A ‘path-integral’ approach to CMB

Based on PHR and L. R. Abramo, *CMB and Random Flights: temperature and polarization in position space,* JCAP06(2013)043

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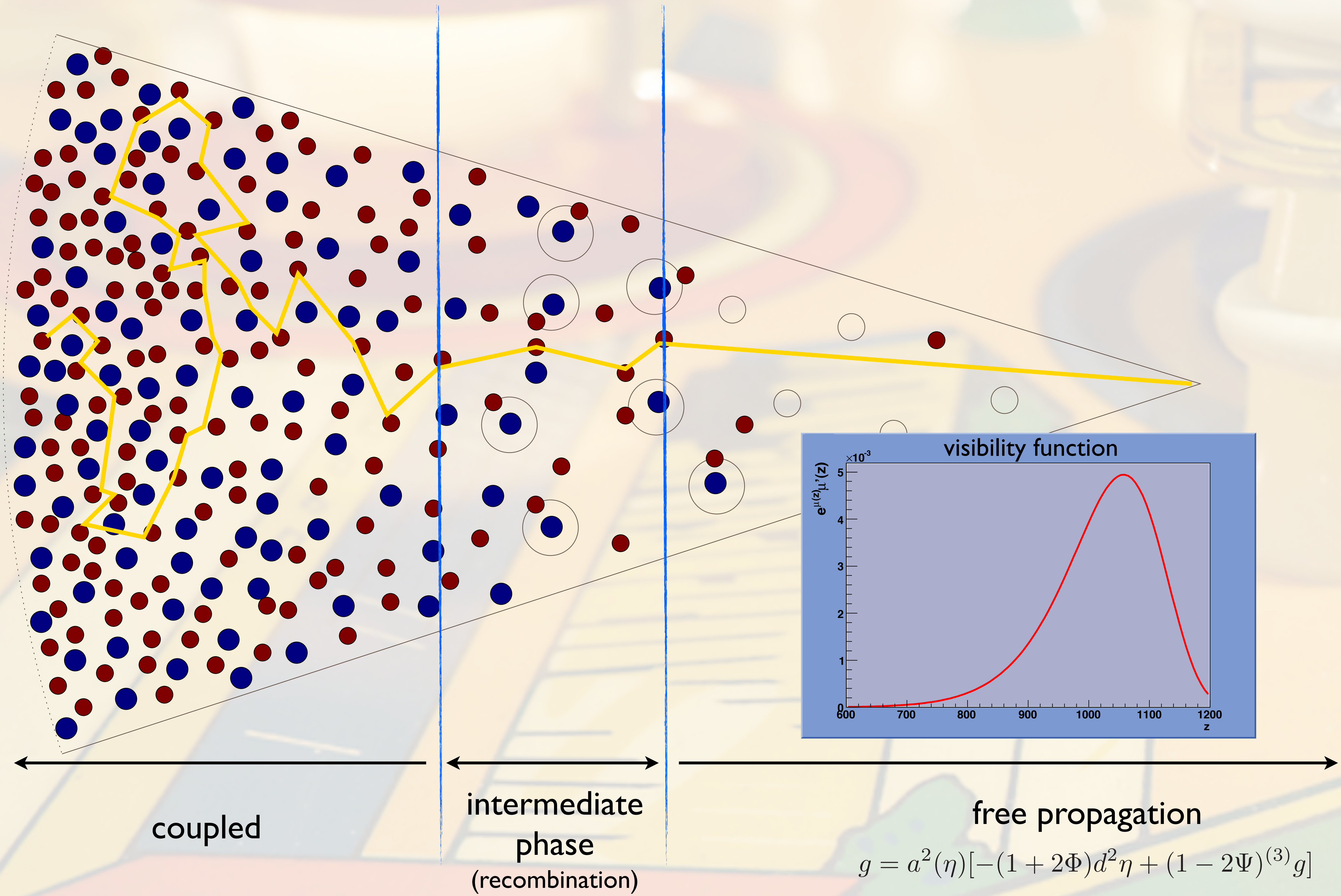
CMB: basics



$$g = a^2(\eta)[-(1+2\Phi)d^2\eta + (1-2\Psi)^{(3)}g]$$

Integral version of Boltzmann's equations

Temperature

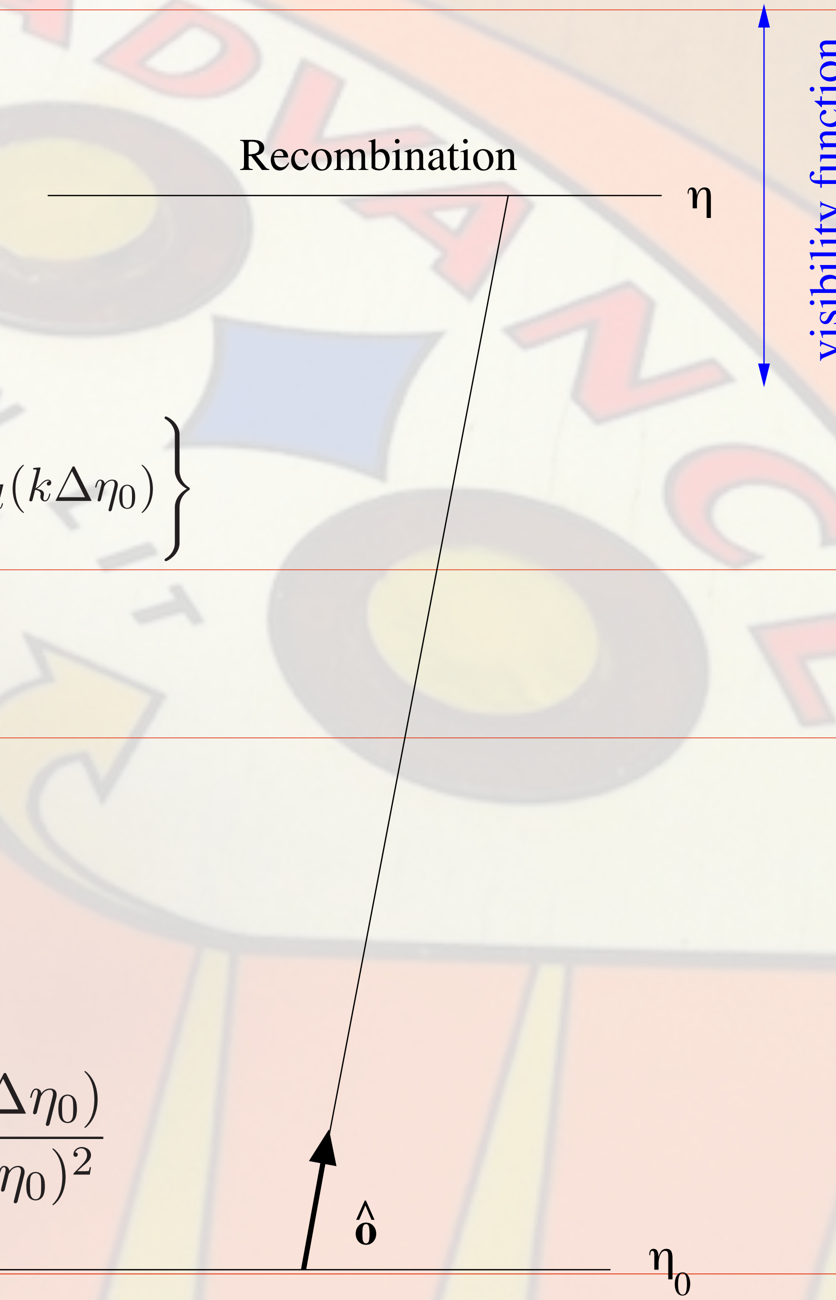
$$\Theta(\mathbf{x}_o,\eta_o,\hat{\mathbf{o}}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\, e^{i\mathbf{k}\cdot\mathbf{x}_o}\, 4\pi \sum_{lm} i^l\, \Theta_l(\mathbf{k},\eta_o)\, \mathrm{Y}^*_{lm}(\hat{\mathbf{k}})\, \mathrm{Y}_{lm}(\hat{\mathbf{o}})$$

