# Conformal symmetry and renormalization

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**Partially based on:** *I.L. Buchbinder and I.Sh., (Oxford UP, 2021); M. Asorey, W.C. Silva, I.Sh. and P.R.B. do Vale, 2202.00154 (EPJC); A.O. Barvinsky, G. Camargo, A.E. Kalugin, N. Ohta and I.Sh. 2308.05251 (PRD); I.Sh., 2310.04131 (EPJC-2024), 2504.01340.*

# Some examples of $4D$ conformal theories

- **General scalar action with conformal nonminimal term**

$$S_{scal} = \frac{1}{2} \int d^4x \sqrt{-g} \left\{ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{6} R\phi^2 \right\} ,$$

**is invariant under local conformal transformation,**

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} \, e^{2\sigma} , \quad \phi \to \phi' = \phi \, e^{-\sigma} , \qquad \sigma = \sigma(x) .$$

*R. Penrose, (Les Houches lectures, Gordon and Breach, 1964); N.A. Chernikov & E.A. Tagirov (Ann. Inst. H. Poincare, 1968).*

- - **Massless spinor, maybe also coupled to vector or axial vector (antisymmetric torsion)**

$$S_{1/2} = \frac{i}{2} \int d^4x \sqrt{-g} \left\{ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right\}$$

**The transformation rules are**

$$\psi \to \psi' = \psi \, e^{k_f \sigma} , \quad \bar{\psi} \to \bar{\psi}' = \bar{\psi} \, e^{k_f \sigma} , \quad k_f = -\frac{3}{2}$$

- **Electromagnetic field**

$$S_{vec} \,=\, \frac{1}{4}\int d^4x\sqrt{-g}\,F_{\mu\nu}\,F^{\mu\nu}\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- **The conformal (Weyl) gravity in the dimension $D = 4$.**

$$S_W \,=\, \int d^4x\sqrt{-g}\,C^2(4)\,,$$

$$C^2 \,=\, C^{\mu\nu\alpha\beta}\,C_{\mu\nu\alpha\beta} \,=\, R^2_{\mu\nu\alpha\beta} - 2R^2_{\mu\nu} + \frac{1}{3}\,R^2\,.$$

**In both cases there are generalizations to arbitrary dimensional spaces, see, e.g.,** *hep-th/0609138 (PRD); 2107.13125 (EPJP).*

- **Fourth derivative scalar**

$$S_4 \,=\, \int d^4x\sqrt{-g}\,\varphi\,\Delta_4\,\varphi\,,$$

**where** $\quad \Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}\,R\Box + \frac{1}{3}\,R_{;\mu}\,\nabla^\mu\,.$

**The transformation law is** $\varphi \to \varphi'\,.$

*E.S. Fradkin & A.A. Tseytlin, PLB,NPB - 1982.*

*S.M. Paneitz, MIT preprint - 1983; SIGMA - 2008*

- **Antisymmetric tensor field coupled to Dirac fermion**

$$S_B = \int d^4x\sqrt{-g}\left\{\frac{1}{2}\big(W_4+\lambda W_1\big) - \frac{1}{2}M^2 B_{\mu\nu} - \frac{1}{4!}\big(f_2 W_2 + f_3 W_3\big)\right\}$$
$$+\, i\int d^4x\sqrt{-g}\,\bar{\psi}\{\gamma^\mu\nabla_\mu - \Sigma^{\mu\nu}B_{\mu\nu} - im\}\psi, \qquad m = M = 0.$$

**where** $\gamma^\mu = e^\mu_a\gamma^a$, $\Sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ **and** $\lambda$ **is an arbitrary nonminimal parameter and** $f_{2,3}$ **are the quartic self-couplings,**

$$W_1 = \sqrt{-g}\,B^{\mu\nu}B^{\alpha\beta}C_{\alpha\beta\mu\nu},$$
$$W_2 = \sqrt{-g}\,(B_{\mu\nu}B^{\mu\nu})^2, \qquad W_3 = \sqrt{-g}\,B_{\mu\nu}B^{\nu\alpha}B_{\alpha\beta}B^{\beta\mu},$$
$$W_4 = \sqrt{-g}\,\big\{(\nabla_\alpha B_{\mu\nu})(\nabla^\alpha B^{\mu\nu}) - 4(\nabla_\mu B^{\mu\nu})(\nabla^\alpha B_{\alpha\nu})$$
$$+\, 2B^{\mu\nu}R^{\alpha}_{\nu}B_{\mu\alpha} - \frac{1}{6}RB_{\mu\nu}B^{\mu\nu}\big\}.$$

**The conformal transformation is** $B_{\mu\nu} \to B'_{\mu\nu} = B_{\mu\nu}\,e^\sigma$.


- *T.P. Branson, Com.Part.Diff.Eqs.* ***7*** *(1982) 393;*
- *J. Erdmenger, hep-th/9704108, CQG.*
- *E.S. Fradkin, and A.A. Tseytlin, NPB 203 (1982) 157.*
- *I.Sh., ... Cheshire Cat Smile ..., arXiv:2310.04131 (EPJC).*

# Important point:

**Antisymmetric tensor field theory is usually associated with the gauge symmetry**

$$\delta b_{\mu\nu} \,=\, \partial_\mu \xi_\nu - \partial_\nu \xi_\mu\,, \qquad \partial_\mu \xi^\mu = 0.$$

