Compact Objects

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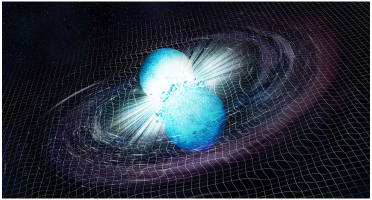
What we are going to talk about

- Neutron stars



The collision of two neutron stars that made waves in spacetime (www.sciencenews.org)

- White dwarfs

A white dwarf (All About Space/Imagine Publishing)

Bibliography

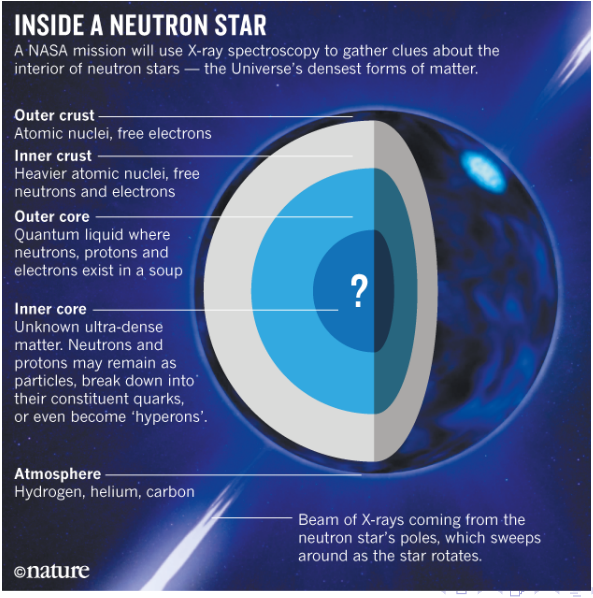
- **Black Holes, White Dwarfs, and Neutron Stars**, S.L. Shapiro and S.A. Teukolsky
- **Compact Stars: Nuclear Physics, particle Physics and General Relativity**, N.K. Glendenning, Springer (2000)
- **Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity**, S. Weinberg (1972) Wiley
- A lot of nice lectures in the Internet, for example have a look on M. Pettini's and K. Kokkotas'
- Review on **Stellar structure in modified gravity**, G. Olmo, D. Rubiera-Garcia, A. Wojnar, to appear this year

A brief history of neutron stars

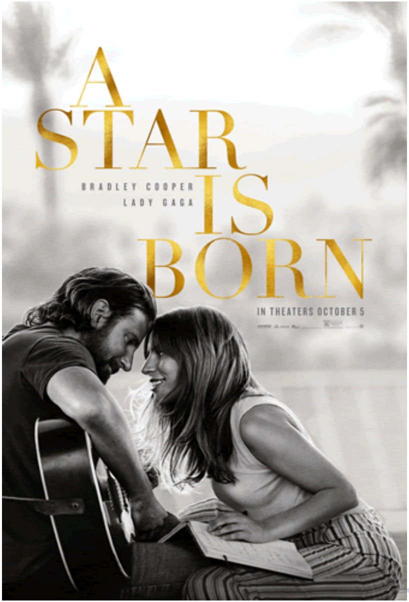
- Theoretical prediction in 1934 by Walter Baade and Fritz Zwicky
- Discovery of pulsars (spinning neutron stars) in 1967 by Jocelyn Bell (**a graduate student!**)
- Thomas Gold and Franco Pacini suggested that pulsars are rotating neutron stars, in 1968

Jocelyn Bell

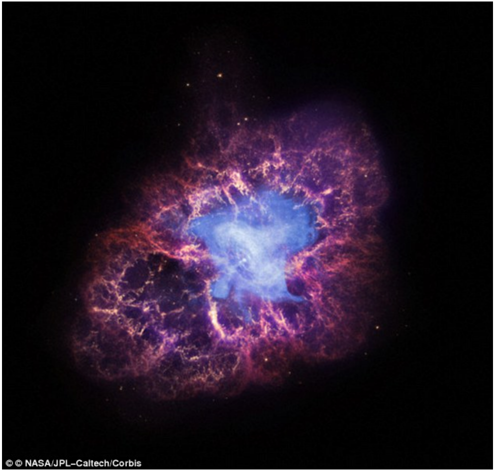
A neutron star: a super-dense, fascinating object



A star is born

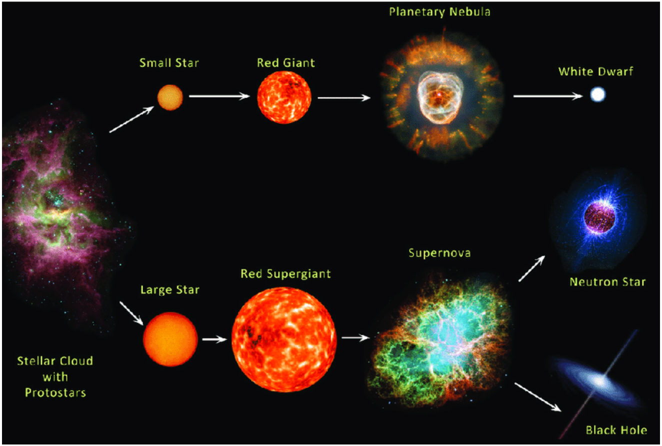


A neutron star is born



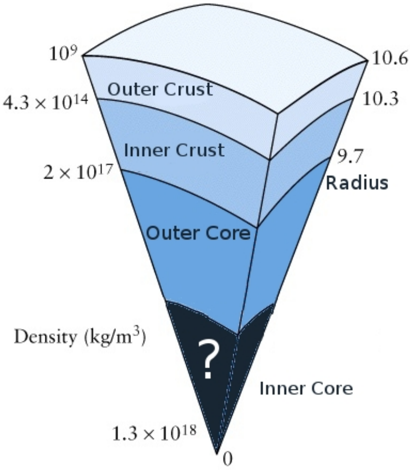
NS is the collapsed core of a massive star ($8 - 25\ M_{\odot}$) left behind after a supernova explosion. This compresses at least to $1.4 - 2M_{\odot}$ into a sphere about $10 - 13$km.

