motion for a metric derivable from the action in 4D.’

How to modify gravity? Modified Theories of Gravitation (MTG) are theories that modify the left-hand side of Einstein's equations.



1 - Introduction/Motivation.

ΛCDM model. Einstein-Hilbert+C.C. action:

$$S_{\mathrm{EH+CC}} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(R-2\Lambda),$$

Variational principle:

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \kappa^2 T_{\mu\nu},$$

- trace

$$R = 4\Lambda - \kappa^2 T.$$

When we consider the FLRW metric:

$$ds^2 = dt^2 - a(t)^2\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\phi^2 + r^2 sin^2(\phi)d\theta^2\right)$$

and the Hubble constant is:

$$H = \frac{\dot{a}}{a}$$

Expanding in terms of the components:

$$\begin{aligned} 3H^2 &= \kappa^2(\rho_m + \rho_r) + \Lambda \\ 2\dot{H} + 3H^2 &= -\kappa^2(p_m + p_r) + \Lambda. \end{aligned}$$

ρ_m anda ρ_r are the matter and radiation density.

2 - Metric Theories (f(R)).

Unlike GRT, the metric theories define the functional in term of a arbitrary function of Ricci scalar f(R).

$$S = \frac{1}{2\kappa^2}\int \sqrt{-g}f(R)d^4x + S_{mat}$$

So, using the variational principle:

$$G_{\mu\nu} = \frac{1}{f'}\left[\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)f' + \frac{g_{\mu\nu}}{2}(f - Rf')\right] + \kappa^2 T_{\mu\nu},$$

$$f' = \frac{df(R)}{dR}.$$

trace:

$$3\Box f'(R) - 2f(R) + f'(R)R = f'(R)\kappa^2 T.$$

We can see the effective Einstein tensor as:

$$G^{\rm eff}_{\mu\nu} = G_{\mu\nu} - \frac{1}{f'}\left[\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)f' + \frac{g_{\mu\nu}}{2}(f - Rf')\right],$$

The term ( ) act like Λ in ΛCDM.

$$\frac{1}{f'}\left[\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)f' + \frac{g_{\mu\nu}}{2}(f - Rf')\right],$$

2 - Metric Theories (Palatini).

Palatini theories: The Palatini's scalar curvature is defined with a metric independent connection: $\delta\mathcal{R} = \hat{\nabla}_\nu(\delta\hat{\Gamma}^\sigma{}_{\mu\sigma}) - \hat{\nabla}_\sigma(\delta\hat{\Gamma}^\sigma{}_{\mu\nu}),$ $\quad \hat{\Gamma}^\alpha_{\mu\nu} = \hat{\Gamma}^\alpha_{\mu\nu}(h_{\mu\nu}).$

The Functional: $$S = \frac{1}{2\kappa^2}\int \sqrt{-g}f(\mathcal{R})d^4x + S_{mat}.$$

Field Equations: $$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}f(\mathcal{R}) = \kappa T_{\mu\nu},$$ $$\hat{\nabla}_\lambda(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}) = 0.$$

and the trace: $$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \kappa T.$$

2 - Metric Theories (Palatini).

Using $\hat{\nabla}_\lambda(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}) = 0.$ we can define the conformal transformation:

$$h_{\mu\nu} = f'(\mathcal{R})g_{\mu\nu} \qquad \longrightarrow \qquad h^{\mu\nu} = \frac{1}{f'(\mathcal{R})}g^{\mu\nu},$$

and after some algebra we can write:

$$\hat{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \frac{1}{2f'(\mathcal{R})}(g_{\mu\beta}f'(\mathcal{R})_{,\nu} + g_{\beta\nu}f'(\mathcal{R})_{,\mu} - g_{\mu\nu}f'(\mathcal{R})_{,\beta}),$$

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + \frac{3}{2f'(\mathcal{R})^2}(\nabla_\mu f'(\mathcal{R}))(\nabla_\nu f'(\mathcal{R})) - \frac{1}{f'(\mathcal{R})}\left(\nabla_\mu\nabla_\nu + \frac{g_{\mu\nu}}{2}\Box\right)f'(\mathcal{R}),$$

$$\mathcal{R} = R + \frac{3}{2(f'(\mathcal{R}))^2}\nabla^\mu f'(\mathcal{R})\nabla_\mu f'(\mathcal{R}) - \frac{3}{f'(\mathcal{R})}\Box f'(\mathcal{R}).$$ trace.

2 - Metric Theories.