*V.I. Ogievetsky, I.V. Polubarinov, Sov. J. Nucl. Phys. 4 (1967) 156.*
*M. Kalb and P. Ramond, PRD 9 (1974) 2273.*

**And this theory is not conformal since its basic element**

$$F_{\mu\nu\lambda} \,=\, \nabla_\mu b_{\nu\lambda} + \nabla_\lambda b_{\mu\nu} + \nabla_\nu b_{\lambda\mu} \,=\, \partial_\mu b_{\nu\lambda} + \partial_\lambda b_{\mu\nu} + \partial_\nu b_{\lambda\mu}.$$

**gives a non-conformal expression (i.e., different from $W_4$)**

$$\mathcal{L}_{inv} \,=\, \frac{1}{3}\,\sqrt{-g}\,F_{\mu\nu\lambda}F^{\mu\nu\lambda}$$

$$\,=\, \sqrt{-g}\Big\{(\nabla_\alpha b_{\mu\nu})^2 - 2(\nabla_\mu b^{\mu\nu})^2 + \frac{1}{3}\,b_{\mu\nu}^2 R - b_{\mu\nu}b_{\alpha\beta}C^{\alpha\beta\mu\nu}\Big\}.$$

**This situation may be that $B_{\mu\nu}$ includes gauge field $b_{\mu\nu}$ plus extra degrees of freedom, which may be ghosts or tachyons.**

**However, only the conformal theory may be renormalizable.**

# Quantum (Semiclassical) Theory

**Introduction:** *Birrell & Davies (1980);*
*Buchbinder & I.Sh. (Oxford Un. Press, 2021).*

**Review (including applications):** *I.Sh. gr-qc/0801.0216, CQG.*

**The two remarkable statements at the quantum level are**

**1. Classical conformal invariance is broken by trace anomaly.**

*D. M. Capper, M. J. Duff and L. Halpern, PRD 10 (1974) 461;*
*S. Deser, M.J. Duff and C. Isham, NPB 111 (1976) 45;*
*M.J. Duff, NPB 125 (1977) 334; Review: hep-th/9308075 (CQG).*

**2. The conformal invariance holds in the one-loop divergences.**

*I.L. Buchbinder, Theor. Math. Phys.* **61** *(1984) 393.*

**Starting from second loop, one should expect non-conformal divergences to emerge.**

**The pertinent question is that how these two things can be combined and how these two things can be used together.**

**Is it possible to have anomalous, albeit renormalizable theory?**

# Anomaly and the anomaly-induced action

**What could be the building blocks for $\delta\Gamma^{(1)}_{div}$? We can distinguish two types of local conformal symmetry in the vacuum sector:**

- **C-type local conformal symmetry:**

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\, e^{2\sigma(x)} \quad \Longrightarrow \quad S_C(g_{\mu\nu}) = S_C(\bar{g}_{\mu\nu})\,. \qquad (C)$$

**Strong condition, as it requires cancelation in all orders in $\sigma$!**

**For $D$-dimensional space this implies cancelation of $1, 2, \dots, D$ powers of $\sigma$ and its derivatives. Examples: $\int C^2$-term, or the $\int F^2_{\mu\nu}$-term in the presence of electromagnetic field, both in $4D$.**

- **N-type local conformal symmetry:**

$$S_N(g_{\mu\nu}) \neq S_N(\bar{g}_{\mu\nu})\,, \qquad \textbf{but} \qquad \frac{1}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta S_N}{\delta g_{\mu\nu}} = 0\,. \qquad (N)$$

**Compared to the C-type, here only linear in $\sigma$ terms should cancel. It is the same as for the global symmetry with $\sigma = const$.**

# Examples of N-type invariants in $4D$, pure metric case

**Vacuum action in semiclassical gravity has the form**

$$S_{vac} = S_{EH} + S_{HD}$$

**where**

$$S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,\{R + 2\Lambda\}\,.$$

**is the Einstein-Hilbert action with a CC.**

$S_{HD}$ **includes higher derivative terms**

$$S_{HD} = \int d^4x\sqrt{-g}\,\{a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2\}\,,$$

**1. Topological term** $E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4\,R_{\alpha\beta}R^{\alpha\beta} + R^2$

**is the integrand of the** 4D **Gauss-Bonnet topological invariant.**

**2. Surface term** $a_3\Box R$.

# Conformal anomaly

**Consider the set of fields $\Phi$ with the conformal weight $k_\Phi$.**
**The Noether identity for the local conformal symmetry**

$$\Big( - 2\, g_{\mu\nu}\, \frac{\delta}{\delta g_{\mu\nu}} + k_\Phi\, \Phi\, \frac{\delta}{\delta \Phi} \Big)\, S(g_{\mu\nu},\, \Phi) \,=\, 0$$

**produces on shell** $\quad - \dfrac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \dfrac{\delta S_{vac}(g_{\mu\nu})}{\delta g_{\mu\nu}} \,=\, T_{(vac)\,\mu}^{\quad\;\;\mu} \,=\, T^\mu_\mu \,=\, 0\,.$

**At quantum level $S_{vac}(g_{\mu\nu})$ is replaced by the EA $\Gamma_{vac}(g_{\mu\nu})$.**

**For free fields only 1-loop order is relevant**

$$\Gamma_{div} \,=\, -\, \frac{1}{n-4} \int d^4x \sqrt{-g}\, \Big\{ \beta_1 C^2 + \beta_2 E + \beta_3 \Box R \Big\}\,.$$