$$\Theta_l(\mathbf{k},\eta_o) = \int_0^{\eta_o} d\eta\, e^{-\mu(\eta)} \Big\{ \mu'(\eta) \Big[\theta_{SW}(\mathbf{k},\eta)\, j_l(k\Delta\eta_0) - k\, V_b(\mathbf{k},\eta)\, j_l'(k\Delta\eta_0)$$
$$+ \frac{1}{2}\Big[\Theta_2(\mathbf{k},\eta) - \sqrt{6}\alpha_2(\mathbf{k},\eta)\Big] \Big[\frac{3}{2} j_l''(k\Delta\eta_0) + \frac{1}{2} j_l(k\Delta\eta_0)\Big]\Big] + (\Psi' + \Phi')(\mathbf{k},\eta)\, j_l(k\Delta\eta_0) \Big\}$$

Polarization

$$\frac{Q+iU}{4I}(\mathbf{x}_o,\eta_o,\hat{\mathbf{o}}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\, e^{i\mathbf{k}\cdot\mathbf{x}_o}\, 4\pi \sum_{lm} i^l\, \alpha_l(\mathbf{k},\eta_o)\, \mathrm{Y}^*_{lm}(\hat{\mathbf{k}})\, {}_2\mathrm{Y}_{lm}(\hat{\mathbf{o}})$$

$$\alpha_l(\mathbf{k},\eta_o) = -\frac{3}{2}\sqrt{\frac{(l+2)!}{(l-2)!}} \int_0^{\eta_o} d\eta\, \mu'(\eta) e^{-\mu(\eta)}\, \frac{1}{2}\Big[\Theta_2(\mathbf{k},\eta) - \sqrt{6}\alpha_2(\mathbf{k},\eta)\Big] \frac{j_l(k\Delta\eta_0)}{(k\Delta\eta_0)^2}$$



Polarization in position space - uncoupled temperature

$$\Theta(\eta_o, \hat{\mathbf{o}}) = \sum_{lm} \theta^{(0)}_{lm}(\eta_o) \mathrm{Y}_{lm}(\hat{\mathbf{o}})$$

$$\theta^{(0)}_{lm} = 2\, i^l \int_0^{\eta_o} d\eta \int \frac{dk}{(2\pi)^{1/2}}\, k^2\, S_{lm}(k,\eta)\, j_l(k\Delta\eta_o)$$

$$S_{lm}(k,\eta) := \int d^2\hat{\mathbf{k}}\, \mathrm{e}^{-\mu(\eta)} \left\{ \mu'(\eta) \left[\Theta_{SW}(\mathbf{k},\eta) - V_b(\mathbf{k},\eta) \frac{\partial}{\partial\eta} \right] + (\Psi' + \Phi')(\mathbf{k},\eta) \right\} \mathrm{Y}^*_{lm}(\hat{\mathbf{k}})$$

$$S_{lm}(k,\eta) = 2\,(-i)^l \int \frac{dX}{(2\pi)^{1/2}}\, X^2\, S_{lm}(X,\eta)\, j_l(k\,X)$$

$$\theta^{(0)}_{lm} = \int_0^{\eta_o} d\eta\, S_{lm}(X = \Delta\eta_0, \eta)$$

Polarization in position space - uncoupled temperature

$$\frac{Q+iU}{4I}(\eta_o,\hat{\mathbf{o}}) = \sum_{lm} \pi_{lm}(\eta_o)\, {}_2\mathrm{Y}_{lm}(\hat{\mathbf{o}})$$

