Dressed and Naked Singularities in 2+1 Dimensions

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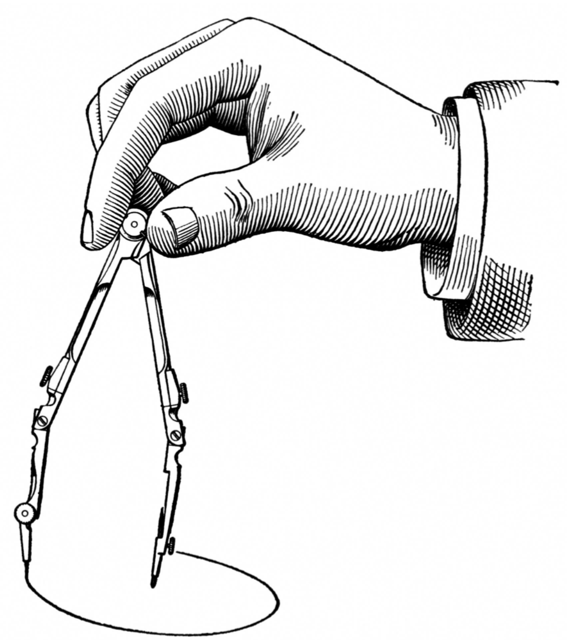
I. What is 2+1 Gravity?

Gravity = Study of the geometry of spacetime (A. Einstein)

Two ingredients (Euclid):



- Metric structure
(length/area/volume, scale)

e^a

$$e^a = e^a_\mu(x)dx^\mu$$

$$[\ g_{\mu\nu}(x) = \eta_{ab}e^a_\mu(x)e^b_\nu(x)\ ]$$

- Affine structure
(parallel transport)

$$\omega^a{}_b = \omega^a{}_{b\mu}(x)dx^\mu$$

$$[\ D_\mu u^a = \partial_\mu u^a + \omega^a{}_{b\mu}u^b\ ]$$

- The fundamental ingredients for a dynamical theory of the spacetime geometry are:

e^a	*vielbein* (1-form)
$\omega^a{}_b$	*connection* (1-form)
$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \omega^c{}_b$	*curvature* (2-form)
$T^a = de^a + \omega^a{}_c e^c$	*torsion* (2-form)
$\varepsilon_{a_1 a_2 \cdots a_D}, \quad \eta_{ab}$	*invariant tensors* (0-forms)

- *By taking successive derivatives no new functionally independent objects are produced:*

$$DR^a_b \equiv 0, \quad DT^a = R^a_b e^b, \quad (dd = 0).$$

- Einstein not only showed how Riemannian geometry describes spacetime. Einstein's equations provide the laws that govern that geometry: what geometries are possible, how spacetime curves, etc.

- The Hilbert-Einstein action is the principle from which Einstein's equations are obtained.

Lorentz invariant *D*-form

$$I[e,\omega] = \int_M L_D(e, de, \omega, d\omega)$$

D-dimensional manifold

The Lagrangian L_D is assumed to be the most general Lorentz invariant D-form constructed with these ingredients in each dimension.

II. What is 3D Gravity?

The most general Lagrangian for 3D gravity is a 3-form made out of e^a, $\omega^a{}_b$ and their exterior derivatives, and *quasi*-invariant under local Lorentz transformations:

$$\boxed{I[e,\omega] = \int \left(\epsilon_{abc}[R^{ab} + \tfrac{\alpha}{3} e^a e^b]e^c + \lambda e^a T_a + \gamma[\omega^a{}_b d\omega^b{}_a + \tfrac{2}{3}\omega^a{}_b \omega^b{}_c \omega^c{}_a]\right)}$$

The field equations read

(*constants*)

$$\boxed{R^{ab} + \Lambda e^a e^b = 0, \qquad T^a + \tau \epsilon^{abc} e_b e_c = 0}$$

All solutions of 3*D* gravity are spacetimes of *constant* (Lorentz) *curvature* and *constant torsion*.