**For the global conf. symmetry the renormalization group tells us**

$$\langle T^\mu_\mu \rangle \,=\, \big\{ \beta_1 C^2 + \beta_2 E + a' \Box R \big\}\;,$$

**where $a' = \beta_3$. In the local case $a'$ is ambiguous.**

**The simplest way to derive the conformal anomaly is using dimensional regularization (Duff, 1977).**

**The expression for divergences**

$$\bar{\Gamma}_{div} = \frac{1}{n-4} \int d^4x\sqrt{g}\,\{\beta_1 C^2 + \beta_2 E + \beta_3 \Box R\} \,.$$

**where (pay attention to the sign pattern)**

$$\begin{pmatrix} \beta_1 \\ -\beta_2 \\ \beta_3 \end{pmatrix} = \frac{1}{360(4\pi)^2} \begin{pmatrix} 3N_0 + 18N_{1/2} + 36N_1 \\ N_0 + 11N_{1/2} + 62N_1 \\ 2N_0 + 12N_{1/2} - 36N_1 \end{pmatrix}.$$

**The renormalized one-loop effective action has the form**

$$\Gamma_R = S + \bar{\Gamma} + \Delta S\,,$$

**where $\bar{\Gamma} = \bar{\Gamma}_{div} + \bar{\Gamma}_{fin}$ is the naive quantum correction to the classical action and $\Delta S$ is a counterterm.**

$\Delta S$ **is an infinite local counterterm which is called to cancel the divergence. It is the only source of non-invariance.**

**The anomalous trace is**

$$T \,=\, \langle T^\mu_\mu \rangle \,=\, -\,\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\,\frac{\delta\,\Gamma_R}{\delta\, g_{\mu\nu}}\Big|_{n=4} \,=\, -\,\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\,\frac{\delta\,\Delta S}{\delta\, g_{\mu\nu}}\Big|_{n=4}\,.$$

**Conformal parametrization of the metric:**

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\cdot e^{2\sigma}\,,\qquad \sigma = \sigma(x)$$

**where $\bar{g}_{\mu\nu}$ is the fiducial metric with fixed determinant.**

**There is a useful relation**

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\,\frac{\delta\, A[g_{\mu\nu}]}{\delta\, g_{\mu\nu}} = \frac{1}{\sqrt{-\bar{g}}}\;\frac{\delta\, A[\bar{g}_{\mu\nu}\, e^{2\sigma}]}{\delta\sigma}\Big|_{\bar{g}_{\mu\nu}\to g_{\mu\nu}\,,\,\sigma\to 0,\, n\to 4} \qquad (*)$$

$$\int d^n x\,\sqrt{-g}\,C^2(n) \,=\, \int d^n x\sqrt{-\bar{g}}\,e^{(n-4)\sigma}\;\bar{C}^2(n)\,.$$

**Then** $$\frac{\delta}{\delta\sigma}\int\frac{d^4x\sqrt{-\bar{g}}}{n-4}\;e^{(n-4)\sigma}\;\bar{C}^2(n)\Big|_{n\to 4} \,=\, \sqrt{-g}\,C^2\,.$$

**The terms with derivatives of $\sigma(x)$ are irrelevant.**

**For global conformal transform this procedure always works,**

**In the simplest case** $\sigma = \lambda = const$**, we immediately arrive at the expression for** $T$ **with** $a' = \beta_3$**.**

$$\langle T^\mu_\mu \rangle \,=\, \frac{1}{(4\pi)^2}\,\big(\,\omega C^2 + bE + c\Box R\,\big)\,.$$

**However the local case** $\sigma(x)$ **is more complicated, e.g.,**

$$\frac{\delta}{\delta g_{\mu\nu}}\,\int d^4x\sqrt{-g}\,\Box R \equiv 0\,.$$

**We have a conflict between global and local conf. anomalies.**

**Or a conflict between formulas and intuitive expectations.**
*M.J. Duff, Class. Quantum. Grav. (1994)*

**Problem resolved:**
*M. Asorey, E. Gorbar & I.Sh., CQG 21 (2003).*

**Classification of terms in anomaly is based on C/N separation: conformal terms vs topological term vs surface terms.**

*S. Deser, M.J. Duff and C.J. Isham, NPB (1976).*

*S. Deser, and A. Schwimmer, PLB (1993), hep-th/9302047.*

*S. Deser, PLB (2000), hep-th/9911129.*

**There is universality of signs of the coefficients of Weyl invariant (C-type) and topological terms.**

**The related property of the renormalization group flows probably holds beyond the one-loop level, at least for some QFT models.**

**This opened a new area which is known as $c$- and $a$-theorems,**

*Z. Komargodski, A. Schwimmer, JHEP (2011), arXiv:1107.3987.*

*Z. Komargodski, JHEP (2012), arXiv:1112.4538.*

*M.A. Luty, J. Polchinski, R. Rattazzi, JHEP (2013), arXiv:1204.5221.*

# Anomaly-induced Effective Action (EA) of vacuum.

**One can use the trace (conformal) anomaly $\langle T^\mu_\mu \rangle$ to derive the finite part of the one-loop EA**

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu} \frac{\delta\, \bar{\Gamma}_{ind}}{\delta g_{\mu\nu}} \,=\, \langle T^\mu_\mu \rangle\,.$$

**In $2D$ the equation to solve is**

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu} \frac{\delta\, \bar{\Gamma}_{ind}}{\delta g_{\mu\nu}} \,=\, \langle T^\mu_\mu \rangle \,=\, \frac{1}{(4\pi)}\, aR\,.$$