Stellar evolution



Earth Blog [2013]: This remnant is crushed so tightly that gravity overcomes the repulsive force between electrons and protons.

Neutron star



The structure is complex: solid crystalline crust about 1 km thick which encases a core of superfluid neutrons and supercondacting protons. Above the crust there is an ocean and atmosphere of much less dense material.

https://apatruno.wordpress.com/neutron-stars/

Relativistic stars in GR: neutron stars

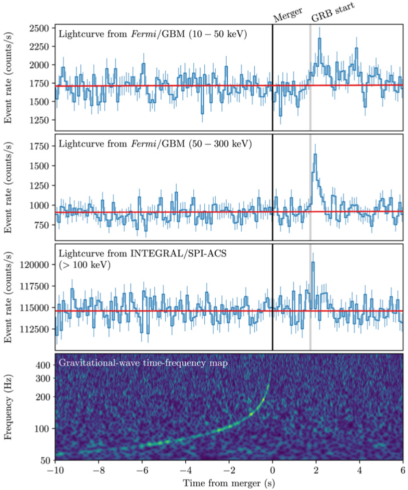
- "Aborted" black holes - stars which failed during the gravitational collapse to become black holes because they were not massive enough
- One of the possible final evolution scenarios of luminous stars.
- The smallest and densest stars known so far.
- A mass of $1.4 - 2$ of solar masses and a radius around $9 - 14$ kilometers.
- Rough assumptions about the neutron stars' composition: neutrons supported by their degeneracy pressure (Pauli exclusion principle).
- So far, we know about 2500+ neutron stars around here and there
- In our Galaxy there can be around 100,000,000 neutron stars

Why are they so interesting? I

- The pressure at the core may be like this that existed at the time of the Big Bang
- The most extreme magnetic fields known in the Universe (up to 10^{16} the strength of Earth's magnetic field)
- They can spin as fast as 716 (!!) times per second
- Suitable to test our theories about the interaction of gravity and matter at the highest densities achievable in the universe.
- NS can help constrain numerous models (both: models of nuclear matter composition & alternative theories of gravity).

Why are they so interesting? II

Ability to rule out certain families of modified gravity theories: the recent observation via gravitational waves and electromagnetic radiation of a neutron star merger.



Abbott, B. P., et al. "Gravitational Waves and Gamma Rays from a Binary Neutron Star Merger: GW170817 and GRB 170817A." 2017, ApJL, 848, L13.

General Relativity in a nutshell

General Relativity is a theory which describes gravity in a geometric way

In order to describe a gravitational system, we need to be supplied with the following

$$\begin{cases} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \\ \nabla^\mu T_{\mu\nu} = 0, \\ \text{Equation of state if vacuum not considered} \end{cases}$$

The main ingredients of the theory:

- A lot of indexes, usually based on Greek alphabet (and it is never enough)
- The metric $g_{\mu\nu}$ which is a solution of the Einstein field eqs
- The connection ∇_μ compatible with the metric $g_{\mu\nu}$
- Ricci curvature tensor $R_{\alpha\beta} = \partial_\rho \Gamma^\rho{}_{\beta\alpha} - \partial_\beta \Gamma^\rho{}_{\rho\alpha} + \Gamma^\rho{}_{\rho\lambda} \Gamma^\lambda{}_{\beta\alpha} - \Gamma^\rho{}_{\beta\alpha} \Gamma^\lambda{}_{\rho\alpha}$
- Christoffel symbols $\Gamma^\mu{}_{\nu\lambda} = \frac{1}{2} g^{\mu\beta} (g_{\beta\nu,\alpha} + g_{\beta\alpha,\nu} - g_{\nu\alpha,\beta})$
- Ricci curvature scalar $R = R_{\alpha\beta} g^{\alpha\beta}$
- Energy-momentum tensor, for example $T_{\mu\nu} = -p g_{\mu\nu} + (p + \rho) u_\mu u_\nu$
- EoS, for example barotropic one $p = p(\rho)$

General Relativity in a nutshell

General Relativity describes gravitational phenomena and some of them are pretty weird

- Describes motion in the Solar System: anomalous perihelion advance of Mercury
- Provides cosmological model (evolution of the Universe)
- Describes light propagation through spacetime
- Black holes
- Gravitational waves
- Compact stars
- ...
- Wormholes - time travels?

How to describe neutron stars

A spherical symmetric object

$$ds^2 = e^{\alpha(r)}dt^2 - e^{\beta(r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2.$$

with matter described by the perfect-fluid energy momentum tensor

$$T_{\mu\nu} = -pg_{\mu\nu} + (p+\rho)u_\mu u_\nu.$$

From the Einstein's field eqs. ($\kappa = 8\pi Gc^{-4}$) and the hydrostatic equilibrium

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}, \quad \nabla^\mu T_{\mu\nu} = 0$$

we are able to write the Tolman-Oppenheimer-Volkoff equations[1]

$$m'(r) = 4\pi r^2 \rho(r),$$
$$\frac{dp}{d\rho}\rho'(r) = -\frac{Gm(r)\rho(r)}{r^2}\frac{\left(1+\frac{p(r)}{\rho(r)}\right)\left(1+\frac{4\pi r^3 p(r)}{Gm(r)}\right)}{1-\frac{2Gm(r)}{r}}.$$

[1] R.C. Tolman, Phys Rev. 55(4):364,1939; J.R. Oppenheimer, G.M. Volkoff, Phys.Rev. 55(4):374,1939

Why TOV equations are so important?