Splitting the connection:

$$\omega^a{}_b = \bar{\omega}^a{}_b + \kappa^a{}_b$$

Torsion-free part (Riemannian)

Contorsion 1-form

where $\boxed{D_{\bar{\omega}} e^a = de^a + \bar{\omega}^a{}_b e^b \equiv 0.}$

Then:

$$\boxed{\begin{aligned} T^a &= \kappa^a{}_b e^b \\ R^a{}_b &= \bar{R}^a{}_b + D_{\bar{\omega}} \kappa^a{}_b + \kappa^a{}_c \kappa^c{}_b \end{aligned}}$$

Lorentz curvature

Torsion-free curvature

This is true for any dimension.

What does it mean for (2+1)-dimensional geometries?

In 2+1 dimensions,

$$\boxed{R^{ab} + \Lambda\, e^a e^b = 0, \qquad T^a + \tau\varepsilon^{abc} e_b e_c = 0}$$

$$T^a = \kappa^a{}_b e^b \Longrightarrow \boxed{\kappa^a{}_b = \tau\varepsilon^a{}_{bc} e^c}$$

$$\Longrightarrow R^a{}_b = \bar{R}^a{}_b + \cancel{D_{\bar{\omega}}\kappa^a{}_b} + \kappa^a{}_c \kappa^c{}_b = \bar{R}^a{}_b + \tau^2 e^a e_b$$

In $3D$ in a spacetime of constant curvature $-\Lambda$ and torsion $-\tau$, it is always possible to choose a torsion-free $\bar{\omega}^a_b$ so that

$$\begin{aligned}\bar{R}^{ab} &= -(\Lambda + \tau^2) e^a\, e^b \\ &= \Lambda'\, e^a\, e^b\end{aligned}$$

From now on we assume the spacetimes to be torsion-free, constant (negative) curvature geometries ($\Lambda = -1/\ell^2$, ℓ=AdS_3 radius).

III. 3D black holes

- Black holes were known to exist in dimensions $D \geq 4$. Solutions of Einstein's equations in $D = 3$ are too simple:

$$R^{ab} = -\ell^{-2} e^a e^b \qquad (*)$$

- A $3D$ bh space should have the same constant curvature for all values of its integration constants (M and J) 🤔

- The black hole solution in $3D$ were a surprise:

Static b.h.:
$J = 0$

$$\boxed{ds^2 = -\left(\frac{r^2}{\ell^2} - M\right) dt^2 + \frac{dr^2}{\left(\frac{r^2}{\ell^2} - M\right)} + r^2 d\phi^2}$$