$$\pi_{lm}(\eta_o) = \sum_{n=0}^{\infty} \pi^{(n)}_{lm}(\eta_o)$$

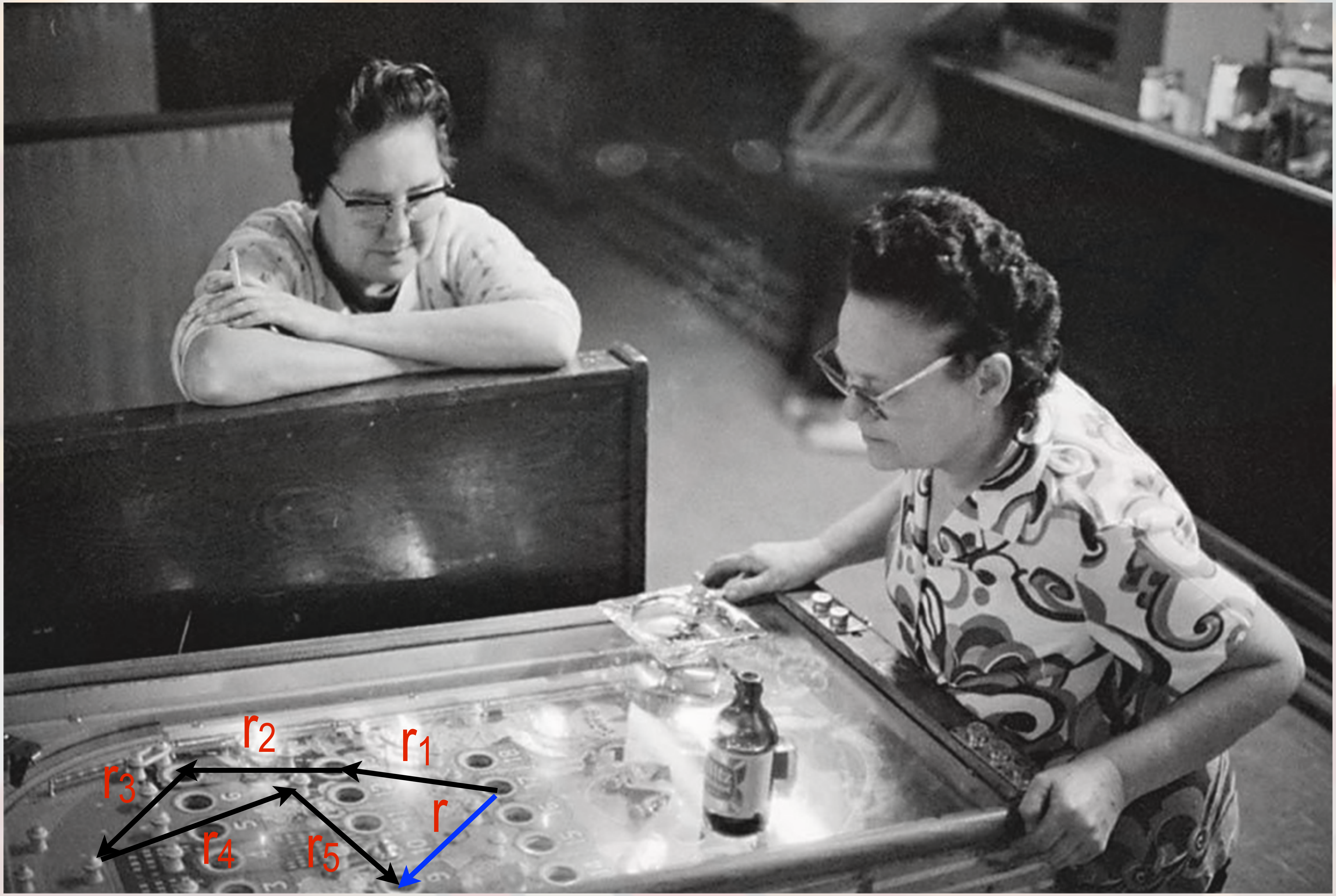
$$\pi^{(1)}_{lm}(\eta_o) = -\frac{3}{2\pi}\sqrt{\frac{(l+2)!}{(l-2)!}}\int_0^{\eta_o} d\eta_1\, g(\eta_1) \int_0^{\eta_1} d\eta \int_0^{\infty} dX\, X^2\, S_{lm}(X,\eta) \int_0^{\infty} dk\, k^2\, j_l(kX)\, \frac{j_l(k\Delta\eta_o)}{(k\Delta\eta_o)^2}\, j_2(k\Delta\eta_1)$$

$$\pi^{(2)}_{lm}(\eta_o) = -\frac{3}{2\pi}\sqrt{\frac{(l+2)!}{(l-2)!}}\int_0^{\eta_o} d\eta_1\, g(\eta_1) \int_0^{\eta_2} d\eta_2\, g(\eta_2) \int_0^{\eta_2} d\eta \int_0^{\infty} dX\, X^2 S_{lm}(X,\eta) \left[9 \int dk k^2 j_l(kX) \frac{j_l(k\Delta\eta_o)}{(k\Delta\eta_o)^2} \frac{j_2(k\Delta\eta_1)}{(k\Delta\eta_1)^2} j_2(k\Delta\eta_2) \right]$$

$$\pi^{(3)}_{lm}(\eta_o) = -\frac{3}{2\pi}\sqrt{\frac{(l+2)!}{(l-2)!}}\int_0^{\eta_o} d\eta_1\, g(\eta_1) \int_0^{\eta_1} d\eta_2\, g(\eta_2) \int_0^{\eta_2} d\eta_3\, g(\eta_3) \int_0^{\eta_3} d\eta \int_0^{\infty} dX\, X^2\, S_{lm}(X,\eta)\;\; 9^2 \int dk\, k^2\, j_l(kX)\, \frac{j_l(k\Delta\eta_o)}{(k\Delta\eta_o)^2}\, \frac{j_2(k\Delta\eta_1)}{(k\Delta\eta_1)^2}\, \frac{j_2(k\Delta\eta_2)}{(k\Delta\eta_2)^2}\, j_2(k\Delta\eta_3)$$

$$\pi^{(n)}_{lm}(\eta_o) = -\frac{3}{2\pi}\sqrt{\frac{(l+2)!}{(l-2)!}}\int_0^{\eta_o} d\eta_1\, g(\eta_1) \times \frac{1}{(n-1)!}\int_0^{\eta_1} d\eta_2 \ldots d\eta_n\, \mathrm{T}\{g(\eta_2)\ldots g(\eta_n)\}$$
$$\times \int_0^{\eta_n} d\eta \int_0^{\infty} dX\, X^2\, S_{lm}(X,\eta)\;\; 9^{(n-1)} \int dk\, k^2\, j_l(kX)\, \frac{j_l(k\Delta\eta_o)}{(k\Delta\eta_o)^2}\, \underbrace{\frac{j_2(k\Delta\eta_1)}{(k\Delta\eta_1)^2} \ldots \frac{j_2(k\Delta\eta_{n-1})}{(k\Delta\eta_{n-1})^2}}_{(n-1)\,times}\, j_2(k\Delta\eta_n)$$

