Back-reaction in AdS Braneworld

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Basic idea

A 3-brane moving in AdS_5 background of the second Randall-Sundrum (RSII) model behaves effectively as a tachyon with the inverse quartic potential.

The RSII model may be extended to include the back reaction due to the radion field. Then the tachyon Lagrangian is modified by the interaction with the radion.

As a consequence, the effective equation of state obtained by averaging over large scales describes a warm dark matter.

Based on the work in colaboration with Garry Tupper

"Warm" Tachyon Matter from Back-reaction on the Brane" arXiv:1302.0955 [hep-th].

Outline

1. Randall–Sundrum Model
2. Gravity in the Bulk – the Radion
3. Dynamical Brane – the Tachyon
4. Field Equations
5. Isotropic Homogeneous Evolution
6. Conclusins & Outlook

1. Randall – Sundrum model

Branewarld cosmology is based on the scenario in which matter is confined on a brane moving in the higher dimensional bulk with only gravity allowed to propagate in the bulk

N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429** (1998)
L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 3370; ibid. 4690

It is usually assumed that extra dimensions are compact and if their size is large enough compared to the Planck scale, such a scenario may explain the large mass hierarchy between the electroweak scale and the fundamental scale of gravity.

The Randall–Sundrum solution to the hierarchy problem is a five dimensional universe containing two four dimensional branes separated in the 5^{th} dimension: the observer's brane and the negative tension brane.

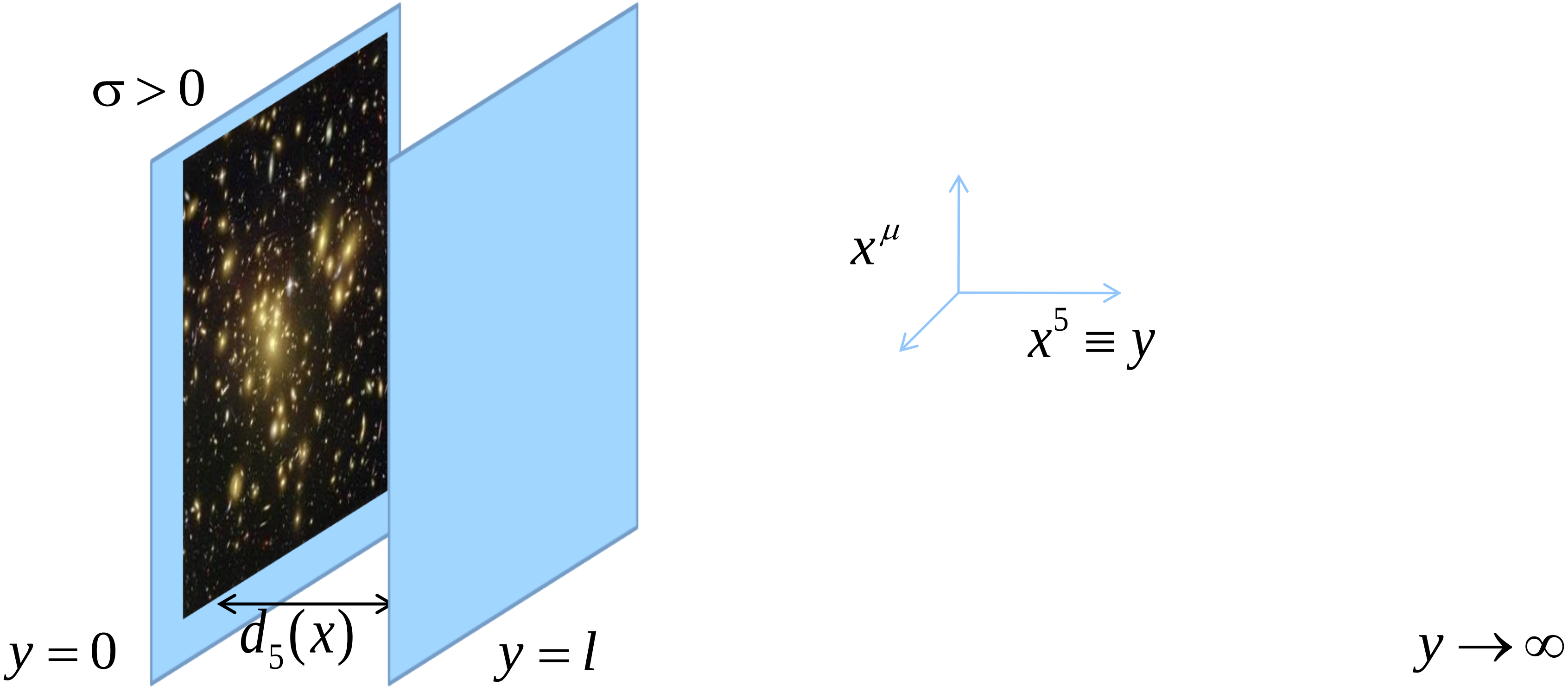
The separation proposed is such that the strength of gravity on observer's brane is equal to the observed four-dimensional Newtonian gravity.