Horizon: $r_+ = \ell\sqrt{M}$

- For $r > 0$ * is satisfied. For $r \gg r_+$ the geometry approaches AdS_3

Spinning 2+1 black hole

$J \neq 0$

$$ds^2 = -f^2(r)dt^2 + \frac{dr^2}{f^2(r)} + r^2(Ndt + d\phi)^2$$

$$f^2 = -M + r^2 + \frac{J^2}{4r^2} \qquad N = -\frac{J}{2r^2}$$

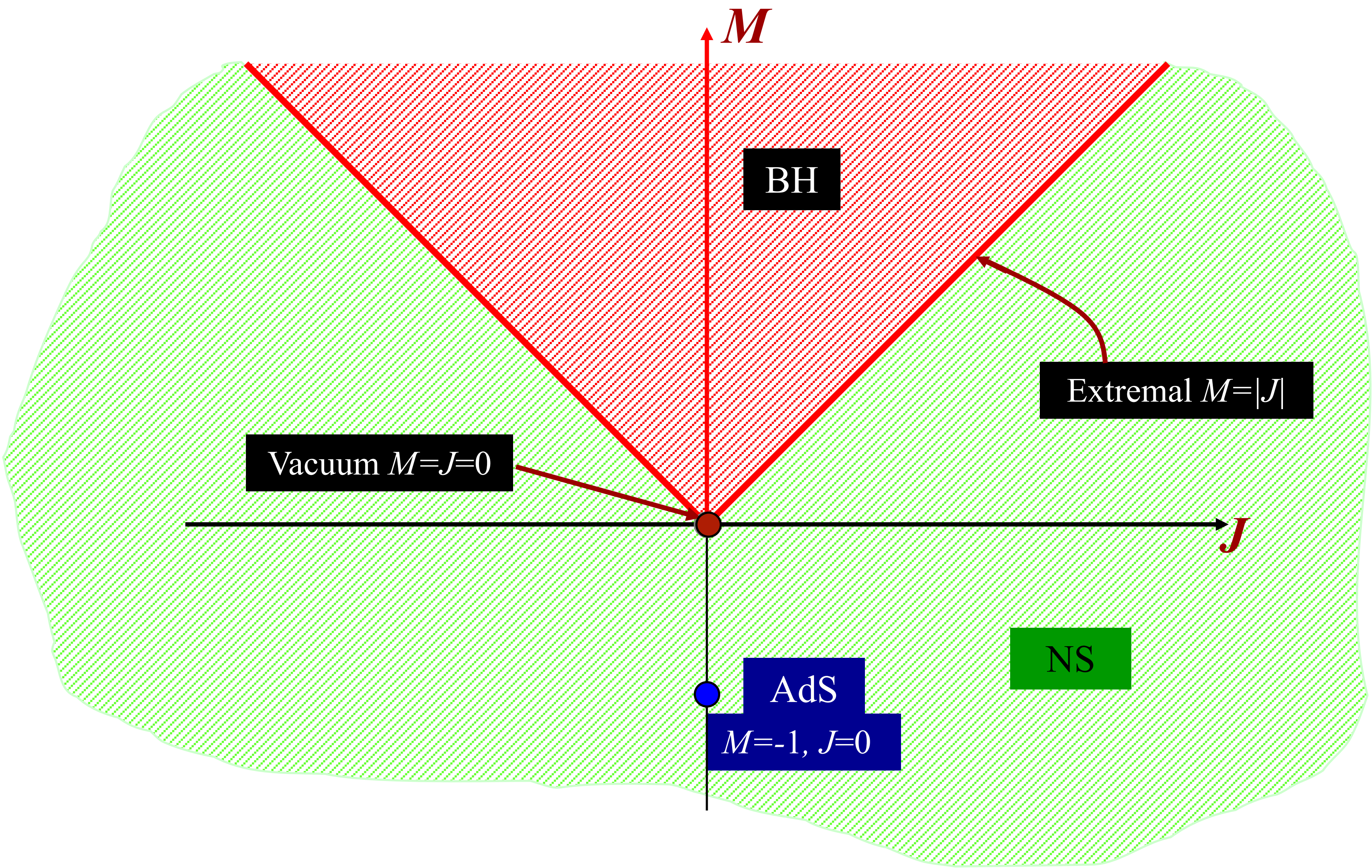
Horizons: $$r_\pm^2 = \frac{M}{2}\left(1 \pm \sqrt{1 - \frac{J^2}{M^2}}\right) \in \mathbb{R}^+ \Leftrightarrow M \geq |J|$$

$M \geq |J| \geq 0$ → BH; horizons $r_\pm \geq 0$

$M = -1, J = 0$ → AdS

$M < |J|, \neq -1$ → Naked singularities

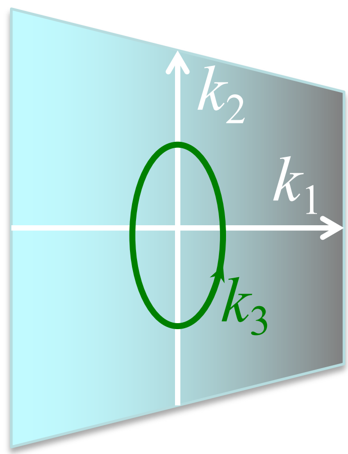
2+1 *black hole states*



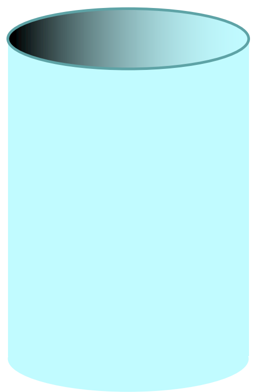
IV. How are 3D BHs made?