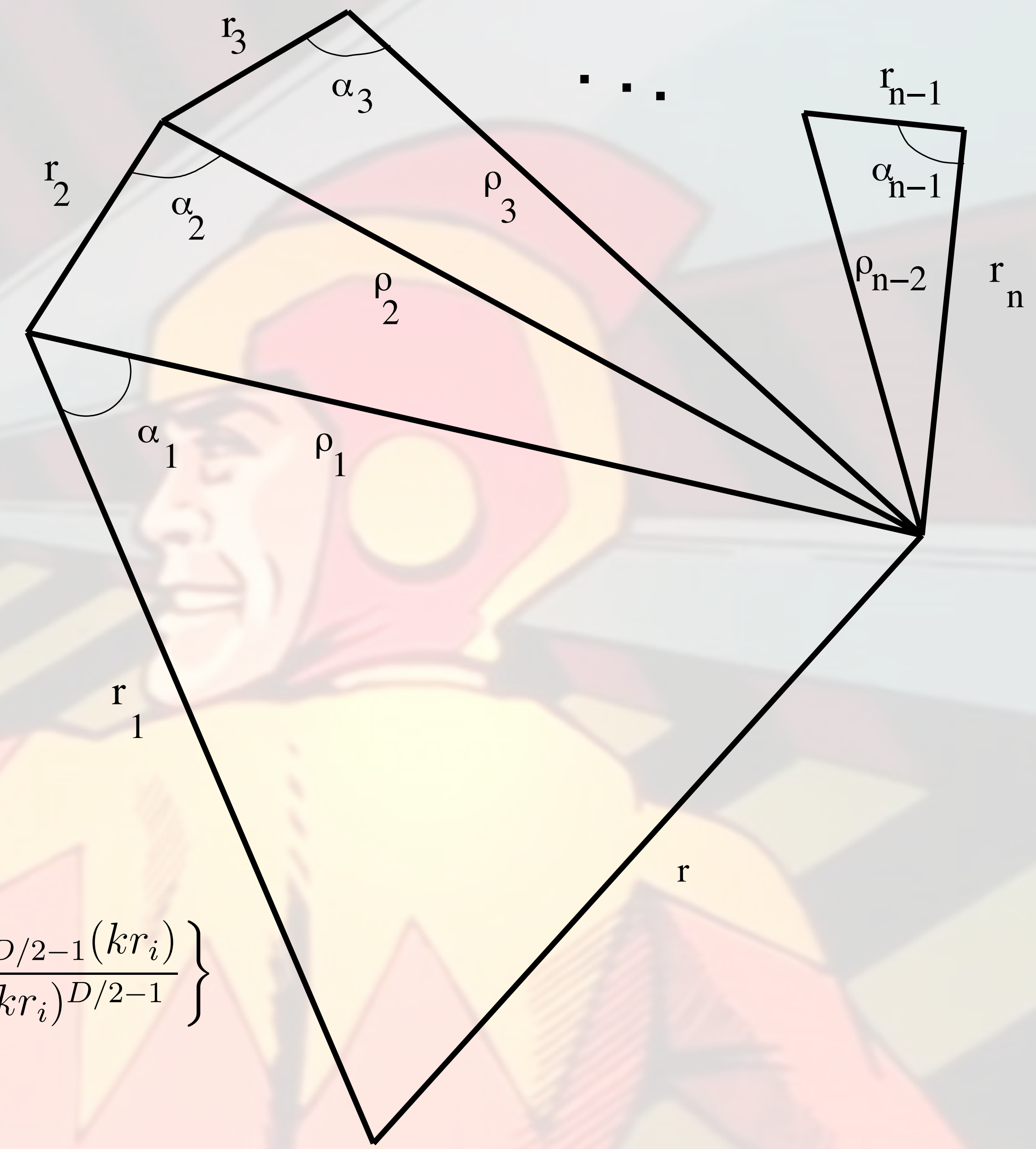
Random Flights



H. Spohn, Commun. Math. Phys., 60, 1978

Random Flights: position of the problem

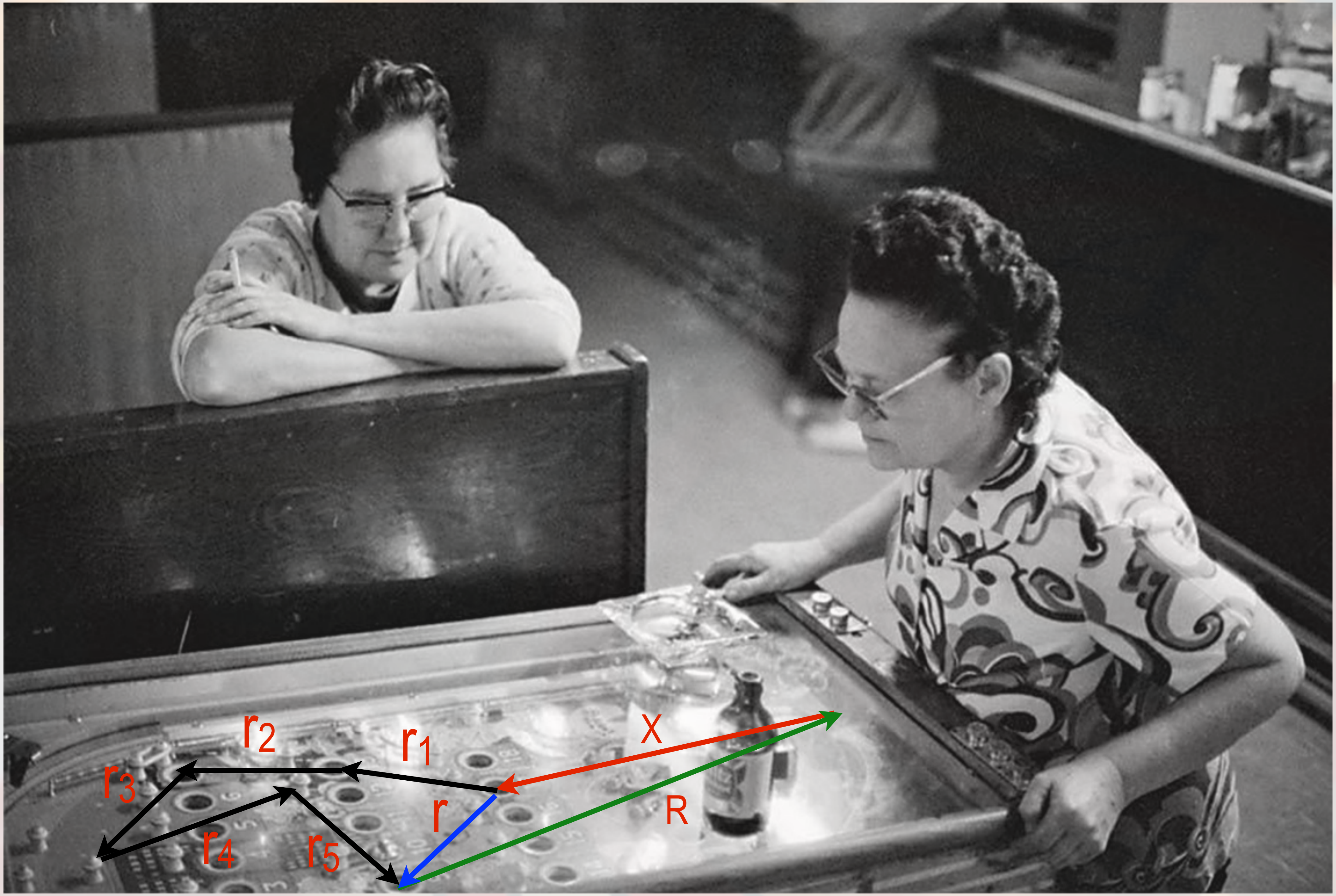
A walker moves with constant speed in a D-dimensional and performs n steps of lengths r_1, r_2, …, r_n. After each step the walker changes isotropically its direction of motion. What is the probability that after n steps, the walker will be at a distance r from the origin?