So we can rewrite the field equations:

$$G_{\mu\nu} = \frac{\kappa^2 T_{\mu\nu}}{f'} + \frac{g_{\mu\nu}}{2}\left(\frac{f}{f'} - \mathcal{R}\right) - \frac{3}{2f'^2}\left[\partial_\mu f' \partial_\nu f' - \frac{g_{\mu\nu}}{2}(\partial f')^2\right] + \frac{1}{f'}(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box) f',$$

Note that:

While in f(R) the trace equation:

$$3\Box f'(R) - 2f(R) + f'(R)R = f'(R)\kappa^2 T.$$

In Palatini:

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) = \kappa T.$$

The equations reveal that f(R) theories possess an extra propagating degree of freedom, in contrast to Palatini theories. This property can be rigorously expressed through the scalar-tensor (S.T.) formalism.

2 - Metric Theories.

Scalar-Tensor equivalence. Scalar fields:

f(R) x Palatini

$$\phi(R) = \frac{\partial f(R)}{\partial R}$$

$$\phi(\mathcal{R}) = \frac{\partial f(\mathcal{R})}{\partial \mathcal{R}}$$

Funcional:

$$S = \frac{1}{2\kappa}\int \sqrt{-g}[\phi R - V(\phi)]d^4x + S_{mat},$$

$$V(\phi) = R\phi - f(R).$$

$$S = \int \sqrt{-g}\left[\phi R + \frac{3}{2\phi}(\partial\phi)^2 - V(\phi)\right] d^4x + S_{mat}$$

$$V(\phi) = \phi\mathcal{R} - f(\mathcal{R}),$$

2 - Metric Theories.

Field equations in f(R):

$$G_{\mu\nu} = \frac{\kappa}{\phi}T_{\mu\nu} - \frac{1}{2\phi}g_{\mu\nu}V(\phi) + \frac{1}{\phi}\left[\nabla_\mu\nabla\nu - g_{\mu\nu}\Box\right]\phi,$$

$$V_\phi = R.$$

$$3\Box\phi + 2V(\phi) - \phi V_\phi = \kappa T.$$

In Palatini:

$$\begin{aligned} G_{\mu\nu} &= \frac{\kappa}{\phi}T_{\mu\nu} - \frac{3}{2\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\lambda\phi\nabla_\lambda\phi\right) + \\ &\quad + \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi) - \frac{V}{2\phi}g_{\mu\nu}, \\ \Box\phi &= \frac{\phi}{3}(R - V') + \frac{1}{2\phi}\nabla^\mu\phi\nabla_\mu\phi. \end{aligned}$$

$$2V - \phi V' = \kappa T.$$

2 - Metric Theories

- There is a substantial body of observational results placing constraints on f(R) and Palatini theories (Ishak, 2019; de Felice, 2010), indicating that they do not provide a viable alternative to GRT $+ \Lambda$.
- Cosmological observations suggest that Palatini's theories are nearly indistinguishable from GRT (Tsujikawa, 2008).
- f(R) theories require the "chameleon mechanism" to prevent the extra degree of freedom from altering post-Newtonian predictions (e.g., in the Solar System). Various astrophysical observations show no evidence of such a mechanism in action (Jain, 2013).

3 - Hybrid Theories.

$$f(R,\mathcal{R}) \;=\; R+\Phi(\mathcal{R}),$$

Supposing:

$$S=\frac{1}{2\kappa^2}\int\sqrt{-g}(R+\Phi(\mathcal{R}))d^4x+S_{mat}.$$

Field equations:

$$R_{\mu\nu}+\Phi'\mathcal{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}(R+\Phi)=\kappa T_{\mu\nu}.$$

$$\hat{\nabla}_c(\sqrt{-g}f_{\mathcal{R}}g^{\mu\nu})=\hat{\nabla}_c(\sqrt{-g}\Phi' g^{\mu\nu})=0,$$

$$\mathcal{R}_{\mu\nu}=R_{\mu\nu}+\frac{3}{2\Phi'^2}\partial_\mu\Phi'\partial_\nu\Phi'-\frac{1}{\Phi'}\left(\nabla_\mu\nabla_\nu+\frac{1}{2}g_{\mu\nu}\Box\right)\Phi',$$

$$\mathcal{R}=R+\frac{3}{2}\left[\frac{1}{\Phi'^2}(\partial\Phi')^2-\frac{2}{\Phi'}\Box\Phi'\right].$$

$$G_{\mu\nu}+\Phi' R_{\mu\nu}=-\frac{3}{2\Phi'}\partial_\mu\Phi'\partial_\nu\Phi'+\left(\nabla_\mu\nabla_\nu+\frac{g_{\mu\nu}}{2}\Box\right)\Phi'+\frac{g_{\mu\nu}}{2}\Phi+\kappa T_{\mu\nu}.$$

3 - Hybrid Theories.