- Basic equations to study relativistic stars
- They describe the interior of a relativistic, static, spherical symmetric star in hydrostatic equilibrium (the gravitational pull is exactly counter-balanced by the interior pressure)
- Numerical solutions of the TOV equations:
    - provide the $m - R$ and $m - \rho$ relations
        - That allows to test gravitational theories
        - as well as the theories of nuclear matter
        - and compare it with the observations of stars
    - allow to study the stability problem: equilibrium configuration can be stable or not (related to the maximum mass of the star)

Compactness

Another relevant parameter that we could find a limit for but we will not do that

- Compactness parameter $\mathcal{C} := M/R$
  - In vacuum, a Schwarzschild black hole $r_s = 2M$ and thus $\mathcal{C} = 1/2$ - absolute limit for any horizonless compact object
  - Upper theoretical limit on $\mathcal{C}$ **independent of the EoS** for a spherical fluid model of a NS:
    Buchdahl limit: the maximum amount of mass that can be enclosed within a sphere without experiencing gravitational collapse is

$$\mathcal{C} < 4/9$$

  - Compact objects violating the Buchdahl limit (it means it should collapse into a black hole)

$$4/9 < \mathcal{C} < 1/2$$

- Typically, a neutron star lies in the range $\mathcal{C} \sim 0.1 - 0.2$
- Any other star: $\mathcal{C} << 0.1$

Pulsars and planets questions

- After collapsing of the star's core, its rotation increases (conservation of angular momentum + significant decrease of the mass)
- Beams of electromagnetic radiation emitted from magnetic poles (visible light, X-rays, gamma-rays)
- Their light, like a lighthouse beam, sweeps across the Earth
- Not all neutron stars are pulsars
- The pulsar emission is derived from the kinetic energy of a rotating NS. Thus, pulsars are spinning down...
- ... and also because they loose energy because of the gravitational wave emission - magnetic fields slightly deform the pulsar which is the source of GWs
- Pulsar in binaries may pull over or accrete the matter from its companion. This may heat the transfered gas and produce X-rays
- Slowly rotating and non-accreting NS are almost undetectable (quickly cooling to 10^6K, light generated in X-rays
- Yes, they can have (and have) planets!

Derivation of the hydrodynamical equilibrium I

A spherical symmetric object

$$ds^2 = e^{\alpha(r)}dt^2 - e^{\beta(r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{1}$$

ruled by the Einstein field equations

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}$$

with matter described by the perfect-fluid energy momentum tensor

$$T_{\mu\nu} = -pg_{\mu\nu} + (p + c^2\rho)u_\mu u_\nu.$$

From the Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ one gets the conservation of the energy-momentum tensor $\nabla_\mu T^{\mu\nu} = 0$, that is:

$$0 = \nabla_\mu T^{\mu\nu} = \frac{\partial p}{\partial x^\nu}g^{\mu\nu} + \frac{\partial}{\partial x^\nu}[(p + c^2\rho)u^\mu u^\nu] + (p + c^2\rho)\Gamma^\mu_{\ \nu\lambda}u^\nu u^\lambda.$$

From the normalization of the observer $u^\mu u^\nu g_{\mu\nu} = c^2$ wrt the metric (1)

$$u^\mu = (ce^{-\frac{\alpha(r)}{2}}, 0, 0, 0)...$$

Derivation of the hydrodynamical equilibrium II

which simplify the conservation equation to

$$0 = \nabla_\mu T^{\mu\nu} = \frac{\partial p}{\partial x^\nu} g^{\mu\nu} + (p + c^2\rho)\Gamma^\mu_{\ 00} u^0 u^0.$$

What we need now are Christoffel symbols $\Gamma^\mu_{\ \nu\lambda}$ for the sph-sym metric.
Exercise: Calculate the Christoffel symbols for the spherical-symmetric metric.
Here you have the formula: $\Gamma^\mu_{\ \nu\alpha} = \frac{1}{2} g^{\mu\beta}(g_{\beta\nu,\alpha} + g_{\beta\alpha,\nu} - g_{\nu\alpha,\beta})$.

$$0 = \frac{\partial p}{\partial x^\nu} g^{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{00}(c^2 p + \rho) e^{-\alpha(r)} \frac{\partial \alpha(r)}{\partial x^\nu} \quad / g_{\mu\lambda}$$

$$\frac{dp}{dr} = -\frac{1}{2}(c^2\rho + p)\frac{d\alpha}{dr}$$

In the Newtonian limit $\rho >> p$ and $g_{00} = e^\alpha \approx 1 + 2U$ the above equation simplifies to the Poisson equation:

$$\frac{\partial p}{\partial r} = -\rho \frac{\partial U}{\partial r}.$$

The Einstein field equations

Let us write the Einstein's field eqs. ($\kappa = 8\pi G c^{-4}$) in a different form:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}, \quad \rightarrow \quad R_{\mu\nu} = \kappa (T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}.)$$

Exercise: Calculate the components of the Ricci tensor for the spherical-symmetric metric. Here you have the formula: $R_{\alpha\beta} = \partial_\rho \Gamma^\rho{}_{\beta\alpha} - \partial_\beta \Gamma^\rho{}_{\rho\alpha} + \Gamma^\rho{}_{\rho\lambda}\Gamma^\lambda{}_{\beta\alpha} - \Gamma^\rho{}_{\beta\alpha}\Gamma^\lambda{}_{\rho\alpha}$.

$$\{\theta\theta\} : 1 - e^{-\beta}[1 + \frac{r}{2}\alpha' - \beta'] = -4\pi r^2 (c^2 \rho - p), \tag{2}$$

$$\{rr\} : \frac{\alpha''}{2} + \frac{(\alpha')^2}{4} - \frac{\alpha'\beta'}{4} - \frac{\beta'}{r} = 4\pi e^{\beta}(c^2\rho - p), \tag{3}$$

$$\{tt\} : \frac{\alpha''}{2} + \frac{(\alpha')^2}{4} - \frac{\alpha'\beta'}{4} - \frac{\alpha'}{r} = -4\pi e^{\beta}(3c^2\rho + p). \tag{4}$$

$$\rightarrow \; r\beta' + e^{\beta} - 1 = 8\pi e^{\beta} r^2 \rho \quad \text{gives} \quad e^{-\beta(r)} = \left(1 - \frac{2Gm}{c^2 r}\right)$$

$$\text{where} \quad \frac{dm}{dr} = 4\pi r^2 \rho.$$

Finally, we obtain the Tolman-Oppenheimer-Volkoff equations

Using the solution $e^{-\beta(r)} = \left(1 - \frac{2Gm}{r}\right)$ to the equation (2)