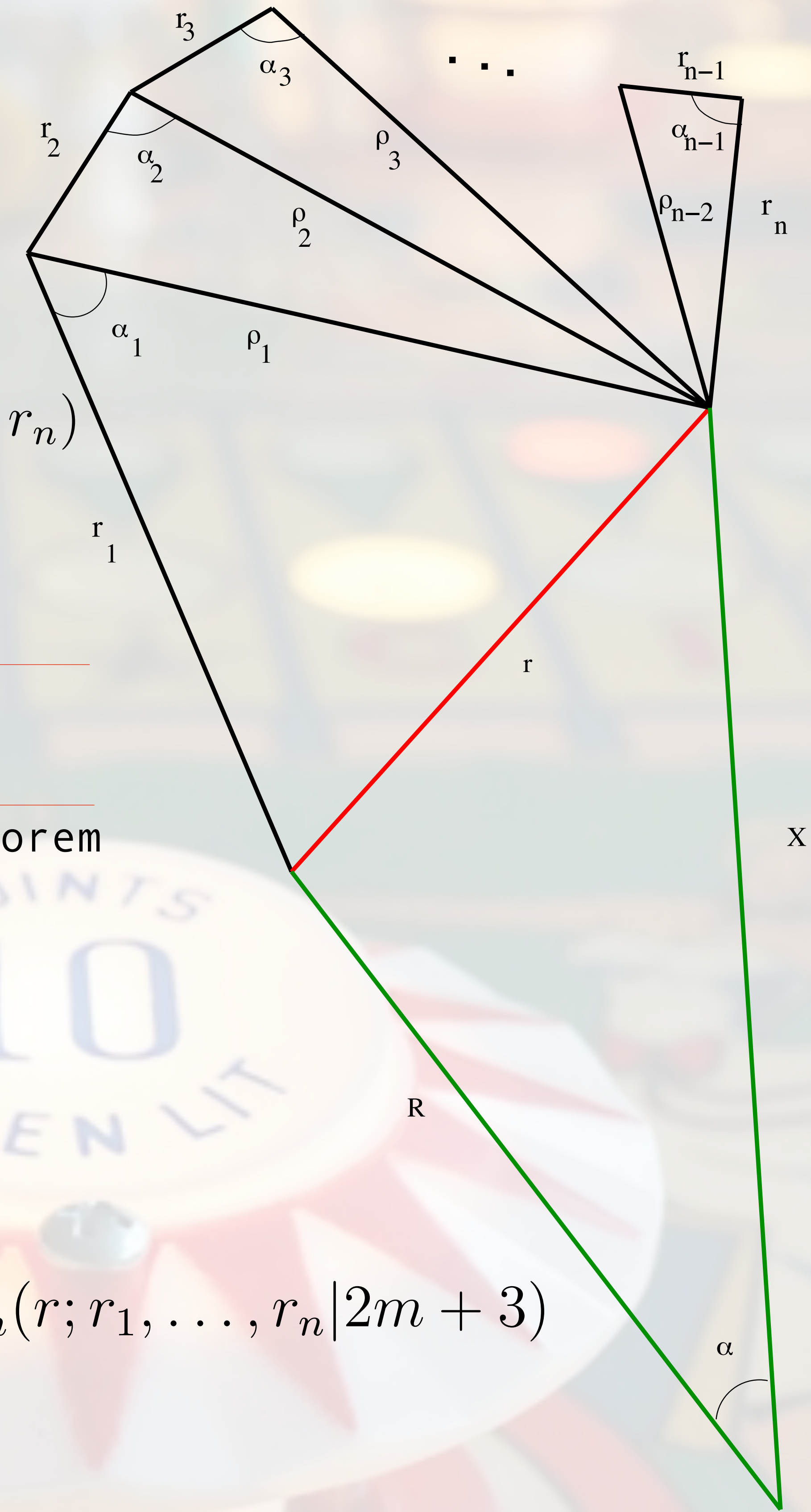
$$p_n(r; r_1, \ldots, r_n|D) := \frac{d}{dr}\left\{ [\Gamma(D/2)]^{n-1} \int_0^\infty dk\, r \left(\frac{kr}{2}\right)^{D/2-1} J_{D/2}(kr) \prod_{i=1}^{n} \frac{J_{D/2-1}(kr_i)}{(kr_i)^{D/2-1}} \right\}$$

$$\frac{\left(\frac{\pi}{2}\right)^{(n+1)/2}}{\left[\Gamma\left(m+\frac{3}{2}\right)\right]^{n-1}} \frac{r_n^m}{r^{m+2}} p_n(r; r_1, \ldots, r_n|2m+3) = \int_0^\infty dk\, k^2\, j_m(kr) \prod_{q=1}^{n-1} \frac{j_m(kr_q)}{(kr_q)^m} j_m(kr_n)$$

Extended Random Flights



Extended random flights



$$H_{lm}(R, X, r_1, ..., r_n) := \int_0^\infty dk\, k^2\, \frac{j_l(k\, R)}{(k\, R)^m}\, j_l(k\, X) \prod_{q=1}^{n-1} \frac{j_m(k\, r_q)}{(k\, r_q)^m}\, j_m(k\, r_n)$$

$$j_l(k\, r_1)\, j_l(k\, r_2) = \frac{(-1)^m}{2} \int_{|r_1 - r_2|}^{r_1 + r_2} dr\, k^m \left(\frac{r_1 r_2}{r}\right)^{m-1} P_l^{-m}(\cos\alpha)\, \sin^m \alpha\, j_m(k\, r)$$

Gegenbauer addition theorem

$$H_{lm}(R, X, r_1, ..., r_n) = \frac{(-1)^m}{2} \frac{\left(\frac{\pi}{2}\right)^{(n+1)/2}}{\left[\Gamma\left(m + \frac{3}{2}\right)\right]^{n-1}} \int_{|R-X|}^{R+X} dr\; \frac{r_n^m}{r^{m+2}}$$

$$\times \frac{1}{R^m} \left(\frac{RX}{r}\right)^{m-1} P_l^{-m}(\cos\alpha)\; \sin^m \alpha\; p_n(r; r_1, \ldots, r_n | 2m+3)$$