S.T. representation:

$$S = \kappa^2 \int \sqrt{-g}\left[R(1+\psi) + \frac{3}{2\psi}(\partial\psi)^2 - V(\psi)\right] d^4x + S_{\rm mat}.$$

$$V(\psi) = -\Phi + \Phi'\mathcal{R} = -\Phi + \psi\mathcal{R},$$

Field equations:

$$(1+\psi)G_{\mu\nu} = \kappa^2 T_{\mu\nu} + (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\,\psi - \frac{3}{2\psi}(\partial_\mu\psi\partial_\nu\psi) + g_{\mu\nu}\left[-\frac{1}{2}V + \frac{3}{4\psi}(\partial\psi)^2\right],$$

$$\Box\psi = -\frac{\kappa^2 T\psi}{3} + \frac{1}{2\psi}(\partial\psi)^2 + \frac{\psi}{3}[2V - (1+\psi)V_\psi].$$

trace:

$$-R(1+\psi) = \kappa^2 T - 3\Box\psi + \frac{3}{2\psi}(\partial\psi)^2 - 2V.$$

$$G^{\rm eff}_{\mu\nu} = (1+\psi)G_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\,\psi + \frac{3}{2\psi}(\partial_\mu\psi\partial_\nu\psi) - g_{\mu\nu}\left[-\frac{1}{2}V + \frac{3}{4\psi}(\partial\psi)^2\right],$$

4 - Cosmology and Dynamical System.

Back to GRT+CC:

$$3H^2 = \kappa^2(\rho_m + \rho_r) + \Lambda$$

$$2\dot{H} + 3H^2 = -\kappa^2(p_m + p_r) + \Lambda.$$

deSitter solution: with $p = \rho = 0$

$$3H^2 = 3\frac{\dot{a}^2}{a^2} = \Lambda. \quad \text{---->} \quad a(t) = e^{\sqrt{\frac{\Lambda}{3}}t}.$$ Stable solution.

Renormalized Dynamical Variables:

$$x = \Omega_m = \frac{\kappa^2\rho_m}{3H^2}, \qquad y = \Omega_r = \frac{\kappa^2\rho_r}{3H^2} \quad \text{e} \quad \Omega_\Lambda = \frac{\kappa^2\rho_\Lambda}{3H^2}.$$

$$1 = x + y + \Omega_\Lambda.$$

4 - Cosmology and Dynamical System.

Example of dynamical systems analysis in GRT:

Dynamical System:

$$
\begin{aligned}
x_{,N} &= x(3x+4y-3) \\
y_{,N} &= y(3x+4y-4).
\end{aligned}
$$

with $N = log(a)$

Critical points:

$$O=(0,0), \qquad R=(0,1) \qquad \text{e} \qquad M=(1,0).$$

$$\omega_{eff} = \frac{p_{eff}}{\rho_{eff}} = -\frac{2}{3}\frac{\dot{H}}{H^2} - 1.$$

$$\omega^R_{eff} = \frac{1}{3}, \qquad \omega^M_{eff} = 0 \qquad \text{e} \qquad \omega^O_{eff} = -1.$$

$$
\begin{aligned}
&\text{Para} \quad O: \quad \lambda_i^O = \{-4,-3\}, \\
&\text{Para} \quad R: \quad \lambda_i^R = \{1,4\} \quad \text{e} \\
&\text{Para} \quad M: \quad \lambda_i^M = \{-1,3\}.
\end{aligned}
$$

4 - Cosmology and Dynamical System.