$$\frac{d\alpha}{dr} = 2\frac{Gm + 4\pi r^3 P}{r(r - 2Gm)}.$$

Recalling that from the conservation law we have got $\frac{dP}{dr} = -\frac{1}{2}(c^2\rho + P)\frac{d\alpha}{dr}$, then

$$\boxed{\frac{dP}{dr} = -\frac{GM\rho}{r^2}\left(1 + \frac{P}{c^2\rho}\right)\frac{\left(1 + \frac{4\pi r^3 P}{Mc^2}\right)}{\left(1 - \frac{2GM}{c^2 r}\right)}}$$

$$\boxed{\frac{dm}{dr} = 4\pi r^2\rho}$$

Example: For a uniform energy density find the exact solution of the TOV

For $\rho =$ constant and from $\frac{dm}{dr} = 4\pi r^2 \rho$ we get immediately

$$m(r) = \frac{4}{3}\pi r^3 \rho \quad \text{for} \quad r \leq R,$$

$$M := m(r = R) = \frac{4}{3}\pi R^3 \rho \quad \text{for} \quad r \geq R.$$

Using this to the TOV eq. $\frac{dP}{dr} = -(c^2\rho + P)\frac{Gm + 4\pi r^3 P}{r(r - 2Gm)}$ one finds

$$\frac{P(r)}{\rho} = \frac{(1 - 2Gmr^2/R^2)^{\frac{1}{2}} - (1 - 2Gm/R)^{\frac{1}{2}}}{3(1 - 2Gm/R)^{\frac{1}{2}} - (1 - 2Gmr^2/R^3)^{\frac{1}{2}}}.$$

Now, use the above solution of the TOV eq. and apply it to $\frac{dP}{dr} = -\frac{1}{2}(c^2\rho + P)\frac{d\alpha}{dr}$.
You will find out that

$$e^{\frac{\alpha(r)}{2}} = \frac{3}{2}\left(1 - \frac{2Gm}{R}\right)^{\frac{1}{2}} - \frac{1}{2}\left(1 - \frac{2Gmr^2}{R^3}\right)^{\frac{1}{2}}$$

Thus, together with the previous solution on $e^{-\beta(r)} = \left(1 - \frac{2Gm}{r}\right)$, the metric is

$$ds^2 = \left(\frac{3}{2}\left(1 - \frac{2Gm}{R}\right)^{\frac{1}{2}} - \frac{1}{2}\left(1 - \frac{2Gmr^2}{R^3}\right)^{\frac{1}{2}}\right)^2 dt^2 - \left(1 - \frac{2Gm}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2.$$

Equation of State

- There are a lot of models of many-body nuclear theory providing different EoS (extrapolation over $\rho_{saturation} \approx 2.8 \times 10^{17}\text{kg/m}^3$)
- Not too much info on the contribution of quarks, hyperons, and phase transitions
- The fulfillment of the weak energy condition, which demands

$$\rho > 0 \text{ and } \rho + P > 0,$$

- The Le Chatelier's principle for microscopic stability of matter (to avoid spontaneous local collapse of matter), which reads

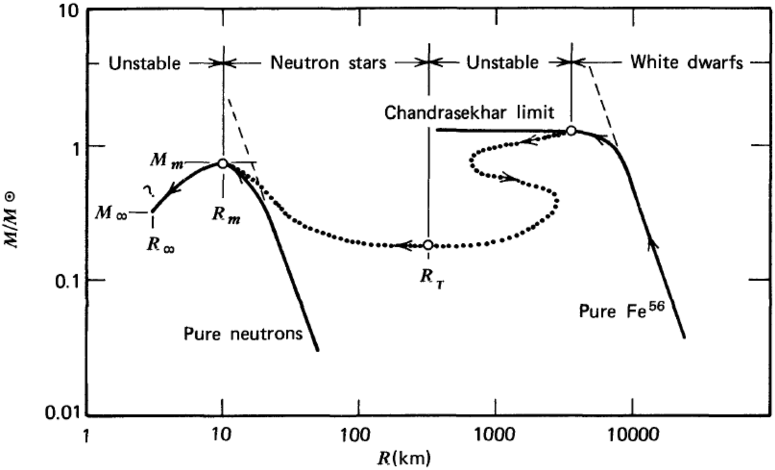
$$P \geq 0 \text{ and } dP/d\rho > 0,$$

- A causality constraint upon the speed of perturbations to be lower than the speed of light, namely,

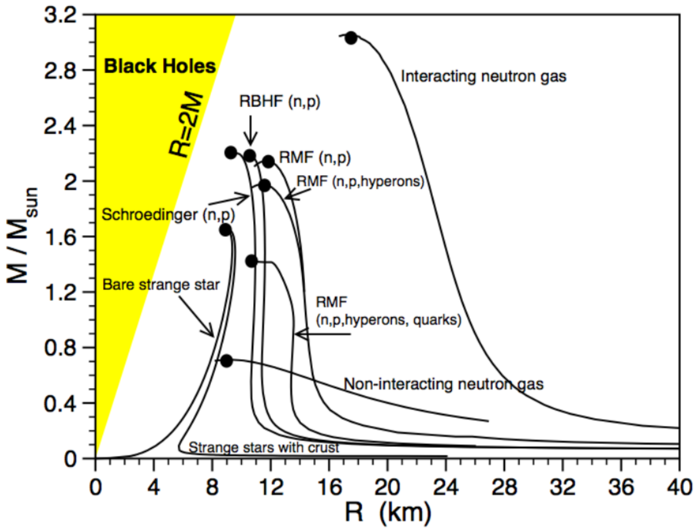
$$c_s \equiv (dP/d\rho)^{1/2} \leq c$$

- Consistency of the output of the corresponding numerical integration of the TOV equations with the maximum neutron star mass observed so far
- Typical boundary conditions: at the star's center $r = 0$ we take $\rho(0) = \rho_c$ (or $P(0) = P_c$) and $m(0) = 0$ while at $r = r_s$ one has $P(r_s) = 0$
- Matching with the external vacuum solution, that is, the Schwarzschild solution

Numerical solutions of TOV equations



Numerical solutions of TOV equations and EoS



$f(R)$ gravity in metric formalism[2]

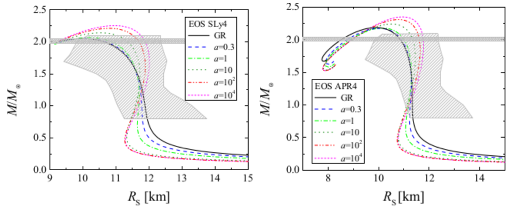


Figure 1. The mass of radius relation for EOS SLy4 (left panel) and APR4 (right panel). Different styles and colors of the curves correspond to different values of the parameter a. The current observational constrains are shown as shaded regions.