Recovering the hierarchy

$$\Theta(\eta_o, \hat{\mathbf{o}}) = \sum_{lm} \theta^{(0)}_{lm}(\eta_o) \mathrm{Y}_{lm}(\hat{\mathbf{o}})$$

$$\theta^{(0)}_{lm} = \int_0^{\eta_o} d\eta \, S_{lm}(X = \Delta\eta_0, \eta)$$

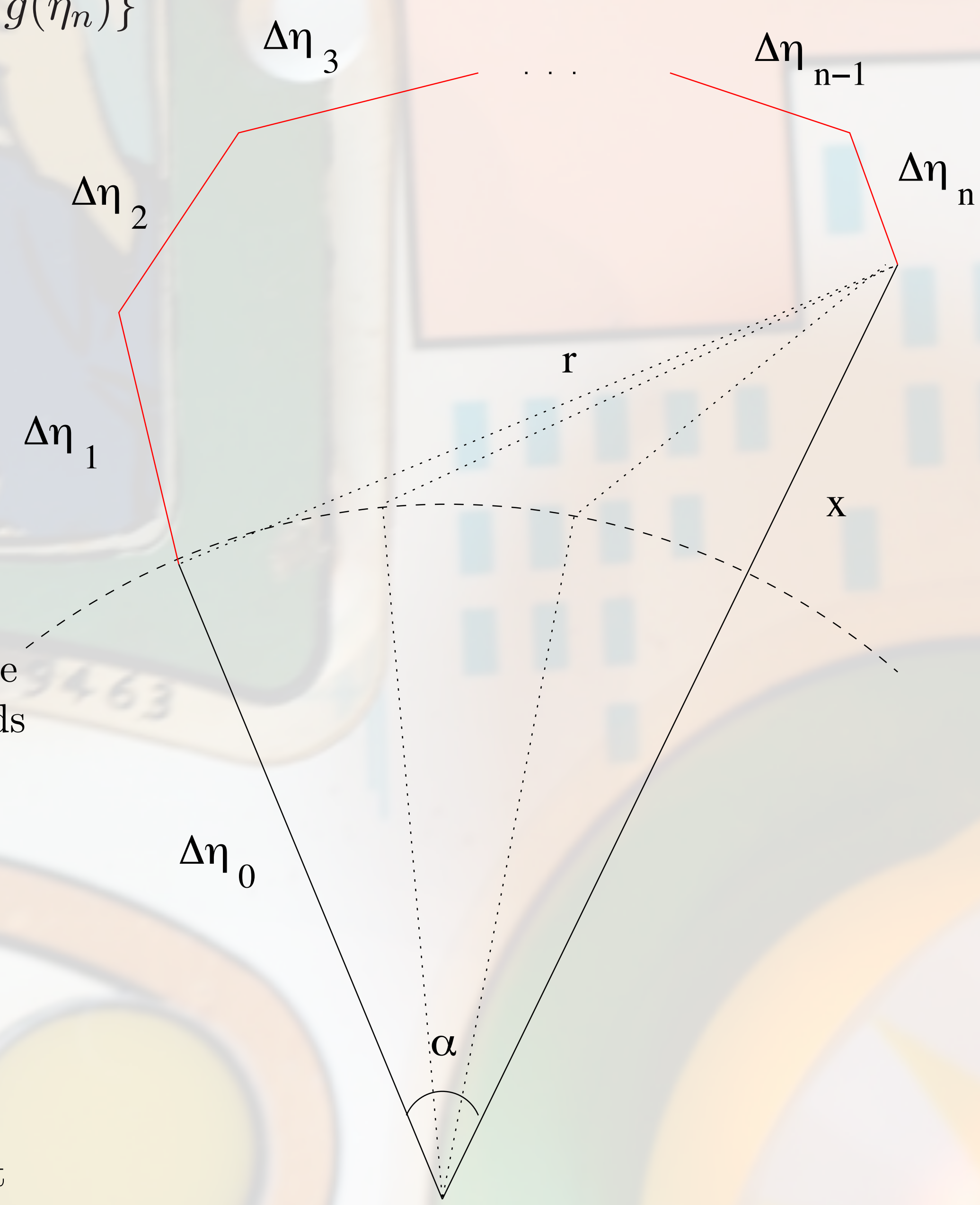
$$\frac{Q + iU}{4I}(\eta_o, \hat{\mathbf{o}}) = \sum_{lm} \pi_{lm}(\eta_o) \, {}_2\mathrm{Y}_{lm}(\hat{\mathbf{o}})$$

$$\pi_{lm} = -\frac{3}{4\pi}\sqrt{\frac{(l+2)!}{(l-2)!}} \int_0^{\eta_o} d\eta_1 \, g(\eta_1) \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_0^{\eta_1} d\eta_2 \ldots d\eta_n \, \mathrm{T}\{g(\eta_2) \, \ldots \, g(\eta_n)\}$$

$$\times \frac{9^{n-1}\left(\frac{\pi}{2}\right)^{(n+1)/2}}{\left[\Gamma\left(\frac{7}{2}\right)\right]^{n-1}} \int_0^{\eta_n} d\eta \int_0^{\infty} dX \, X^2 \, S_{lm}(X, \eta) \int d(\cos\alpha) \left(\frac{X\Delta\eta_n}{r^3}\right)^2 P_l^{-2}(\cos\alpha) \, \sin^2\alpha \; p_n(r; \Delta\eta_1, \ldots, \Delta\eta_n \,|\, 7)$$

Interpretation

$$\pi_{lm} = -\frac{3}{4\pi}\sqrt{\frac{(l+2)!}{(l-2)!}}\int_0^{\eta_o} d\eta_1\, g(\eta_1) \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\int_0^{\eta_1} d\eta_2 \ldots d\eta_n\, \mathrm{T}\{g(\eta_2)\ \ldots\ g(\eta_n)\}$$
$$\times \frac{9^{n-1}\left(\frac{\pi}{2}\right)^{(n+1)/2}}{\left[\Gamma\left(\frac{7}{2}\right)\right]^{n-1}} \int_0^{\eta_n} d\eta \int_0^{\infty} dX\, X^2\, S_{lm}(X,\eta)$$
$$\times \int d(\cos\alpha) \left(\frac{X\,\Delta\eta_n}{r^3}\right)^2 P_l^{-2}(\cos\alpha)\, \sin^2\alpha\ p_n(r;\Delta\eta_1,\ldots,\Delta\eta_n\,|\,7)\ .$$



i. e.,

Average over all possible shapes of the polygon with given sides;

Compute the source term contributions mediated by the weight given the the average shape of the polygon. We should remark that the integration over x no longer extends to infinity: x is limited by the sum of intermediate steps!

Average the lengths of the intermediate steps weighted by the visibility functions;

Sum for all possible number of intermediate steps;