Identifications by Killing vectors respect *local geometry*:

k_1, k_2, k_3: isometries

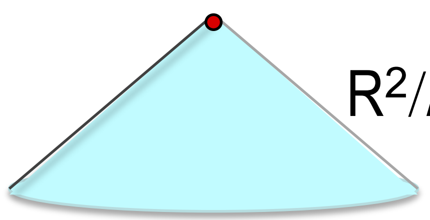


R^2



R^2/k_1

k_1 has no fixed points
→ no singularities



R^2/k_3

r=0 fixed point of k_3
→ conical singularity

- Consider the constant negative curvature pseudosphere

$\underline{S^{2,2}:}$ $$\boxed{-(x^0)^2+(x^1)^2+(x^2)^2-(x^3)^2=-1}$$

This is invariant under $SO(2,2)$; 6 generators, 6 independent Killing vectors (global isometries).

$$k=\tfrac{1}{2}k^{ab}(x_a\partial_b-x_b\partial_a)=\tfrac{1}{2}k^{ab}J_{ab}\in so(2,2)$$

- Identifying by k results in a manifold with the same local geometry but less symmetry.

- Since $so(2,2)\simeq so(2,1)\times so(2,1)$, identifying by k produces a manifold with at least 2 globally defined Killing vectors that commute with k: $\{k,k_\perp\}$.

- Fixed points of k produce conical singularities
- Timelike k produces closed timelike curves (☠!)

Black hole identifications

AdS_{2+1} $\quad -(x^0)^2 + (x^1)^2 + (x^2)^2 - (x^3)^2 = -1$

- $k_{\pm} = r_+ J_{31} - r_- J_{02}$ Generic (spinning) bh, $r_+ > r_- \geq 0$
- $k_{Ext} = r_+(J_{03} - J_{12}) + \frac{1}{2}(J_{31} + J_{02} + J_{01} + J_{23})$

  Extremal bh, $r_+ = r_- > 0$, $M = J \neq 0$

- $k_{Ext} = \frac{1}{2}(J_{31} + J_{02} + J_{01} + J_{23})$

  Zero mass bh, $r_+ = r_- = 0$, or $M = J = 0$

These are all non-compact elements of $so(2,2)$

No fixed points $\Rightarrow$ *No conical singularities*

$k \cdot k > 0$ $\Rightarrow$ *No closed timelike curves*

- The BH geometries are *locally* AdS (but globally not so).

V. Extended spectrum of the 3D BH

Other related spacetimes

The geometries $ds^2 = -f^2(r)dt^2 + \frac{dr^2}{f^2(r)} + r^2(Ndt + d\phi)^2$ (*)

with $f^2(r) = -M + r^2 + \frac{J^2}{4r^2}$ and $N = \frac{J}{2r^2}$

are locally also constant curvature spaces even if $M < |J|$, and the horizons

$$r_\pm^2 = \frac{M}{2}\left(1 \pm \sqrt{1 - \frac{J^2}{M^2}}\right)$$

take complex values. In that case, (*) no longer describes black holes because there are no horizons. The singularity at $r = 0$ is *naked*.

What kind of singularity is that?

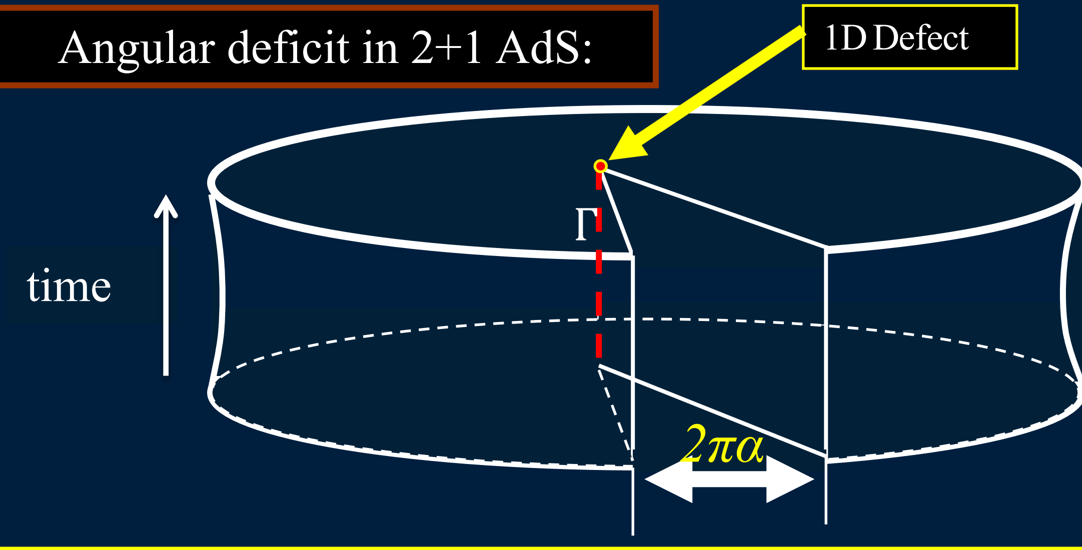
These naked singularities also result from identifications by a spacelike Killing vector k in AdS_3.