$O = (0,0), \qquad R = (0,1) \qquad \text{e} \qquad M = (1,0).$

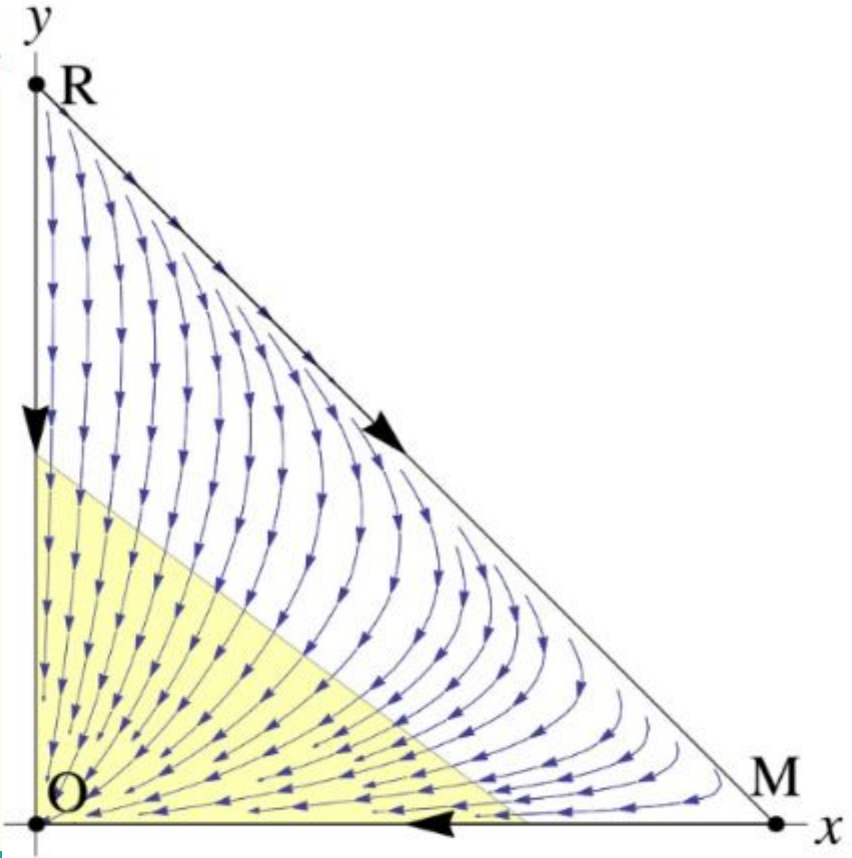
Para $\quad O: \quad \lambda_i^O = \{-4, -3\},$

Para $\quad R: \quad \lambda_i^R = \{1, 4\} \quad$ e

Para $\quad M: \quad \lambda_i^M = \{-1, 3\}.$

$$\omega_{eff} = \frac{p_{eff}}{\rho_{eff}} = -\frac{2}{3}\frac{\dot{H}}{H^2} - 1.$$

$\omega_{eff}^R = \frac{1}{3}, \qquad \omega_{eff}^M = 0 \qquad \text{e} \qquad \omega_{eff}^O = -1.$



4 - Cosmology and Dynamical System.

Cosmology in f(R,□):

$$
\begin{aligned}
(1+\psi)3H^2 &= \kappa^2(\rho_r+\rho_m)+\frac{V}{2}-3H\dot{\psi}-\frac{3}{4\psi}(\dot{\psi})^2 \\
(1+\psi)(-3H^2-2\dot{H}) &= \kappa^2 p_r+2H\dot{\psi}+\ddot{\psi}-\frac{V}{2}-\frac{3}{4\psi}(\dot{\psi})^2.
\end{aligned}
$$

$$
\ddot{\psi} = -3H\dot{\psi}+\frac{\dot{\psi}}{2\psi}-\frac{2H^2\psi}{1}\left(2+\frac{\dot{\psi}}{H^2}\right)+\frac{2V_\psi\psi}{3}-\kappa(\rho_{\Gamma}-3p_{\Gamma})
$$

In terms of density:

$$
\begin{aligned}
(\psi+1)3H^2 &= \kappa^2(\rho_c+\rho_r+\rho_m)=\kappa^2\rho_{eff}, \\
(\psi+1)\dot{H} &= -\frac{\kappa^2}{2}(\rho_{eff}+p_{eff})
\end{aligned}
$$

with

$$
\kappa^2\rho_c = \frac{V}{2}-3H\dot{\psi}-\frac{3}{4\psi}(\dot{\psi})^2.
$$

4 - Cosmology and Dynamical System.

Considering vacuum,

$$3(\psi+1)H^2 = -3\dot{\psi}H - \frac{3\dot{\psi}^2}{4\psi} + \frac{V}{2}$$

$$2(\psi+1)\dot{H} = H\dot{\psi} + \frac{3\dot{\psi}^2}{2\psi} - \ddot{\psi},$$

$$\ddot{\psi} = -3H\dot{\psi} + \frac{\dot{\psi}^2}{2\psi} - \frac{\psi}{3}[2V - (\psi+1)V_\psi].$$

and ψ as constant, we have in deSitter state:

$$H_{dS} = \sqrt{\frac{V_{dS}}{6(\psi_{dS}+1)}}.$$

$$\frac{\psi_{dS}}{3}[2V_{dS} - (\psi_{dS}+1)V_{\psi_{dS}}] = 0.$$

$$\psi_{dS} = 0.$$

$$\frac{2}{(\psi_{dS}+1)} = \frac{V'_{dS}}{V_{dS}}$$

4 - Cosmology and Dynamical System.