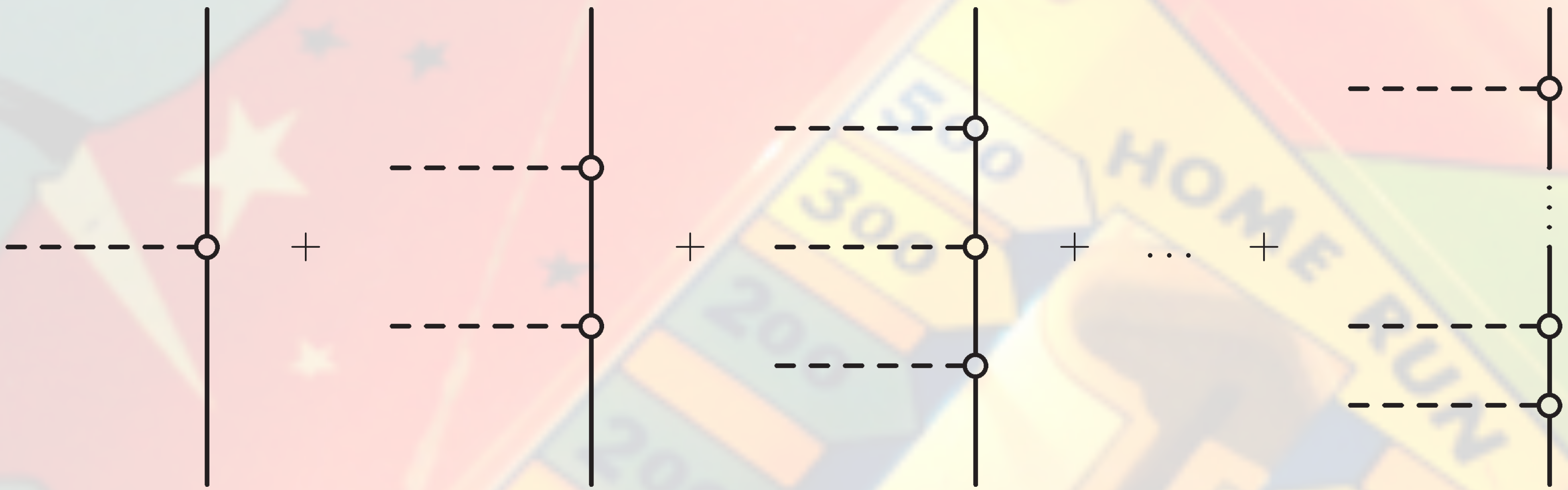
7=2 + 2(2.2 + 1);

Upper limit in the integration in X assures that sources are computed in the support of the observers past light cone.

Diagrams

- Solid lines mean polarization, and dashed lines mean temperature;
- The vertical lines to the right of the diagrams represent the observable being calculated (time runs upward);
- The interception of two lines determine what sources are being considered for a given observable.

$\pi_{lm}(\eta_o) =$



Note: Diagrams with more legs correspond to higher powers of the visibility function.

Convergence

We shall consider the special case when $\Delta\eta_1 = \Delta\eta_2 = \ldots = \Delta\eta_n =: \Delta$ in the limit $n \to \infty$.
It can be shown (Watson, A Treatise on the Theory of Bessel Functions) that:

$$p_n(r; \Delta, \ldots, \Delta \,|\, 7) \leq \frac{1}{r} \left(\frac{7r^2}{2n\Delta^2} \right)^{7/2} \mathrm{e}^{-\frac{7r^2}{2n\Delta^2}}$$

and then, for $\mathrm{r}/_{\Delta}$ finite:

$$\lim_{n\to\infty} p_n(r; \Delta, \ldots, \Delta \,|\, 7) = \delta(r) \,.$$

In this case we show that:

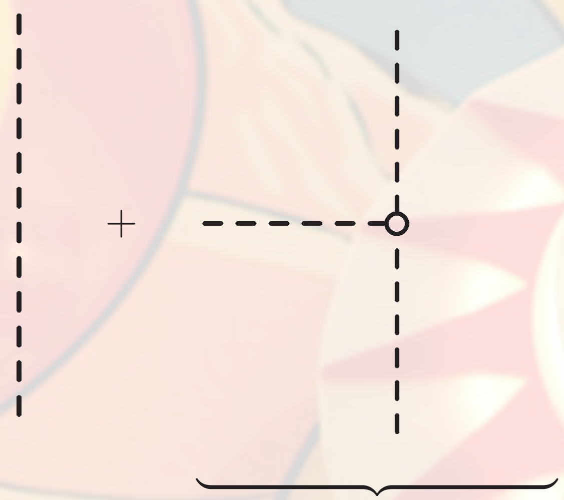
$$\mathcal{S}_{lm} = \lim_{n\to\infty} \int_0^{\eta_o - \eta} dx x^2 S_{lm}(x, \eta) \int d(\cos\alpha) \left(\frac{x\Delta}{r^3} \right)^2 P_l^{-2}(\cos\alpha)(\sin\alpha)^2 \, p_n(r; \Delta, \ldots, \Delta|7)$$

$$= \frac{\pi}{30} \frac{\Delta^2}{(\Delta\eta_0)^2} S_{lm}(\Delta\eta_0, \eta) \to 0 \,.$$

We conclude, then, that high order terms (corresponding to large number of intermediate scatterings) are suppressed in the series expansion, as one would expect from Boltzmann's H-theorem.

Recoupling the temperature

$\bar{\theta}^{(1)}_{lm} =$

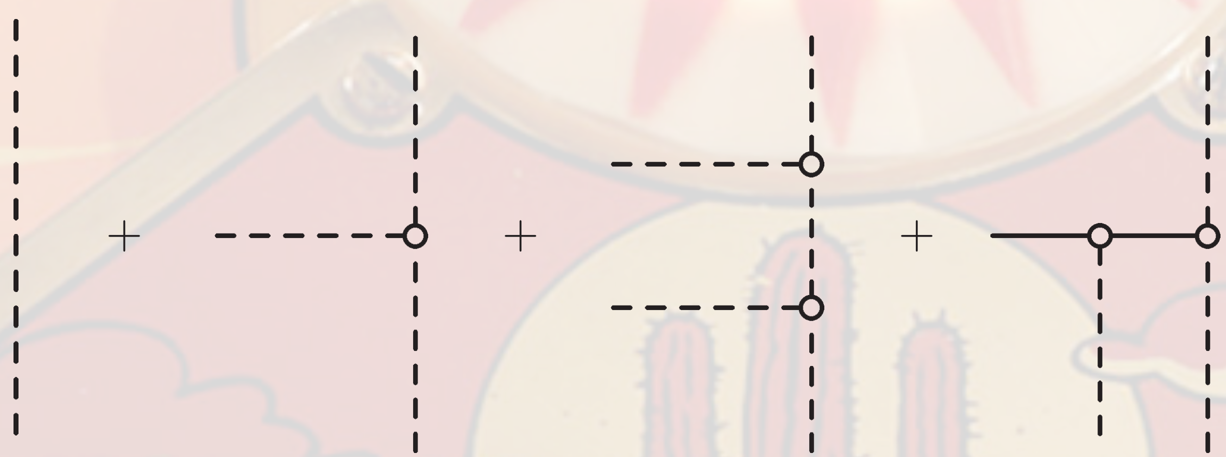


$H_{l2}, H_{(l+1)2}$

Complicator:

$$\int dk\, k^2\, j_l(kX) \left[1 + 3\frac{\partial^2}{\partial(k\Delta\eta_o)^2}\right] j_l(k\Delta\eta_o)\, j_2(k\Delta\eta_1)$$