**In $4D$ we meet the equation**

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu} \frac{\delta\, \bar{\Gamma}_{ind}}{\delta g_{\mu\nu}} \,=\, \langle T^\mu_\mu \rangle \,=\, \frac{1}{(4\pi)^2}\, \big( \omega C^2 + bE + c\Box R \big)\,.$$

*R.J. Riegert, PLB 134 (1984) 56.*
*E.S. Fradkin and A.A. Tseytlin, PLB 134 (1984) 187.*

# Particular dimensions

$\underline{2D}$. **There are no C-type invariants and only one N-type invariant,**

$$S_2(g_{\mu\nu}) = \int d^2x\sqrt{-g}\,R \quad\Longrightarrow\quad \langle T^\mu_\mu\rangle = aR\,,$$

**Integration of anomaly is simple and yields the Polyakov action**

$$\Gamma_{ind}(g_{\mu\nu}) = \frac{a}{4}\int d^2x\sqrt{-g}\;R\,\frac{1}{\Box}\,R\,.$$

*A.M. Polyakov, Phys. Lett. B (1981).*

$\underline{4D}$. **There is one C-type invariant, and two N-type invariants,**

$$\langle T^\mu_\mu\rangle = -\omega C^2 - bE - c\Box R\,,$$

**where**

$$\begin{pmatrix}\omega\\ b\\ c\end{pmatrix} = \frac{1}{360(4\pi)^2}\begin{pmatrix} +\,3N_0 + 18N_{1/2} + 36N_1\\ -\,N_0 - 11N_{1/2} - 62N_1\\ +\,2N_0 + 12N_{1/2} - 36N_1\end{pmatrix}.$$

**In $4D$ we can obtain the non-local covariant solution for the anomaly-induced effective action of vacuum.**

**First one has to establish the relations**

$$\sqrt{-g}C^2 = \sqrt{-\bar{g}}\bar{C}^2\,, \qquad \sqrt{-\bar{g}}\,\bar{\Delta}_4 = \sqrt{-g}\,\Delta_4\,,$$

$$\sqrt{-g}\Big(E - \frac{2}{3}\Box R\Big) \;=\; \sqrt{-\bar{g}}\Big(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R} + 4\bar{\Delta}_4\sigma\Big) \qquad \textbf{!!}$$

**and also introduce the Green function**

$$\sqrt{-g}\Delta_4\, G(x,y) = \delta(x,y)\,.$$

**Using these formulas we find, for a functional $A(g_{\mu\nu}) = A(\bar{g}_{\mu\nu})$,**

$$\frac{\delta}{\delta\sigma}\int_x A\,\Big(E - \frac{2}{3}\Box R\Big)\Big| \;=\; 4\sqrt{-g}\Delta_4\, A\,,$$

**where** $\displaystyle\int_x = \int d^4x\sqrt{-g(x)}$ **and** $\displaystyle\Big| = \Big|_{\bar{g}_{\mu\nu}\to g_{\mu\nu}}$ **.**

**As a consequence, we obtain**

$$\frac{\delta}{\delta\sigma(y)}\,\iint\limits_{x\,z}\,\frac{1}{4}\,C^2(x)\,G(x,z)\,\left(E-\frac{2}{3}\Box R\right)_z\bigg|$$

$$=\int d^4x\sqrt{-\bar{g}(x)}\,\bar{\Delta}_4(x)\,\bar{G}(x,y)\,\bar{C}^2(x)\bigg|\;=\;\sqrt{-g(y)}\,C^2(y)\,.$$

**Hence, the part of $\Gamma_{ind}$ which is responsible for $T_\omega = -\omega C^2$, is**

$$\Gamma_\omega = \frac{\omega}{4}\,\iint\limits_{x\,y}\,C^2(x)\,G(x,y)\left(E-\frac{2}{3}\Box R\right)_y.$$

**Similarly, one can check that the variation $T_b = -b\,(E-\frac{2}{3}\Box R)$ is produced by the term**

$$\Gamma_b = \frac{b}{8}\,\iint\limits_{x\,y}\,\left(E-\frac{2}{3}\Box R\right)_x G(x,y)\left(E-\frac{2}{3}\Box R\right)_y.$$

**Finally, we can use the simple relation**

$$-\frac{2}{\sqrt{-g}}\,g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}}\int_x R^2(x) \;=\; 12\,\Box R\,.$$

**to establish the remaining local constituent of $\Gamma_{ind}$:**

$$\Gamma_c = -\frac{3c+2b}{36(4\pi)^2}\int_x R^2(x)\,.$$

**The general covariant solution for $\Gamma_{ind}$ is the sum,**

$$\begin{aligned}
\Gamma_{ind} &= S_c[g_{\mu\nu}] \;-\; \frac{3c+2b}{36(4\pi)^2}\int_x R^2(x) \\
&+ \frac{\omega}{4}\int_x\int_y C^2(x)\,G(x,y)\,\Big(E-\frac{2}{3}\Box R\Big)_y \\
&+ \frac{b}{8}\int_x\int_y \Big(E-\frac{2}{3}\Box R\Big)_x\,G(x,y)\,\Big(E-\frac{2}{3}\Box R\Big)_y\,.
\end{aligned}$$