Changing variables:

$$\psi = s\chi^2, \qquad \dot{\psi} = 2s\chi\dot{\chi}, \qquad \ddot{\psi} = 2s\dot{\chi}^2 + 2s\chi\ddot{\chi}, \qquad V(\psi) = V(\chi), \quad \text{e} \quad V_\psi = \frac{V_\chi}{2s\chi},$$

t

o avoid indetermination problems.

$$\begin{aligned}
3H^2(s\chi^2+1) &= -6sH\chi\dot{\chi} - 3s\dot{\chi}^2 + \frac{V(\chi)}{2}, \\
2\dot{H}(s\chi^2+1) &= 8sH\chi\dot{\chi} + 4s\dot{\chi}^2 + \frac{s\chi^2}{3}\left[2V(\chi) - \frac{V_\chi(s\chi^2+1)}{2s\chi}\right], \\
\ddot{\chi} &= -3H\dot{\chi} - \frac{\chi}{6}\left[2V(\chi) - \frac{V_\chi(s\chi^2+1)}{2s\chi}\right].
\end{aligned}$$

4 - Cosmology and Dynamical System

Renormalized dynamical variables:

$$x = \chi, \qquad y = \frac{\dot{\chi}}{H} \quad \text{e} \quad z = \frac{V}{6H^2}. \qquad u(x) = \frac{V'(x)}{V(x)}.$$

$$\begin{aligned} \frac{\dot{H}}{H^2} &= \frac{1}{2(sx^2+1)}[8sxy + 4sy^2 + 4sx^2z - u(x)zx(sx^2+1)] \\ \frac{\ddot{\chi}}{H^2} &= -3y - 2xz + \frac{u(x)z}{2s}(sx^2+1). \end{aligned}$$

$$sx^2 + 2sxy + sy^2 + 1 = s(x+y)^2 + 1 = z,$$

$$\begin{aligned} \frac{\dot{H}}{H^2} &= -\frac{u(x)x}{2} - \frac{s}{2}(x+y)^2(u(x)x - 4), \\ \frac{\ddot{\chi}}{H^2} &= -3y + 2x + \frac{u}{2s} + u(x)x^2 + u(x)xy + \frac{u(x)y^2}{2} + \frac{xs}{2}(x+y)^2(u(x)x-4). \end{aligned}$$

$$\begin{aligned} x_{,N} &= F(x,y) \\ y_{,N} &= G(x,y). \end{aligned}$$

$$N = log(a)$$

4 - Cosmology and Dynamical System.

Dynamical system:

$$
\begin{aligned}
x_{,N} &= y, \\
y_{,N} &= \frac{\ddot{\chi}}{H} - y\frac{\dot{H}}{H^2},
\end{aligned}
$$

$$
\begin{aligned}
x_{,N} &= y \\
y_{,N} &= -3y - 2x + \frac{u}{2} + ux^2 + \frac{3}{2}uxy + \frac{1}{2}uy^2 + \frac{1}{2}(x+y)^3(ux-4).
\end{aligned}
$$

With these critical points: $P_0 = (0,0)$ e $P_u = (x_u, 0)$,

$$F(x,y) = 0$$

$$G(x,y) = 0$$

$$u(x_u) = \frac{4x_u}{x_u^2+1}.$$

4 - Cosmology and Dynamical System.

In (0,0), the linearized matrix is:

$$M_0 = \begin{bmatrix} 0 & 1 \\ -2 + \frac{u_x(0)}{2} & -3 \end{bmatrix},$$

And the eigenvalues:

$$\lambda_{i0} = \left\{ -\frac{1}{2} \left(3 \pm \sqrt{1 + 2u_x(0)} \right) \right\}$$

	λ_1	λ_2	Stable/Unstable	stab/unstab charact.
$u_x > 4$	> 0	< 0	Unstable	Saddle
$u_x = 4$	$= 0$	< 0	Center Manifold	Saddle or stable
$-\frac{1}{2} < u_x < 4$	$\neq \lambda_2 < 0$	$\neq \lambda_1 < 0$	Stable	Improper node (assintotic. stab.)
$u = -\frac{1}{2}$	$= \lambda_2 < 0$	$= \lambda_1 < 0$	Stable	Jordan node (assintotic. stab.)
$u_x < -\frac{1}{2}$	$\mathbb{R}(\lambda_1) < 0$	$\mathbb{R}(\lambda_2) < 0$	Stable	Vortex (assintotic. stab.)