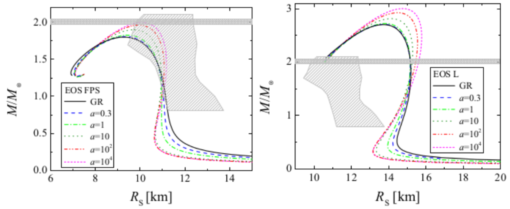


Figure 2. The mass of radius relation for EOS FPS (left panel) and L (right panel). Different styles and colors of the curves correspond to different values of the parameter a. The current observational constrains are shown as shaded regions.

[2] S. Yazadjiev, D. Doneva, K. Kokkotas, K. Staylov, 2014

What is a maximal mass of a neutron star?

- Highly depends on the EoS and theory of gravity
- The observations provide:
  - The radius bound $r_S \geq 9.60^{+0.14}_{-0.03}$km [3]
  - The heaviest compact pulsars:
    - ⋆ PSR J0348+0432, with $2.01 \pm 0.04 M_\odot$ [4].
    - ⋆ PSR J1614-2230, with $1.97 \pm 0.04 M_\odot$ [5].
    - ⋆ PSR J2215+5135, with $2.27^{+0.17}_{-0.15} M_\odot$ [6].
  - Several EoS ruled out in context of GR.

[3] A.Bauswein et al, Astrophys.J. 859 (2017) L34
[4] J. Antoniadis et al., Science 340, 6131 (2012)
[5] F. Crawford et al, Astrophys. J. 652 (2006) 1499
[6] M. Linares at al, The Astrophys. J., 859 (2018) 54

White dwarfs

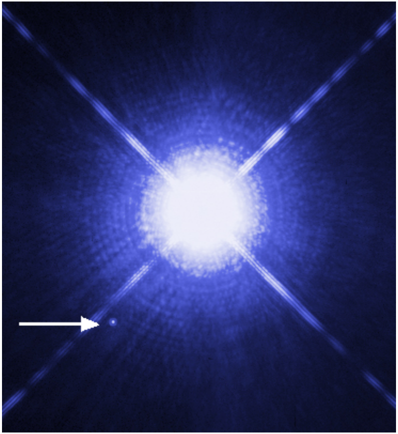
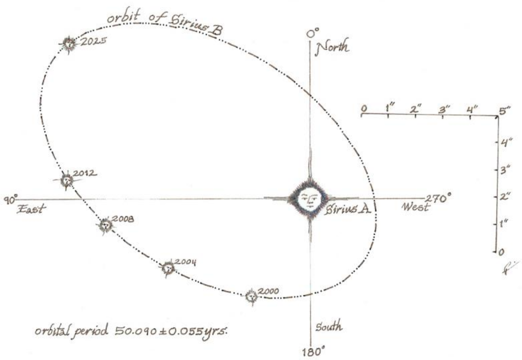


Image of Sirius A and Sirius B (white dwarf) at 8.6 light years

From Hubble Space Telescope

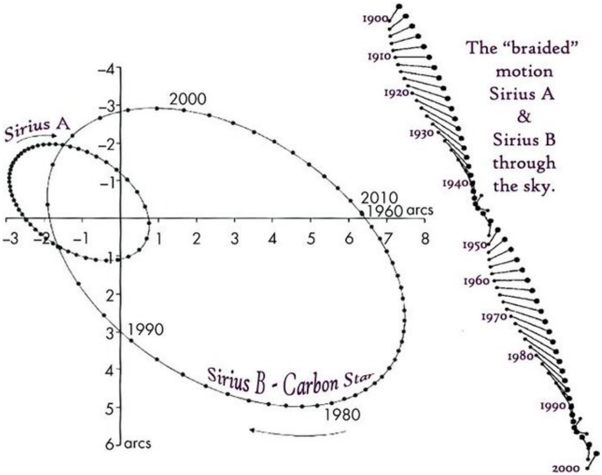
Sirius A and Sirius B orbits



The orbits of Sirius A and B

The first prediction made by F.W. Bessel (1784-1846), discovered by A. Clark (1832-1897), from Royal Astronomical Society of Canada

Sirius A and Sirius B orbits



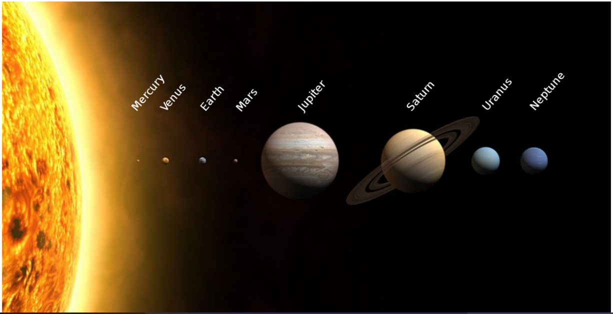
The orbits of Sirius A and B about the common center of mass of the binary system, and their projection on the sky

What is a white dwarf?