**One can rewrite this expression using auxiliary scalars.**

## The nonlocal terms can be recast into symmetric form

$$\Big(E - \frac{2}{3}\Box R\Big)_x \,G(x,y)\,\Big[\frac{\omega}{4}C^2 - \frac{b}{8}(E - \frac{2}{3}\Box R)\Big]_y$$

$$= \;\frac{b}{8}\iint\limits_{x\,y}\Big(E - \frac{2}{3}\Box R - \frac{\omega}{b}C^2\Big)_x G(x,y)\Big(E - \frac{2}{3}\Box R - \frac{\omega}{b}C^2\Big)_y$$

$$-\;\frac{\omega^2}{8b}\;\iint\limits_{x\,y}\;C_x^2\,G(x,y)\,C_y^2\,.$$

## Thus we arrive at the local covariant expression for EA

$$\Gamma_{ind}\;=\;S_c[g_{\mu\nu}] - \frac{3c+2b}{36(4\pi)^2}\int\limits_x R^2(x) + \int\limits_x\Big\{\frac{1}{2}\,\varphi\Delta_4\varphi - \frac{1}{2}\,\psi\Delta_4\psi$$

$$+\,\frac{\omega}{8\pi\sqrt{-b}}\,\psi\,C^2 + \varphi\,\Big[\,\frac{\sqrt{-b}}{8\pi}\,\Big(E - \frac{2}{3}\,\Box R\Big) - \frac{\omega}{8\pi\sqrt{-b}}\,C^2\,\Big]\,\Big\}\;.$$

**The above form of EA is the best one for $\Gamma_{ind}$.**
*I.Sh. and A.Jacksenaev, Phys. Lett. B (1994)*

**Similar expression has been independently introduced by**
*P. Mazur & E. Mottola, 1997-1998-2001.*

**Comments:**

**1) Imposing boundary conditions on the two auxiliary fields $\varphi$ and $\psi$ is equivalent to defining boundary conditions for the Green functions $G(x,y)$.**

**2) Introducing the new term $\int C_x^2 G(x,y) C_y^2$ into the action may be viewed as redefinition of the conformal functional $S_c[g_{\mu\nu}]$.**

**However, writing the non-conformal terms in the symmetric form, essentially modifies the four-point function.**
**Using $\psi$ we preserve the UV structure generated by anomaly.**

# General perspective: a few mysteries remains

- **Why the beta-functions for conformal and topological terms, which come from fields of all spins, have the same sign?**

- **And why there exists the "main formula"**

$$\sqrt{-g}\Big(E - \frac{2}{3}\Box R\Big) = \sqrt{-\bar{g}}\Big(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R} + 4\bar{\Delta}_4\sigma\Big) \quad ??$$

**And why the "magic"** $2/3$ **coefficient? It there some general mathematical rule behind this occurrence?**

- **Since our knowledge is restricted, we need further examples.**
- **But before that let us consider some general features.**

$\forall D$. **Several C-type invariants, and two kinds of N-type invariants,**

$$T = \langle T^\mu_\mu \rangle = c_r\, W^r_D + a\, E_D + \Xi_D\,,$$

**with the sum over** $r$. **Here** $W^r_D$ **are conformal invariant terms and**

$$E_D = D^{-1}\,\varepsilon^{\rho_1\cdots\rho_D}\,\varepsilon^{\sigma_1\ldots\sigma_D}\,R_{\rho_1\sigma_1\rho_2\sigma_2}\cdots R_{\rho_{D-1}\sigma_{D-1}\rho_D\sigma_D}.$$

**Example: $\underline{6D}$. There are three C-type terms,**

$$W_6^1 = C_{\mu\nu\rho\sigma}\, C^{\mu\alpha\beta\nu}\, C_{\alpha\cdot\cdot\beta}^{\ \ \rho\sigma}\,,$$

$$W_6^2 = C_{\mu\nu\rho\sigma}\, C^{\rho\sigma\alpha\beta}\, C_{\alpha\beta\cdot\cdot}^{\ \ \ \mu\nu}\,,$$

$$W_6^3 = C_{\mu\rho\sigma\lambda}\Big(\delta^\mu_\nu + 4R^\mu_\nu - \frac{6}{5}\,R\delta^\mu_\nu\Big)\,C^{\nu\rho\sigma\lambda} \,+\, \nabla_\mu J^\mu\,,$$

**where** $$J_\mu \,=\, \big(4R_\mu^{\cdot\lambda\rho\sigma}\nabla^\nu + 3R^{\nu\lambda\rho\sigma}\nabla_\mu\big)R_{\nu\lambda\rho\sigma}$$

$$+\,\Big(\frac{1}{2}R\nabla_\mu - R^\nu_\mu\nabla_\nu\Big)R + R^{\nu\lambda}\big(\nabla_\nu R_{\lambda\mu} - 5\nabla_\mu R_{\nu\lambda}\big).$$

**This is a more complicated case. There are also several N-type terms, including one topological and others total derivatives.**

*F. Bastianelli, S. Frolov, & A. Tseytlin, JHEP (2000), hep-th/0001041.*

**For a general dimension, $\Xi_D$ is a linear combination of surface terms, $\Xi_D = \sum\gamma_k\chi_k$. The numerical coefficients $a$, $c$, $\gamma_k$ depend on the number of massless conformal fields of different spins.**

**Our purpose is to find the anomaly-induced EA $\Gamma_{ind}$ , such that**

$$-\frac{2}{\sqrt{-g}}\,g_{\mu\nu}\,\frac{\delta\Gamma_{ind}}{\delta g_{\mu\nu}} \,=\, T\,.$$