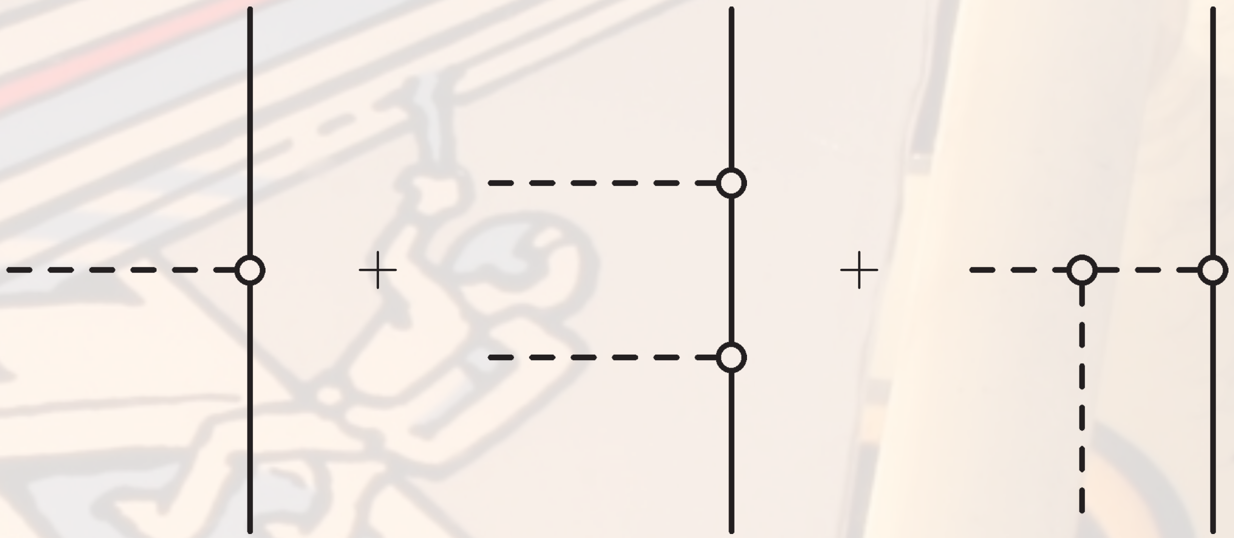
$\bar{\theta}^{(2)}_{lm} =$



$$\bar{\theta}^{(n)}_{lm} = \sum_{q=0}^{n} \theta^{(n)}_{lmq}(\Delta\eta_0, \Delta\eta_1, \ldots, \Delta\eta_n, x)\, H_{(l+q)2}(\Delta\eta_0, x, \Delta\eta_1, \ldots, \Delta\eta_n)$$

Polarization at second order

$\bar{\pi}^{(2)}_{lm} =$



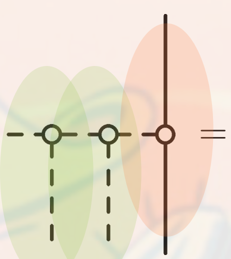
$$\bar{\pi}^{(n)}_{lm} = \sum_{q=0}^{n-1} \pi^{(n)}_{lmq}(\Delta\eta_0, \Delta\eta_1, \ldots, \Delta\eta_n, X)\; H_{(l+q)2}(\Delta\eta_0, x, \Delta\eta_1, \ldots, \Delta\eta_n)$$

General rule for diagrams

$$\text{Contribution} = \int_0^{\eta_o} d\eta_1\, g(\eta_1)\ \frac{1}{(n-1)!}\int_0^{\eta_1} d\eta_2\ \ldots\ d\eta_n\, \mathrm{T}\{g(\eta_2)\ \ldots\ g(\eta_n)\}$$

$$\int_0^{\eta_n} d\eta \int dX\, X^2\, S_{lm}(X,\eta)$$

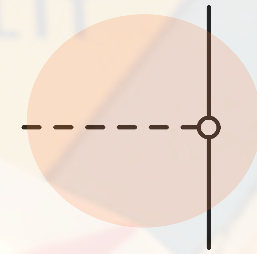
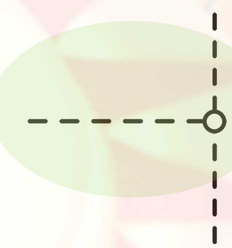
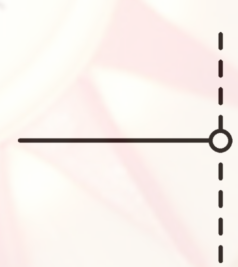
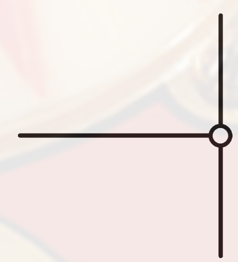
$$\int dk\, k^2\ j_l(kX)\ [\ \text{Diagram}\ ]\, j_2(k\Delta\eta_n)$$



$$= -\frac{1}{2}\frac{1}{2}\frac{3}{2\pi}\sqrt{\frac{(l+2)!}{(l-2)!}}\int_0^{\eta_o} d\eta_1\, g(\eta_1)\int_0^{\eta_1} d\eta_2\, g(\eta_2)\int_0^{\eta_2} d\eta_3\, g(\eta_3)\int_0^{\eta_3} d\eta$$

$$\times \int dX\, X^2\, S_{lm}(X,\eta)\left\{\int dk\, k^2\, j_l(kX)\,\frac{j_l(k\Delta\eta_0)^2}{(k\Delta\eta_0)}\right.$$

$$\left.\times\left[1+3\frac{\partial^2}{\partial(k\Delta\eta_1)^2}\right] j_2(k\Delta\eta_1)\left[1+3\frac{\partial^2}{\partial(k\Delta\eta_2)^2}\right] j_2(k\Delta\eta_2)\, j_2(k\Delta\eta_3)\right\}$$

diag. segment	numerical factor	Bessel function
	$-\frac{3}{2\pi}\sqrt{\frac{(l+2)!}{(l-2)!}}$	$\frac{j_l(k\Delta)}{(k\Delta)^2}$
	$\frac{1}{2}$	$\left[1+3\frac{\partial^2}{\partial(k\Delta)^2}\right] j_l(k\Delta)$
	1	$\frac{j_2(k\Delta)}{(k\Delta)^2}$
	1	$\frac{j_2(k\Delta)}{(k\Delta)^2}$
	$\frac{1}{2}$	$\left[1+3\frac{\partial^2}{\partial(k\Delta)^2}\right] j_2(k\Delta)$
	9	$\frac{j_2(k\Delta)}{(k\Delta)^2}$

Conclusions

The Boltzmann's equations for CMB can be described in position space, where the causal structure becomes evident;

In position space CMB temperature fluctuations and polarization can be given in terms of a series expansion in terms of the number of scatterings during recombination;

Contributions from high order terms are suppressed.

(Figures http://www.pinballcreative.co.uk)