4 - Cosmology and Dynamical System.

In (x_u,0), the linearized matrix is:

$$M_0 = \begin{bmatrix} 0 & 1 \\ 2(x_u^2 - 1) + \frac{u_x(x_u)}{2}(1 + x_u^2)^2 & -3 \end{bmatrix}$$

And the eigenvalues:

$$\lambda_{ui\pm} = \left\{ -\frac{1}{2}\left(3 \pm \sqrt{1 + 8x_u^2 + 2u_x + 4x_u^2 u_x + 2x_u z'^4 u_x}\right)\right\}$$

	λ_1	λ_2	Stable/Unstable	stab/unstab charact.
$u_x > -\frac{8(x_u-1)}{(x_u^2+1)^2}$	> 0	< 0	Unstable	Saddle
$u_x = -\frac{8(x_u-1)}{(x_u^2+1)^2}$	$= 0$	< 0	Center Manifold	Saddle or stable
$-\frac{8}{2(x_u^2+1)^2} < u_x < -\frac{8(x_u-1)}{(x_u^2+1)^2}$	$\neq \lambda_2 < 0$	$\neq \lambda_1 < 0$	Stable	Proper node (assintotic. estab.)
$u = -\frac{8}{2(x_u^2+1)^2}$	$= \lambda_2 < 0$	$= \lambda_1 < 0$	Stable	Jordan node (assintotic. estab.)
$u_x < -\frac{8}{2(x_u^2+1)^2}$	$\mathbb{R}(\lambda_1) < 0$	$\mathbb{R}(\lambda_2) < 0$	Stable	Vortex (assintotic. stab.)

$$
\begin{aligned}
3H^2(s\chi^2+1) &= -6sH\chi\dot{\chi} - 3s\dot{\chi}^2 + \frac{V(\chi)}{2}, \\
2\dot{H}(s\chi^2+1) &= 8sH\chi\dot{\chi} + 4s\dot{\chi}^2 + \frac{s\chi^2}{3}\left[2V(\chi) - \frac{V_\chi(s\chi^2+1)}{2s\chi}\right], \\
\ddot{\chi} &= -3H\dot{\chi} - \frac{\chi}{6}\left[2V(\chi) - \frac{V_\chi(s\chi^2+1)}{2s\chi}\right].
\end{aligned}
$$

5 - Testing Some Models.

Quadratic model:

$$
\Phi(\mathcal{R}) = \alpha + \beta\mathcal{R}^2,
$$
$$
f(R,\mathcal{R}) = R + \alpha + \beta\mathcal{R}^2
$$

Dynamical variables

$$
\begin{aligned}
\psi &= s\chi^2 = 2\beta\mathcal{R}, \\
\mathcal{R}(\chi) &= \frac{s\chi^2}{2\beta}, \\
V(\chi) &= \frac{\chi^4 - 4\alpha\beta}{4\beta} \\
V_\chi(\chi) &= \frac{\chi^3}{\beta} \\
u(\chi) &= \frac{4\chi^3}{\chi^4 - 4\alpha\beta}, \\
u_{dS}(\chi) &= \frac{4s\chi_{dS}}{s\chi_{dS}^2 + 1}.
\end{aligned}
$$

In deSitter state:

$$
H_{dS}^2 = \frac{V_{dS}}{6(s\chi_{dS}^2+1)} = -\frac{\alpha}{6}.
$$

$$
\begin{aligned}
s\chi_{dS}^2 &= 0, \\
&\text{ou} \\
s\chi_{dS}^2 &= -4\alpha\beta \quad \text{para } \chi \neq 0.
\end{aligned}
$$

5 - Testing Some Models.