RSII – model

The 5-dim bulk is ADS_5/Z_2 with the line element

$$ds^2_{(5)} = g_{(5)MN}(X)dX^M dX^N = e^{-2ky} g_{\mu\nu}(x) dx^\mu dx^\nu - dy^2$$

Observers reside on the positive tension brane at *y=0*

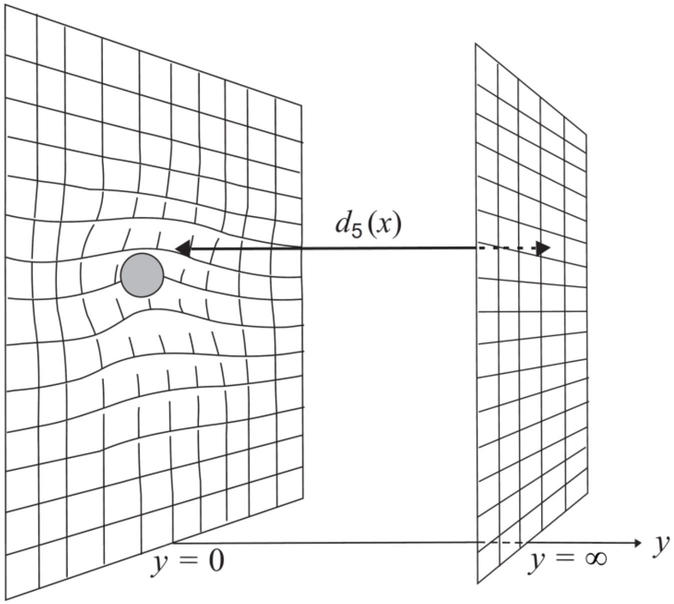


In the second Randall–Sundrum model (RSII) the negative tension brane is pushed off to infinity in the fifth dimension and Planck mass scale is determined by the curvature of the five-dimensional space-time rather then the size of the fifth dimension.

Hence, the model solves the mass hierarchy problem without resorting to finite extra dimensions and provides an alternative to compactification.

Radion

The Randall-Sundrum solution corresponds to an empty brane at $y=0$. Placing matter on this observer brane changes the bulk geometry; this is encoded in the *radion* field related to the variation of the physical interbrane distance $d_5(x)$.



2. Gravity in the Bulk – the Radion

The naive AdS_5 geometry of RSII is distorted by the *radion*.
To see this, consider the total gravitational action in the bulk

$$S = S_{\text{bulk}} + S_{\text{GH}} + S_{\text{brane}}$$

where

$$S_{\text{bulk}} = \frac{1}{2K_5}\int d^5x\sqrt{g_{(5)}}\left[-R_{(5)} - 2\Lambda_5\right]$$

We choose a coordinate system such that

$$g_{(5)\mu 5} = 0, \quad \mu = 0,1,2,3$$

and we assume a general metric which admits Einstein spaces of constant 4-curvature with the line element

$$ds^2_{(5)} = g_{(5)MN}(X)dX^M dX^N = \Psi^2(x,y)g_{\mu\nu}(x)dx^\mu dx^\nu - \varphi^2(x,y)dy^2,$$

At fixed x^μ, $d_5 = \int dy\varphi$ is the distance along the fifth dimension.