- A stellar core remnant composed mostly of electron-degenerate matter
- It is a very dense object: with a mass of the Sun while its volume is comparable to that of Earth
- It is composed of carbon and oxygen
- A faint luminosity comes from the emission of stored thermal energy - no fusion takes place
- It cannot support itself by the heat generated by fusion against gravitational collapsed
- It is supported by electron degeneracy pressure making it to be extremely dense
- This provides a maximum mass of such a star: Chandrasekhar limit $1.44 M_{\odot}$ which we are going to calculate

A very dense star

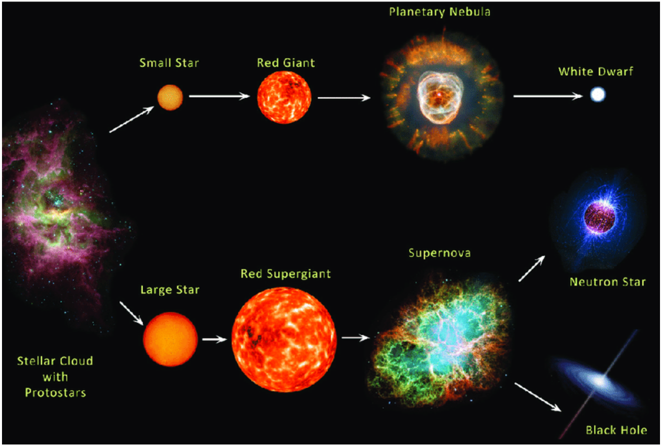
- Mass of the Sun in a volume smaller than the Earth
- The acceleration due to gravity at the surface of Sirius B is $\sim 4 \times 10^5$ times greater than on Earth
- White dwarf material is so dense... a teaspoon would weight 16 tons on Earth and 6.4 million tons on the surface of the white dwarf!
- Despite the name, they come in all colors: with surface temperatures ranging from over 150000 K to barely under 4000 K



How a white dwarf is born?

- It is one of the final evolutionary state of stars whose:
  - mass is not high enough to become a neutron star (about 10 solar masses)
  - period of hydrogen-fusing ends
  - next step is to expand to a red giant and to fuse helium to carbon and oxygen
- Since the core temperature is insufficient to fuse carbon (1 billion K), carbon and oxygen will start building up at its center
- Soon after it sheds its outer layers and forms a planetary nebula, leaving behind the core - a white dwarf is born
- Possibly a white dwarf will evolve into a cold black dwarf (time needed: longer than 13.8 billion years since likely there is no black dwarf yet in the Universe)

Stellar evolution



Earth Blog [2013]: White dwarf is a stellar core remnant of a star of about 10 solar masses.

A "TO DO" list:

- Calculate the density of states of a white dwarf
  - Consider the electron degeneracy for non- and relativistic WD
  - Discuss the classical and quantum regime as well as Fermi momentum
  - Obtain the polytropic equation of state (dependence on density only)
- Calculate the total energy to find the Chandrasekhar mass
- Consider the size of the white dwarfs

Density of states of white dwarfs

No internal supply of energy - the pressure to support white dwarfs against the pull of gravity is given by electron degeneracy.

Let us consider the density of states for free electrons:
How many free-electron states fix into a box of volume $V = L^3$?

- The wave vectors of the free-electron quantum states can only take certain values
- If the electron wave function is $\psi \propto e^{i\mathbf{k}\cdot\mathbf{x}}$, where $\mathbf{k} = (k_x, k_y, k_z)$, the requirement of periodicity implies

$$k_x = n_x \frac{2\pi}{L}, \quad n_x = 1, 2, \ldots$$

- Thus, the allowed states lie on a lattice with spacing $2\pi/L$. The density of states in the k-space is

$$dN = g \frac{L^3}{(2\pi)^3} d^3k, \quad (d^3k \equiv dk_x dk_y dk_z)$$

where g is a degeneracy factor for spin

Density of states of white dwarfs

- From the de Broglie relation between the electron's momentum to its wave vector $\mathbf{p} = \hbar \mathbf{k}$ we may convert dN in k space to that in momentum space

$$dN = g\frac{L^3}{(2\pi)^3}d^3k \quad \rightarrow \quad dN = g\frac{L^3}{(2\pi\hbar)^3}d^3p$$

- The number density of particles (per unit volume) with momentum states in the range of d^3p

$$\boxed{dn = g\frac{1}{(2\pi\hbar)^3}f(p)d^3p}$$

where $f(p)$ - the number of particles in the box with that particular wave function (**called the occupation number of the mode**)

  - for bosons: $f(p)$ is unrestricted
  - for fermions (electrons with spin angular momentum $\hbar/2$) obeying Pauli exclusion principle

$$f(p) \leq 1$$

Density of states of white dwarfs

The criterion $f(p) \leq 1$ imposes a restriction on how dense an electron gas can be before it has to be treated in a manner very different from the classical one. Let us compare:

- The distribution of momenta of the particles treated as Maxwellian distribution (each component of velocity have a Gaussian distribution with standard deviation):

$$\Psi(v) = \frac{1}{(2\pi\sigma^2)^{3/2}} e^{-\frac{v^2}{2\sigma^2}} d^3 v$$

  where v is the particle velocity, $\sigma^2 = kT/m$ is squared root of the dispersion in velocities. To obtain a number density of particles in a given range of momentum space, multiply it by the total density of particles, n, and use $p = mv$:

$$\boxed{dn = \frac{n}{(2\pi mkT)^{3/2}} e^{-\frac{p^2}{2mkT}} d^3 p}$$

- with the obtained general expression for the number density of particles in momentum space

$$\boxed{dn = g \frac{1}{(2\pi\hbar)^3} f(p) d^3 p,}$$

Density of states of white dwarfs

We deduce that there is a *critical density* at which the classical law would yield $f(p) > 1$ (at $p = 0$ which is where $dn = \frac{n}{(2\pi mkT)^{3/2}} e^{-\frac{p^2}{2mkT}} d^3p$ peaks):

$$n_{\text{crit}} = \frac{g}{(2\pi)^{3/2}} \frac{(mkT)^{3/2}}{\hbar^3}$$

At a fixed density, the gas will be in the classical regime at high values of T... but quantum effects become important as $T \to 0$!