If k is a *compact* element of SO(2,2) (rotations), it leaves a *fixed point* → (Naked) Conical singularity (conical defect)

The resulting quotient space (orbifold) AdS_3/k; is less symmetric than AdS_3 due to the presence of the fixed points.

These solutions are stationary, localized, spherically symmetric and all of them have a naked singularity at $r = 0$. Far away they are all similar and share the same asymptotics (AdS_3).

Angular deficit in 2+1 AdS:



Identification in the spatial plane generates a conical singularity at the fixed points of the Killing vector:

$$\xi = -2\pi\alpha\,(x_2\partial_3 - x_3\,\partial_2)$$
$$= -2\pi\alpha\,\partial_\varphi$$

Angular defect ⟺ δ-like curvature singularity at $r = 0$

… rescaling the coordinates,

$$ds^2 = -(r^2 - M)dt^2 + (r^2 - M)^{-1}dr^2 + r^2 d\phi^2 \quad (\ell = 1)$$

Looks like a static black hole with negative mass, $M = -(1-\alpha)^2$, related to the angular deficit, $\Delta\phi = 2\pi\alpha$.

Special cases:

$\alpha = 0 \ (M = -1) \rightarrow$ no deficit (AdS spacetime)

$\alpha = 1 \ \ (M = 0) \rightarrow$ maximum deficit (*Vacuum* bh)

For $-1 < M < 0$, these are ***naked*** but otherwise harmless conical singularities

The identification vector $k = x_1\partial_2 - x_2\partial_1 = \partial\varphi$ is spacelike and has a fixed point at $r = 0$. Identifying with it produces a conical singularity.

The resulting conical geometry has infinite curvature at the apex of the cone,

$$\boxed{R^{ab} = \Lambda\, e^a\, e^b + \Delta\delta(\Gamma)[\delta_1^a\delta_2^b - \delta_2^a\delta_1^b]}$$