Dynamical system and critical points:

$$x_{,\tau} = y(-4\alpha\beta + x^4),$$

$$y_{,\tau} = 12\alpha\beta y + 8\alpha\beta x + 2[x^3 + 4\alpha\beta(x+y)^3] + x^3(2x^2 + 3xy + 2y^2).$$

$$P_1 = (0,0),$$

$$P_{2\pm} = (\pm\sqrt{-4\alpha\beta}, 0),$$

ω_{eff}:

$$\frac{\dot{H}}{H^2}|_{P_0} = \frac{\dot{H}}{H^2}|_{P_u} = 0, \longrightarrow \omega_{\text{eff}}|_P = -1.$$

5 - Testing Some Models.

$$
\begin{aligned}
3H^2(s\chi^2+1) &= -6sH\chi\dot{\chi} - 3s\dot{\chi}^2 + \frac{V(\chi)}{2}, \\
2\dot{H}(s\chi^2+1) &= 8sH\chi\dot{\chi} + 4s\dot{\chi}^2 + \frac{s\chi^2}{3}\left[2V(\chi) - \frac{V_\chi(s\chi^2+1)}{2s\chi}\right], \\
\ddot{\chi} &= -3H\dot{\chi} - \frac{\chi}{6}\left[2V(\chi) - \frac{V_\chi(s\chi^2+1)}{2s\chi}\right].
\end{aligned}
$$

Exponential model:

$$
\Phi(\mathcal{R}) = \alpha(1+e^{\beta\mathcal{R}}),
$$

$$
f(R,\mathcal{R}) = R + \alpha(1+e^{\beta\mathcal{R}}).
$$

Dynamical variables

$$
\begin{aligned}
s\chi^2 &= \alpha\beta e^{\beta\mathcal{R}}, \\
\mathcal{R}\beta &= \log\frac{s\chi^2}{\alpha\beta}, \\
V(\mathcal{R}) &= -\alpha[1+e^{\beta\mathcal{R}}(1-\beta\mathcal{R})], \\
V(\chi) &= -\alpha\left[1+\frac{s\chi^2}{\alpha\beta}\left(1-\log\frac{s\chi^2}{\alpha\beta}\right)\right], \\
V_\chi &= 2s\chi\mathcal{R} = \frac{2s\chi\log\frac{s\chi^2}{\alpha\beta}}{\beta}.
\end{aligned}
$$

$$
u(\chi) = \frac{2\chi\log\frac{\chi^2}{\alpha\beta}}{\chi^2\log\frac{\chi^2}{\alpha\beta} - \alpha\beta - \chi^2}.
$$

In deSitter state:

$$
\chi_{dS} = 0
$$

$$
\frac{2(\chi_{dS}^2+\alpha\beta)}{\chi_{dS}^2-1} + \log\alpha\beta = \log\chi_{dS}^2.
$$

5 - Testing Some Models.

Dynamical system and critical points

$$x_{,N} = y,$$

$$y_{,N} = \frac{1}{x^2 + \alpha\beta - x^2 \log\frac{x^2}{\alpha\beta}} \{-(\alpha\beta + x^2)(2x + 2x^3 + 3y + 6x^2 y + 6xy^2 + 2y^3) + $$
$$+ x \log\frac{x^2}{\alpha\beta}\left[-1 + x^4 + 3x^3 y - y^2 + xy^3 + x^2(2 + 3y^2)\right]\}$$

$$P_0 = (0,0),$$

$$P_u = (x_u, 0),$$

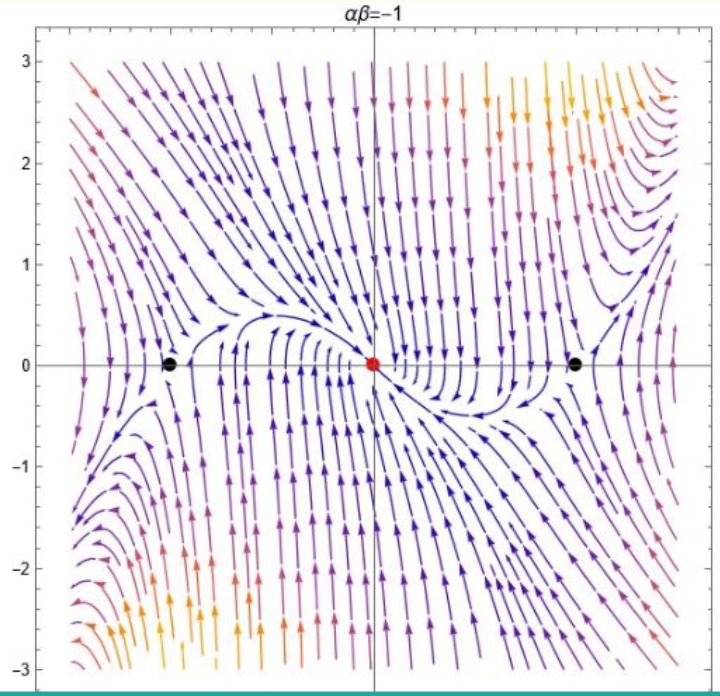
$$\log\frac{x_u^2}{\alpha\beta} = \frac{2(x_u^2 + \alpha\beta)}{x_u^2 - 1},$$

ω_{eff}: $\frac{\dot{H}}{H^2}|_{P_0} = \frac{\dot{H}}{H^2}|_{P_u} = 0, \longrightarrow \omega_{\text{eff}}|_P = -1.$

4 - Testing Some Models.