Let us now integrate the number density of particles in momentum space

$$n = g \frac{1}{(2\pi\hbar)^3} \int f(p) d^3p$$

in the limit of zero temperature. It means that the states are occupied only up to the Fermi momentum p_F

$$n = g \frac{1}{(2\pi\hbar)^3} \int_0^{p_F} d^3p = g \frac{1}{(2\pi\hbar)^3} \frac{4\pi}{3} p_F^3$$

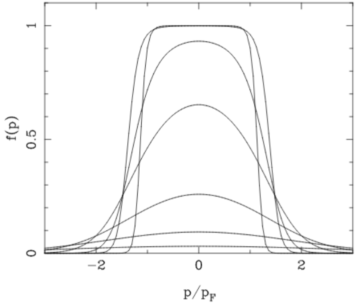
Fermi momentum related to the particle density

$$\boxed{p_F = 2\pi\hbar \left(\frac{3}{4\pi g}\right)^{1/3} n^{1/3}}$$

The occupation number for a gas of fermions as a fuction of their density relative to the critical density (from $n/n_{\text{crit}} = 0.03$ to $n/n_{\text{crit}} = 0.30$)

For the lowest densities (highest T) we have almost exactly the classical Maxwellian velocity distribution.

For densities well above critical, the occupation number tends to a "top-hat" distribution: unity for momenta less than the Fermi momentum, and zero otherwise



Credit to M. Pettini

What is the pressure exerted by degenerate electrons?

Pressure can be thought of as a flux of momentum. Thus, if the number density of electron is n_e, then the flux of electrons in the x-direction is $n_e v_x$. Approximately:

$$P_e \sim p_x n_e v_x.$$

The contribution to the total pressure in the x-direction from all electrons with momentum p_x is just

$$dP_x = p_x v_x dn_{e,x},$$

where $dn_{e,x}$ - the number density of electrons with x-momentum in the range p_x to $p_x + dp_x$. Using $dn = g\frac{1}{(2\pi\hbar)^3} f(p) d^3 p$ we obtain the total pressure in x-direction

$$P = P_x = \frac{g}{(2\pi\hbar)^3} \int p_x v_x f(p) d^3 p.$$

Let us rewrite it in spherical polar coordinates in momentum space:

$$\int p_x v_x dp_x dp_y dp_z = \frac{1}{3} \int (p_x v_x + p_y v_y + p_z v_z) dp_x dp_y dp_z = \frac{1}{3} \int p \cdot v 4\pi p^2 dp$$

so that

$$\boxed{P = \frac{g}{3} \frac{1}{(2\pi\hbar)^3} \int_0^\infty p \cdot v\, f(p) 4\pi p^2 dp.}$$

Non-relativistic speeds of electrons: $p \cdot v = p^2 / m_e$

$$P_e = \frac{g}{3}\frac{1}{(2\pi\hbar)^3}\int_0^\infty p \cdot v\, f(p) 4\pi p^2 dp = \frac{g}{30\pi^2\hbar^3 m_e} p_F^5$$

Since we need to write the pressure as a function of density $\rho_e = m_e n_e$, let us use $p_F = 2\pi\hbar\left(\frac{3}{4\pi g}\right)^{1/3} n^{1/3}$ to write:

$$p_F = \left(\frac{6\pi^2\hbar^3\rho_e}{g m_e}\right)^{1/3}.$$

Then, we get the **polytropic equation of state**

$$P_e = \frac{g}{30\pi^2\hbar^3}\left(\frac{6\pi^2\hbar^3}{g}\right)^{5/3}\rho_e^{5/3} m_e^{-8/3} \equiv K_1 \rho_e^{5/3}$$

where $K_1 = \frac{\pi^2\hbar^2}{5 m_e^{8/3}}\left(\frac{6}{g\pi}\right)^{2/3}$.

Relativistic speeds of electrons: $p \cdot v = pc$

Let us look again at $p_F = 2\pi\hbar\left(\frac{3}{4\pi g}\right)^{1/3} n^{1/3}$. At high densities, the Fermi momentum reaches relativistic values: some particles are forced into momentum states with velocities approaching the speed of light c. Thus,

$$P_e = \frac{4\pi g}{3(2\pi\hbar)^3}\int_0^{p_F} pcp^2 dp = \frac{gc}{24\pi^2\hbar^3}p_F^4,$$

or, using again $p_F = \left(\frac{6\pi^2\hbar^3\rho_e}{gm_e}\right)^{1/3}$ we obtain

$$P_e = \frac{gc}{24\pi^2\hbar^3}\left(\frac{6\pi^2\hbar^3\rho_e}{gm_e}\right)^{4/3} \equiv K_2\rho_e^{4/3},$$

where $K_2 = \frac{\pi\hbar c}{4m_e^{4/3}}\left(\frac{6}{g\pi}\right)^{1/3}$.

Observations

Non-relativistic degenerate gas

$$P_e = K_1 \rho_e^{5/3}$$

Relativistic degenerate gas

$$P_e = K_2 \rho_e^{4/3}$$

- In degenerate gas, the pressure depends only on density and it is independent of temperature
- K_1 and K_2 depend on mass so the mass of other particle can contribute. However, for protons: $m_p / m_e = 1836$ - we may neglect their influence

Estimation of the energy density of degenerate gas

To do it, we integrate over momentum space including a term $\epsilon(p)$ - the energy per mode

$$U = g\frac{1}{(2\pi\hbar)^3}\int_0^\infty \epsilon(p)f(p)4\pi p^2 dp.$$

In the zero-temperature limit, $f(p) = 1$ up to the Fermi momentum and $f(p) = 0$ at all other values.

- Relativistic case, $\epsilon(p) = pc$:

$$U_e = \frac{g}{4(2\pi\hbar)^3}4\pi c p_F^4 = \frac{3}{4}\left(\frac{6\pi^2}{g}\right)^{1/3}\hbar c n_e^{4/3}$$

- Non-relativistic case, $\epsilon(p) = p^2/2m_e$

$$U_e = \frac{3\hbar^2}{10 m_e}\left(\frac{6\pi^2}{g}\right)^{2/3} n_e^{5/3}$$

Total energy

Gravitational energy is proportional to $-M^2/R$ while the kinetic one supplied by the degenerate electrons $E_K \propto U_e V \propto n_e^{4/3} V \propto M^{4/3}/R$

$$E_{tot} = \frac{(AM^{4/3} - BM^2)}{R}, \quad A, B \quad \text{constants}$$

It gives a critical mass. Thus

- For masses smaller than this limit, $E_{tot} > 0$ and will be reduced by making the star expand until the electrons reach the relativistic regime. **Stable white dwarf**.
- For masses larger than this limit, the binding energy increases without limit and the star shrinks - gravitational collapse unstoppable.