The 5-dim bulk action may be put in the form

$$S_{bulk} = \frac{1}{K_5}\int d^4x\sqrt{-g}\int dy\left[-\frac{R}{2}\Psi^2\varphi - 3g^{\mu\nu}(\Psi\varphi)_{,\mu}\Psi_{,\nu} + 6\frac{\Psi^2(\partial_y\Psi)^2}{\varphi} - \Lambda_5\Psi^4\varphi\right]$$

where R is the 4-d Ricci scalar made out of the 4-d metric $g_{\mu\nu}$
We have omitted the total y-derivative term because it is cancelled by the Gibbons – Hawking term in the action

For consistency with Einsten's equations we require

$$R_{(5)\mu 5} = 0$$

In addition, we impose the *'Einstein frame'* gauge condition $\Psi^2\varphi = W(y)^2$ so the coefficient of R in S_{bulk} is a function of y only

We arrive at the 5-dimensional line element

$$ds_{(5)}^2 = \left[\phi(x) + W(y)^2\right] g_{\mu\nu}(x)\, dx^\mu dx^\nu - \left[\frac{W(y)^2}{\phi(x) + W(y)^2}\right]^2 dy^2$$

where the field ϕ represents the radion.

J.E. Kim, G. Tupper, and R. Viollier, PLB **593** (2004)

By choosing $W(y) = e^{-ky}$ and neglecting the radion we recover the AdS$_5$ geometry. Hence, the choice $W(y) = e^{-ky}$ corresponds to the RSII model.

This metric is a solution to Einsten's equations provided

$$k^2 = -\frac{\Lambda_5}{6}$$

precisely as it is in the RSII model

Integration over y fom 0 to ∞ yields

$$S_{\text{bulk}} = \int d^4x\sqrt{-g}\left\{-\frac{R}{2kK_5} + \frac{3}{4kK_5}\frac{g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}}{\phi(1+\phi)} + \frac{6k}{K_5}(1+2\phi)\right\}$$

With the RSII fine tuning

$$\sigma_0 = -\sigma_l = \frac{6k}{K_5} \equiv \sigma_{\text{RS}}$$

the last term may be canceled by the two brane actions

$$S_{\text{brane}}|_{y=0} + S_{\text{brane}}|_{y=l} = -\sigma_0\int d^4x\sqrt{-g}(1+\phi)^2 - \sigma_l\int d^4x\sqrt{-g}(e^{-2kl}+\phi)^2$$

Then, the bulk action takes a simple form

$$S_{bulk} = \int d^4x \sqrt{-g}\left(-\frac{R}{16\pi G} + \frac{1}{2} g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}\right)$$