**Integration of anomaly requires modified topological invariant**

$$\tilde{E}_D = E_D + \sum \alpha_k \chi_k\,, \qquad (*)$$

**where the values of $\alpha_k$ are chosen to provide the special conformal property of the "improved" topological term.**

**Under the local conformal transformation** $\quad g_{\mu\nu} = \bar{g}_{\mu\nu}\,e^{2\sigma(x)}$
**it should be** $\quad \sqrt{-g}\tilde{E}_D \,=\, \sqrt{-\bar{g}}\big(\bar{\tilde{E}}_D + D\bar{\Delta}_D\sigma\big) \quad \dagger,$ **and**
$\Delta_D = \Box^{D/2} + \dots$ **is a conformal operator acting on an inert scalar.**

**The existence of such a modified topological term** (*)
**for the general** $D$ **is an important, albeit unproved conjecture.**

*F. Ferreira, I.Sh, P. Teixeira, EPJ+* **131** *(2016) , 1507.03620.*

**An explicit solution of** $\dagger$ **in** $6D$:
*K.-j. Hamada, hep-th/0012053 (Prog. Theor. Phys.).*
*F. Ferreira, I.Sh, arXiv:1702.06892 (PLB); 1812.01140 (PRD).*

**To integrate anomaly, one has to find local metric-dependent Lagrangians $\mathcal{L}_i$ providing that, with some coefficients $c_{ik}$ , there is an identity (always works, but no explanation known)**

$$
-\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta}{\delta g_{\mu\nu}}\, \sum_i c_{ik} \int_x \mathcal{L}_i \,=\, \chi_k\,.
$$

**After that, using a Green function**

$$
\sqrt{-g}\,\Delta_D^x\, G(x,x') \,=\, \delta^D(x,x')\,, \qquad G \,=\, \bar{G}\,.
$$

**The non-local solution for the anomaly-induced EA is**

$$
\Gamma_{ind} \,=\, S_c + \iint_{x\,y} \Big\{ \frac{1}{4}\, c_r\, W_D^r \,+\, \frac{b}{8}\, \tilde{E}_D(x) \Big\}\, G(x,y)\, \tilde{E}_D(y)
$$

$$
+ \sum_k \left(\gamma_k - \alpha_k\right) \sum_i c_{ik} \int_x \mathcal{L}_i\,.
$$

**The modification of the coefficients $\gamma_k \to \gamma_k - \alpha_k$ takes place since part of the surface terms were absorbed into $\tilde{E}_D$.**

**One can construct a local covariant presentation with the use of two auxiliary fields**

$$
\bar{\Gamma} \,=\, S_c \,+\, \sum_k \left(\gamma_k - \alpha_k\right) \sum_i c_{ik} \int_x \mathcal{L}_i
$$

$$
+\,\frac{1}{2}\int_x \Big\{ \varphi \Delta_D \varphi - \psi \Delta_D \psi + \sqrt{-b}\,\varphi\,\tilde{E}_D + \frac{1}{\sqrt{-b}}\left(\psi - \varphi\right)\, c_r\, W_D^r(x) \Big\}\,.
$$

**Here we assumed $b < 0$, as in the $4D$ case.**
**For the opposite sign one can modify $\tilde{E}_D \to -\tilde{E}_D$.**

**It is remarkable that the leading UV part of the vacuum effective action in an arbitrary even dimension can be given in such an extremely simple and general form.**

**However, this universal representation exists only under the assumption of the universal conformal property of the topological term in an arbitrary dimension.**

Does anomaly mean the conformal theories are necessary nonrenormalizable?

**1. Renormalizability of the theory depends on the form of the counterterms required to cancel divergences in the given order.**

**2. UV divergences are removed by adding local counterterms. The conformal models are not supposed to be different.**

**3. Anomaly-induced action includes nonlocal and local terms.**

**4. It is unlikely that nonlocal terms from the subdiagrams produce local divergent terms in the superficial integration. Thus, the nonlocal terms in the effective action may be not important for renormalizability.**

**5. The local terms may depend on the renormalization scheme. In principle, there may be a scheme providing the absence of nonconformal local terms, even in higher loops.**

**6. It is clearly important to understand the ambiguities in the local terms, in the one-loop approximation and beyond.**

# Examples of the ambiguities of local terms

**I. Dimensional regularization.**

**The source of arbitrariness is that the term $C^2$ can be extended for $n \neq 4$ as $C^2(d)$ where $d = n + \gamma(n-4)$ with arbitrary $\gamma$. E.g.,**

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta}{\delta g_{\mu\nu}}\, \frac{\mu^{n-4}}{n-4} \int d^n x \sqrt{-g}\, \beta_1\, C^2(4)\Big|_{n\to 4} = C^2 - \frac{2}{3}\,\Box R\,.$$

**In general, $\Box R$ term in the anomaly and, consequently, the finite local $\int_x R^2$ term in the effective action, depend on $\gamma$.**

*M. Asorey, E. Gorbar, I.Sh., hep-th/0307187 (Class. Quant. Grav.)*

**The standard solution to this problem is to "add a finite conterterm" $\int_x R^2$ and eliminate this term once and forever.**

**This operation is possible because $\int_x R^2$ term belongs to the vacuum sector of the effective action and does not break the conformal symmetry in the sector of quantum fields.**