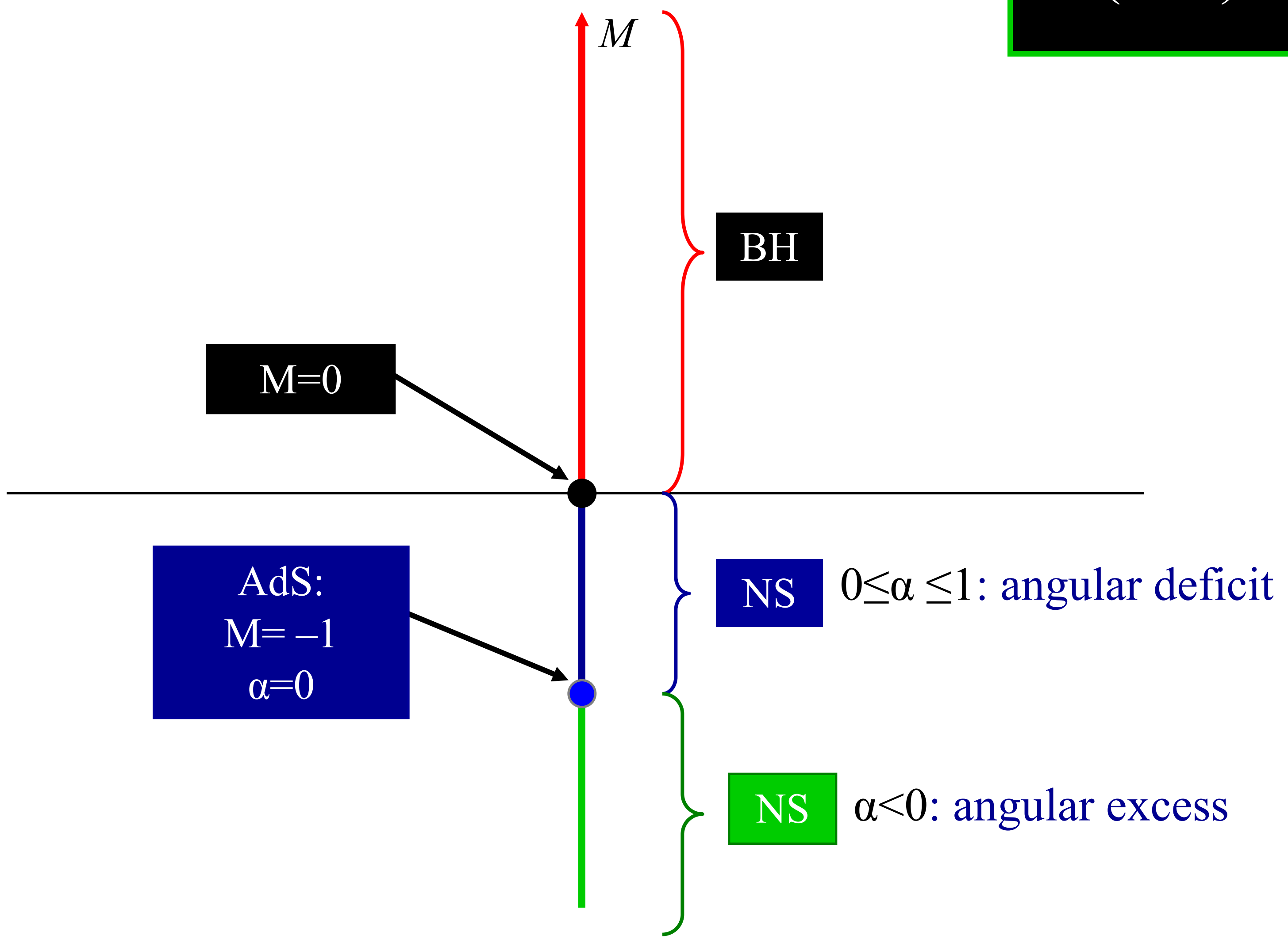
Deficit angle $\Delta = 2\pi\alpha$

2-form Dirac delta supported at $r = 0$

This singularity is not dressed by a horizon.

2+1 *BH-NS spectrum*

(J=0)