where we have introduced the 4-dimensional gravitational constant

$$G = \frac{K_5 k}{8\pi}$$

and the canonically normalized radion

$$\Phi = \sqrt{\frac{3}{4\pi G}} \sinh^{-1}\left(\sqrt{\phi}\right)$$

The appearance of a massless mode – *the radion* – causes 2 effects

1. Matter on observer's brane sees the (induced) metric

$$\tilde{g}_{\mu\nu} = (1+\phi)\, g_{\mu\nu}$$

2. The physical distance to the AdS_5 horizon at coordinate infinity

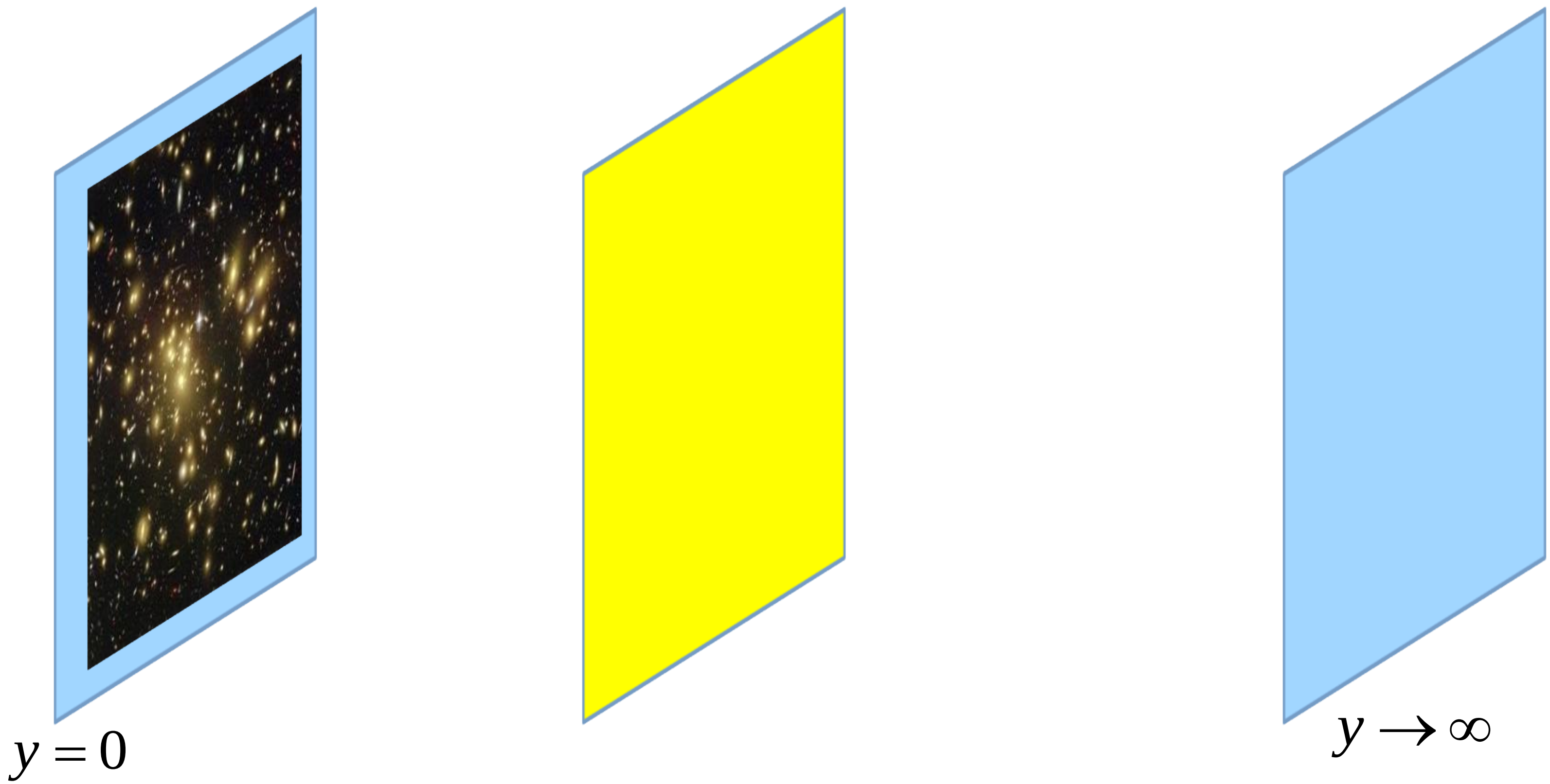
$$d_5 = \frac{1}{2k} \ln\left(\frac{1+\phi}{\phi}\right)$$

is no longer infinite so the physical size of the 5-th dimension is of the order $k^{-1} \sim l_{\mathrm{Pl}}$ although its coordinate size is infinite

3. Dynamical Brane – the Tachyon

Consider a 3-brane moving in the 5-d bulk spacetime with metric

$$ds^2_{(5)} = (e^{-2ky} + \phi) g^{\mu\nu} dx^{\mu} dx^{\nu} - \left(\frac{e^{-2ky}}{e^{-2ky} + \phi} \right)^2 dy^2$$



The points on the brane are parameterized by $X^M = \left(x^\mu, Y(x)\right)$. The 5-th coordinate Y is treated as a dynamical field. The brane action

$$S_{brane} = -\sigma \int d^4x \sqrt{-\det g_{\mu\nu}^{ind}}$$

with the induced metric

$$g_{\mu\nu}^{\text{ind}} = g_{(5)MN} X^M_{,\mu} X^N_{,\nu} = \left(\frac{e^{-2kY}}{e^{-2kY} + \phi} \right)^2 \left[\frac{(e^{-2kY} + \phi)^3}{(e^{-2kY})^2} g_{\mu\nu} - Y_{,\mu} Y_{,\nu} \right]$$

yields

$$S_{brane} = -\sigma \int d^4x \sqrt{-g} (e^{-2kY} + \phi)^2 \left(1 - \frac{(e^{-2kY})^2}{(e^{-2kY} + \phi)^3} g^{\mu\nu} Y_{,\mu} Y_{,\nu} \right)^{1/2}$$