Critical mass - Chandrasekhar limit

For simplicity: $\rho = constant$. Then, in this approximation (Exercise):

$$E_K = \left(\frac{243\pi}{128g}\right)^{1/3} \frac{\hbar c}{R} \left(\frac{M}{\mu m_p}\right)^{4/3}, \quad E_V = -\frac{3}{5}\frac{GM^2}{R}$$

Equating the two terms:

$$M_{crit} = \frac{3.7}{\mu^2}\left(\frac{2}{g}\right)^{1/2}\left(\frac{\hbar c}{G}\right)^{3/2} m_p^{-2} \approx \frac{7}{\mu^2} M_\odot, \quad \text{for} \;\; g = 2$$

- For stars which have burnt the initial fuel into elements heavier than Helium $\mu \simeq 2$, the critical mass is $M_{crit} = 1.75 M_\odot$
- More precise calculations taking into account the density profile within the white dwarf give $\boxed{M_{crit} = 1.44 M_\odot}$

Mass-Volume relation of white dwarfs

Let us consider the non-relativistic case: the energy density $U_e \propto n_e^{5/3}$ so $E_K = CM^{5/3}/R^2$, and $E_V = -BM^2/R$. The equilibrium radius is found by imposing the condition $dE_{tot}/dr = 0$ which gives:

$$R = \frac{2C}{B} M^{-1/3}$$

Since $V \propto R^3$

$$\boxed{M_{wd} V_{wd} = \text{constant}}$$

The more massive white dwarfs the smaller they are

- A consequence of being supported from electron degeneracy pressure
- The electrons must be more closely confined to generate the larger degeneracy pressure to support a more massive star
- Equation $R = \frac{2C}{B} M^{-1/3}$ is non-relativistic - the volume of WD could get infinitely small!! **Relativistic effects must be taken into account**
- Exercise Putting numerical values:

$$\boxed{R \simeq 5975 \left(\frac{M}{M_{crit}}\right)^{-1/3} \text{km}, \quad \rho \simeq 10^{6.6} \left(\frac{M}{M_{crit}}\right)^2}$$

More on polytropic equation of state

Although the polytropic EoS is a **toy model**, the use of it in some situations is well-motivated

$$P = K\rho^{\Gamma}$$

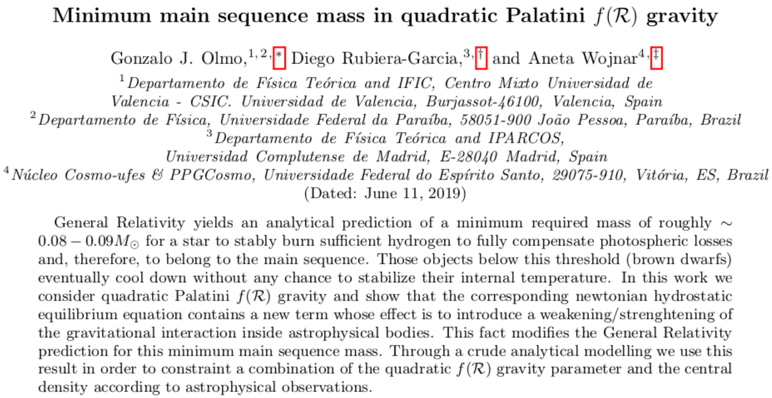
- A first "check" of your (modified) TOV equations
- As we calculated in details, some values of the polytropic index $\Gamma = \{\frac{4}{3}, \frac{5}{3}\}$ describe white dwarfs
- Different parts of the NS can be given by different polytropic phases

  $$P(\rho) = \sum_i K_i \rho^{\Gamma_i}$$

  with particular K_i and $\rho_{i-1} < \rho_i < \rho_i$ smoothly matched to each other at each transition density ρ_i
- A very good approximation for low mass stars, such as brown dwarfs for example

Non-relativistic stars are also very interesting

... because they allow to test theories of gravity!!


Minimum main sequence mass in quadratic Palatini $f(\mathcal{R})$ gravity

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General Relativity yields an analytical prediction of a minimum required mass of roughly $\sim 0.08-0.09M_\odot$ for a star to stably burn sufficient hydrogen to fully compensate photospheric losses and, therefore, to belong to the main sequence. Those objects below this threshold (brown dwarfs) eventually cool down without any chance to stabilize their internal temperature. In this work we consider quadratic Palatini $f(\mathcal{R})$ gravity and show that the corresponding newtonian hydrostatic equilibrium equation contains a new term whose effect is to introduce a weakening/strenghtening of the gravitational interaction inside astrophysical bodies. This fact modifies the General Relativity prediction for this minimum main sequence mass. Through a crude analytical modelling we use this result in order to constraint a combination of the quadratic $f(\mathcal{R})$ gravity parameter and the central density according to astrophysical observations.

Non-relativistic case

$$\frac{dp}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \frac{\left(1+\frac{p(r)}{\rho(r)}\right)\left(1+\frac{4\pi r^3 p(r)}{Gm(r)}\right)}{1-\frac{2Gm(r)}{r}}$$

To get the Newtonian equation, use $p << \rho$ together with $4\pi r^3 p << m$ and $\frac{2Gm}{r} << 1$ which provides

$$-r^2 p' = Gm(r)\rho(r)$$

Dividing it by ρ and differentiating the above equation with respect to r give us

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dp(r)}{dr}\right) = -4\pi G r^2 \rho(r).$$

Using the standard definitions of dimensionless variables $\left(n = \frac{1}{\Gamma - 1}\right)$

$$r = r_c \xi, \quad \rho = \rho_c \theta^n, \quad p = p_c \theta^{n+1}, \quad r_c^2 = \frac{(n+1)p_c}{4\pi G \rho_c^2}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d}{d\xi}\theta\right) = -\theta^n$$

It has the exact solutions for $n = \{0, 1, 5\}$; numerical methods also used.