Spinning naked singularities

These static cones can also be boosted. The result looks familiar

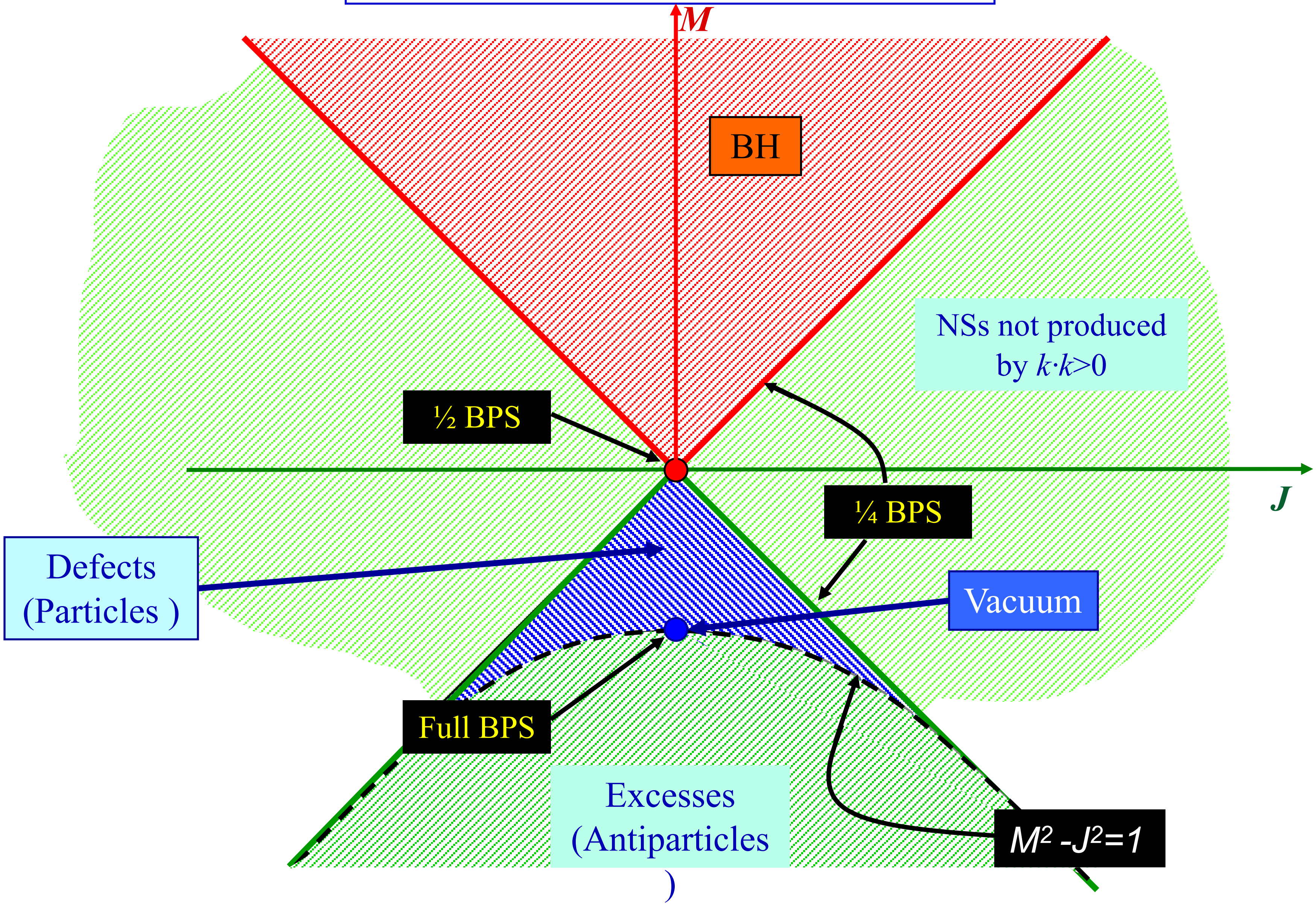
$$ds^2 = -f^2(r)dt^2 + \frac{dr^2}{f^2(r)} + r^2(Ndt + d\phi)^2$$

with $f^2(r) = -M + r^2 + \frac{J^2}{4r^2}$ and $N = \frac{J}{2r^2}$

One formula covers all spherically symmetric, stationary, asymptotically AdS_3 spacetimes

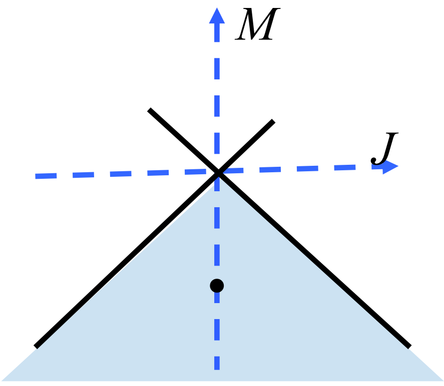
- Generic black holes $M \geq |J|$
- Extremal black holes $M = |J|$
- Zero-mass black hole $M = 0 = J$
- Spinning conical singularities $M < -|J|$
- Overspinning geometries $|M| < |J|$
- AdS vacuum $M = -1, J = 0$

2+1 *BH-NS spectrum*



Conical singularity identifications

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 - (x^3)^2 = -1$$



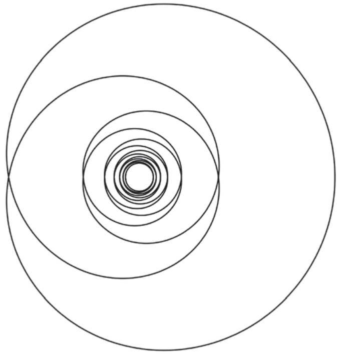
- Generic spinning cone, $M < -|J| \neq 0$