# Examples of the ambiguities of local terms

**II. Covariant Pauli-Villars (PV) regularization.**

**PV regularization requires introducing scalar fields with different masses and nonminimal parameters**

$$S_{\rm reg} = \sum_{i=0}^{N} \int d^4x \sqrt{-g}\, \Big\{ \frac{1}{2}\, g^{\mu\nu} \partial_\mu \varphi_i \partial_\nu \varphi_i + \frac{\xi_i}{2}\, R\, \varphi_i^2 + \frac{m_i^2}{2}\, \varphi_i^2 \Big\}$$

**The PV fields** $\varphi_i$ **are massive** $m_i = \mu_i M \neq 0$ **and may have either bosonic or fermionic statistics.**

**One can choose masses and statistics in such a way that all divergences in the theory are cancelled. This regularization implies that** $M \to \infty$ **at the end.**

**The anomaly can be obtained in this limit using the nonlocal form factors derived in** *E.V. Gorbar, I.Sh., hep-ph/0210388, JHEP.*

**It turns out that the result for the** $\Box R$**- term in the anomaly depends on the choice of** $\xi_i$.

**The effective action of metric and external scalar field.**
**The covariant nonlocal form is**

$$\Gamma_{ind} = S_c - \int\limits_x \Big(\frac{2b+3c}{36}R^2 + \frac{\beta_\tau}{6}R\Phi^2\Big) - \frac{1}{8b}\iint\limits_{x,y} \; Y_x G(x,y) Y_y$$

$$+ \frac{b}{8}\iint\limits_{x,y} \Big(E_4 - \frac{2}{3}\Box R + \frac{1}{b}Y\Big)_x G(x,y)\Big(E_4 - \frac{2}{3}\Box R + \frac{1}{b}Y\Big)_y$$

**with** $\quad Y(g_{\mu\nu},\Phi) \,=\, wC^2 - \gamma_\Phi\Big[(\nabla\Phi)^2 + \frac{1}{6}\,R\Phi^2\Big] + \frac{1}{4!}\,\widetilde{\beta}_\lambda\Phi^4.$

**Here the local terms have the mentioned ambiguity.**
*M. Asorey, W.C. Silva, I.Sh., P. do Vale, arXiv:2202.00154 (EPJC).*

**There is also local representation with two auxiliary fields**

$$\Gamma_{ind} \,=\, S_c[g_{\mu\nu},\Phi] - \int_x \Big\{\frac{2b+3c}{36}R^2 + \frac{\beta_\tau}{6}R\Phi^2\Big\} + \int_x \Big\{\frac{1}{2}\varphi\Delta_4\varphi$$

$$- \frac{1}{2}\psi\Delta_4\psi + \frac{\sqrt{-b}}{2}\varphi\Big(E_4 - \frac{2}{3}\Box R + \frac{1}{b}Y\Big) + \frac{1}{2\sqrt{-b}}\,\psi Y\Big\}.$$

**Different from the vacuum sector, the ambiguities in the $R\Phi^2$- term affect the invariance of the quantum fields and produce nonconformal divergences in higher loops, see e.g.**

*S.J. Hathrell, Ann. Phys. 139 (1982) 136.*

**Thus, the hope to have conformal theory renormalizable is essentially based on the ambiguity in these terms.**

**One has to elaborate a special scheme of renormalization and fix the ambiguities such that in any order of the loop expansion there would be no local nonconformal terms.**

**Such a scheme should require elaborating some regularization with a complete fine-tunings for the relevant coefficients.**

**Only in this special scheme one can think on nonperturbative universality properties of the anomaly coefficients $a$, $c$.**

**In conformal quantum gravity the situation is equivalent even without scalar field. $R^2$ ambiguous, theory nonrenormalizable.**

# Anomaly-induced effective action in the IR

**To derive the IR limit from the anomaly we need a set of additional assumptions**

*M. Giannotti and E. Mottola, arXiv:0812.0351 (PRD).*
*M. Asorey, W.C. Silva, I.Sh., P. do Vale, arXiv:2202.00154 (EPJP).*

**i) All matter fields are approximately massless and conformal.**

**ii) Scalar terms** $\Phi^4$, $X_c$ are dominating over the curvature terms

$$\left|\Phi^2\right| \gg \left|R_{....}\right| \qquad \text{and} \qquad \left|(\nabla\Phi)^2\right| \gg \left|R^2_{....}\right|$$

**for all components of the curvature tensor.**

**iii) As always in GR, the IR limit means the gravitational field is weak, including the linear order in the metric perturbations over the flat background dominating.**

**This implies** $|\Box R| \gg |R^2_{....}|$ **for all curvature contractions.**

# Anomaly-induced effective action in the IR

**The non-local structures in the induced action reduce to**

$$G = \Delta_4^{-1} = \Big(\Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}R^{;\mu}\nabla_\mu\Big)^{-1} \approx \frac{1}{\Box^2}.$$

**The leading terms are those with $\Phi$ and $\Box R$, hence**

$$E_4 - \frac{2}{3}\Box R + \frac{1}{b}Y \approx -\frac{2}{3}\Box R - \frac{1}{b}\Big(\gamma_\Phi X_c - \frac{1}{4!}\widetilde{\beta}_\lambda\Phi^4\Big),$$

**where** $X_c \equiv \frac{1}{2}\Big[(\nabla\Phi)^2 + \frac{1}{6}R\Phi^2\Big]$.

**Finally,**

$$\Gamma_{ind,IR} \approx \frac{b}{18}\int_x R^2 - \frac{1}{6}\int_x\int_y \Big(\frac{1}{4!}\widetilde{\beta}_\lambda\Phi^4 - \gamma_\Phi X_c\Big)_x \Big(\frac{1}{\Box}\Big)_{x,y} R(y)$$