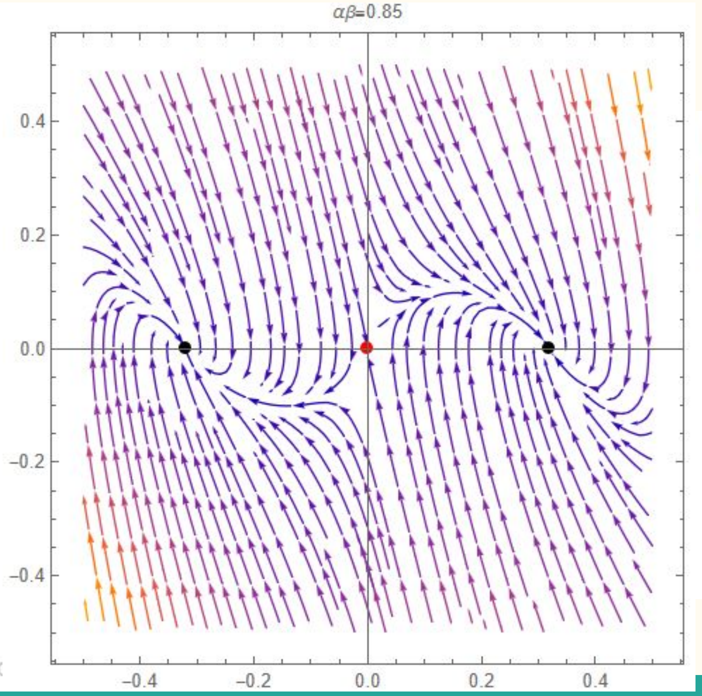
Phase portrait:

Quadratic



Red (0,0)
Black (2,0)
Black(-2,0)

Exponential



Red(0,0)
Black(0.32,0)
Black(-0.32,0)

6 - Future Perspective:

$$
\begin{aligned}
x_{,N} &= y, \\
y_{,N} &= \frac{1}{2}[-x - 3y + (x+y)(\Omega_r - 3z) - s(x+y)^3 + uzs(sx^2+1) + uxyz], \\
z_{,N} &= z[3 - 3z + \Omega_r - s(x+y)^2 + uy + uxz], \\
\Omega_{r,N} &= \Omega_r[-1 + \Omega_r - 3z - s(x+y)^2 + uxz],
\end{aligned}
$$

- Numerical integration/Dynamical analysis of cosmological equations:

$$
\begin{aligned}
\ddot{\chi} &= -3H\dot{\chi} - \frac{\chi V}{3} + \frac{V_\chi}{12s}(s\chi^2+1)] - \frac{\chi\kappa^2\rho_m}{6}, \\
3H^2(s\chi^2+1) &= \kappa^2(\rho_m + \rho_r) + \frac{V}{2} - 6sH\chi\dot{\chi} - 3s\dot{\chi}^2 \\
2\dot{H}(s\chi^2+1) &= 8sH\chi\dot{\chi} + 4s\dot{\chi}^2 + \frac{\chi}{6}[4s\chi V - V_\chi(s\chi^2+1)] - \kappa^2\left[\rho_m\left(1 - \frac{s\chi^2}{3}\right) + \frac{4\rho_r}{3}\right].
\end{aligned}
$$

Considering matter and radiation content.

- Attractive characteristic theories:

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}\xi^\mu\xi^\nu,$$

- Raychaudhury's equation.

$$\mathcal{M}_{\xi^\mu} \equiv -R_{\mu\nu}\xi^\mu\xi^\nu.$$

-Main curvature.

Imposing the S.E.C:

Ricci tensor in hybrid models -

$$
\begin{aligned}
R_{\mu\nu}\xi^\mu\xi^\nu \geq \; & \frac{H^2}{\psi+1}\{\frac{x^2}{2}(2\xi^0\xi^0+1)\left[s + (x+y)^2 + 4xuz(sx^2+1) - 5sz - s\Omega_r\right] - \\
& (4\xi^0\xi^0 - 1)(-2sxy - sy^2) - 3z\}.
\end{aligned}
$$

7 - References.

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