Id. vector: $\xi(\beta_+, \beta_-) = \beta_+ J_{12} + \beta_- J_{03}$

$$\beta_\pm = \sqrt{-M + J} \pm \sqrt{-M - J}$$

These are compact elements of $so(2,2)$

- $r = 0$ fixed point ⟹ *Conical singularity*
- $\xi \cdot \xi > 0$ ⟹ *No closed timelike curves*

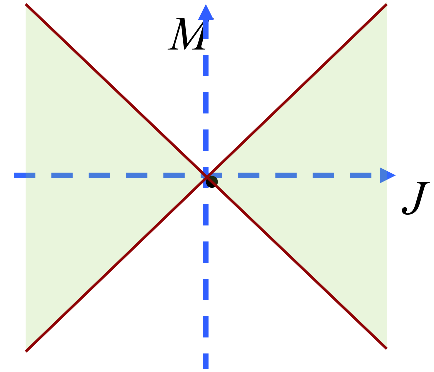


Generic timelike orbits
[PRD **100**, 024026 (2019)]

Overspinning naked singularities ($|J| > |M|$)

These geometries are also obtained by identifications:

$$\xi = a\underbrace{(J_{03} + J_{12})}_{\text{Rotation}} + b\underbrace{(J_{02} + J_{31})}_{\text{Boost}} \sim \partial_\theta$$

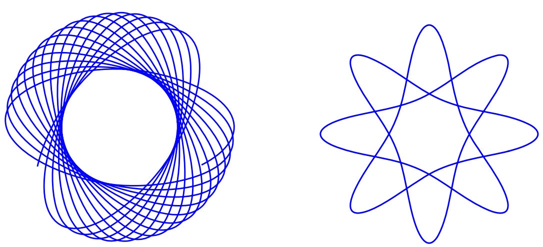


where

$$a \pm ib = \frac{1}{2}\left(\sqrt{-M+J} \pm \sqrt{-M-J}\right)$$

These are noncompact elements of so(2,2)

- $r = 0$ fixed point ➡ *Naked Singularity*
- $\xi\cdot\xi > 0$ ➡ *No CTCs*



Timelike orbits
[PRD **D104**, 044023(2021)]

What is the source at $r = 0$?

This can be answered evaluating the parallel transport of a vector around $r = 0$. The parallel transport operator is given by the holonomy

$$W = \exp \oint A$$

Here $A = \frac{1}{2} k^{ab} J_{ab}$ is an element of the $so(2,2)$ algebra and the loop is an infinitesimal closed path surrounding the origin. The result is

$$j(x) = \delta(\Gamma)\, \mathrm{K}$$

where K is a finite element of $SO(2,2)$.

$\mathrm{K}_{M>J}$ = Boost (M, J)

$\mathrm{K}_{M<-|J|}$ = Rotation (M, J)

$\mathrm{K}_{|J|>|M|}$, $\mathrm{K}_{|J|=|M|}$ $\mathrm{K}_{|J|=|M|=0}$: (Don't miss Escola Plínio 2022)

VI. Summary

- Identifications by a Killing vector $k = -2\pi\alpha\, \partial_\varphi$ in AdS_3 produce naked and dressed singularities at $r = 0$.

- There is a curvature singularity, proportional to the angular deficit angle $2\pi\alpha$, e.g.,

$$\boxed{\bar{R}^{ab} + e^a e^b = 2\pi\alpha\delta^{[ab]}_{[12]}\delta(\Sigma_{12})d\Omega^2_\Sigma}$$

- There is a 1-1 correspondence between the values of *M* and *J* and the KV used in the identification.

- All of these black holes & naked singularities are solutions of Einstein's equations.

- These seem to be all possible 2+1 geometries with this topology and global isometries without closed timelike curves.
- What part of all quantum states in 2+1 are labelled by *M* and *J*?

Obrigado!

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