**which is potentially interesting for various applications.**

# Anomaly-induced effective action in the IR

**In the symmetry-breaking regime $\Phi \approx v = const$.**
**Then we get, in the IR induced action,**

$$-\frac{1}{6}\iint_{xy}\Big(\frac{1}{4!}\,\widetilde{\beta}_\lambda v^4 \,-\, \frac{1}{6}\,\gamma_\Phi R v^2\Big)_x \Big(\frac{1}{\Box}\Big)_{x,y} R(y) \;\sim\; \int R\frac{1}{\Box}R.$$

**This kind of a theory was extensively discussed in cosmology.**
**One can derive it as part of the low-energy induced action.**

**Finally, assuming conformal transformation**

$$g_{\mu\nu} = \bar{g}_{\mu\nu}e^{2\sigma}, \qquad \Phi = \bar{\Phi}e^{-\sigma}$$

**in the linear in $\sigma$ order, we get the one-loop effective potential**

$$V_{eff}^{(1)} \;=\; \frac{1}{4!}\Big(\lambda \,+\, \frac{1}{2}\,\widetilde{\beta}_\lambda \ln\frac{\Phi^2}{\mu^2}\Big)\Phi^4 \;-\; \frac{1}{12}\Big(1 \,+\, \gamma_\Phi \ln\frac{\Phi^2}{\mu^2}\Big)R\Phi^2,$$

$$\widetilde{\beta}_\lambda = \beta_\lambda + 4\lambda\gamma_\Phi.$$

# Antisymmetric tensor field theory

**The unusual feature of the antisymmetric tensor field theory is that there is no local conformal symmetry in the flat limit,**

$$S_{flat} = \int d^4x \,\Big\{ \sum_{k=1}^{N} i\bar{\psi}_k \big(\gamma^\mu \partial_\mu - g\Sigma^{\mu\nu} B_{\mu\nu} - im\big)\psi_k \,+\, \frac{1}{2}(\partial_\alpha B_{\mu\nu})^2$$

$$- \,2(\partial_\mu B^{\mu\nu})^2 - \frac{1}{2}M^2 B^2_{\mu\nu} - \frac{f_2}{8}(B^2_{\mu\nu})^2 - \frac{f_3}{8} B_{\mu\nu} B^{\nu\alpha} B_{\alpha\beta} B^{\beta\mu} \Big\}\,,$$

**but the "conformal" restriction on the coefficients of the two kinetic terms, i.e., $(\partial_\alpha B_{\mu\nu})^2$ and $(\partial_\mu B^{\mu\nu})^2$, are expected to hold even beyond the one-loop approximation.**

*L.V. Avdeev and M.V. Chizhov, PLB 321 (1994), hep-th/9312062.*

**This situation resembles the smile of the Cheshire Cat (local conf. symmetry), that remains seen when the Cat himself gone.**

**Since the renormalization of the curvature-independent terms is the same in flat and curved spaces, the relation between $(\nabla_\alpha B_{\mu\nu})^2$ and $(\nabla_\mu B^{\mu\nu})^2$ holds in both curved and flat spacetimes.**

# Renormalization group equations

**The general arguments presented above are confirmed by the explicit one-loop calculations in the fermionic sector. We rewrite**

$$S_B = \int d^4x\sqrt{-g}\,\Big\{\frac{1}{2}\big(W_4 + \lambda W_1\big) - \frac{1}{2}M^2 B_{\mu\nu} - \frac{1}{4!}\big(f_2 W_2 + f_3 W_3\big)\Big\}$$
$$+\; i\int d^4x\sqrt{-g}\,\bar{\psi}\{\gamma^\mu\nabla_\mu - g\Sigma^{\mu\nu}B_{\mu\nu} - im\}\psi$$

**and integrate over fermions The restrictions in the form of**

$$W_4 = \sqrt{-g}\,\big\{(\nabla_\alpha B_{\mu\nu})(\nabla^\alpha B^{\mu\nu}) - 4(\nabla_\mu B^{\mu\nu})(\nabla^\alpha B_{\alpha\nu})$$
$$+\, 2B^{\mu\nu}R_\nu^{\alpha}B_{\mu\alpha} - \frac{1}{6}RB_{\mu\nu}B^{\mu\nu}\big\}.$$

**were confirmed and we also met UV asymptotic freedom for** $g$**,**

$$\mu\,\frac{dg^2}{d\mu} = -\,\frac{8}{3(4\pi)^2}\,g^4 \qquad\Longrightarrow\qquad \lim_{\mu\to\infty} g(\mu) = 0.$$

**For other couplings** $\lambda,\ f_{1,2}$ **there are UV stable fixed points.**

*I.Sh., arXiv:2310.04131 (EPJC-2024).*

# Conclusions

- **Integrating conformal anomaly is very efficient and economic way to derive the effective action of vacuum.**

- **The original method admits generalizations, including external scalars, vectors, torsion or the antisymmetric field.**

- **In the case of background metric and scalar (also vector, etc), one can obtain the covariant expression for the effective action of vacuum. This expression has non-local and local parts. The local part is typically ambiguous.**

- **Another new development is the connection between anomaly and the IR limit of the effective action (e.g., effective potential), with potentially interesting applications.**

- **In case of antisymmetric field, conformal symmetry controls renormalization, even when there is no conformal symmetry.**