Changing *Y* to a new field $\theta = e^{kY}/k$ we obtain the effective brane action

$$S_{\rm brane} = -\sigma \int d^4x \sqrt{-g}\, \frac{(1+k^2\theta^2\phi)^2}{k^4\theta^4} \sqrt{1 - \frac{g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}}{(1+k^2\theta^2\phi)^3}}$$

In the absence of the radion field ϕ we have a pure undistorted AdS_5 background and

$$S^{(0)}_{\rm brane} = -\int d^4x \sqrt{-g}\, \frac{\sigma}{k^4\theta^4} \sqrt{1 - g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}}$$

This action describes a *tachyon* with inverse quartic potential.

Tachyon as CDM

More generally an effective Born-Infeld type lagrangian

$$\mathcal{L} = -V(\theta)\sqrt{1 - g^{\mu\nu}\theta_{,\mu}\theta_{,\nu}}$$

for the tachyon field θ describes unstable modes in string theory

A. Sen, JHEP **0204** (2002); **0207** (2002).

The typical potential has minima at $\theta = \pm\infty$. Of particular interest is the inverse power law potential $V \propto \theta^{-n}$.

For $n > 2$, as the tachyon rolls near minimum $p \to 0^-$ very quickly and one thus apparently gets pressure-less matter (dust) or cold dark matter.

L.R. Abramo and F. Finelli, PLB **575**(2002).

For $n=0$, i.e., $V=V_0$, one gets the Chaplygin gas

$$p = -\frac{V_0}{\rho}$$

the first definite model for a DE/DM unification

A. Kamenshchik, U. Moschella, V. Pasquier, PLB **511** (2001)
J.C. Fabris, S.V.B. Goncalves, P.E. de Souza, GRG **34** (2002)
N.B., G.B. Tupper, R.D. Viollier, PLB **535** (2002)

In general any tachyon model can be derived as a map from the motion of a 3-brane moving in a warped extra dimension.

Tachyon has been heavily exploited in almost any cosmological context: as inflaton, DM, DE

E.g., for positive power law potential

$$V(\theta) = V_0 \theta^{2n}$$

One can get dark matter *and* dark energy as a single entity i.e., another model for DE/DM unification

N. B., G. Tupper, R.Viollier, *Cosmological tachyon condensation*. Phys. Rev. D 80 (2009)

The dynamical brane causes two back-reaction effects

1. The geometric tachyon is seen on our brane as a form of matter and hence it affects the bulk geometry in which it moves.
2. The back-reaction qualitatively changes the geometric tachyon: the tachyon and radion form a composite substance with a modified equation of state.

4. Field Equations

Instead of the tachyon field defined previously as $\theta = e^{kY}/k$ it is convenient to introduce a new field

$$\Theta(x) = 3e^{-2kY(x)} = \frac{3}{k^2\theta(x)^2}$$

Then the combined radion and brane Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} - \frac{\lambda}{\ell^2}\psi^2\sqrt{1-\ell^2\frac{g^{\mu\nu}\Theta_{,\mu}\Theta_{,\nu}}{\psi^3}}$$

where

$$\psi = 2\Theta + 6\phi, \qquad \lambda = \frac{\sigma}{6k^2}, \qquad \ell = \frac{\sqrt{6}}{k}$$

Where we define canonical conjugate momentum fields

$$\pi_{\Phi}^{\mu} = \frac{\partial L}{\partial \Phi_{,\mu}} = g^{\mu\nu}\Phi_{,\nu} \qquad \pi_{\Theta}^{\mu} = \frac{\partial L}{\partial \Theta_{,\mu}} = \frac{\lambda}{\psi}\frac{g^{\mu\nu}\Theta_{,\nu}}{\sqrt{1-\ell^{2}g^{\mu\nu}\Theta_{,\mu}\Theta_{,\nu}/\psi^{3}}}$$

For timelike $\Phi_{,\nu}$ and $\Theta_{,\nu}$ we may also define the norms

$$\pi_{\Phi} = \sqrt{g_{\mu\nu}\pi_{\Phi}^{\mu}\pi_{\Phi}^{\nu}} \qquad \pi_{\Theta} = \sqrt{g_{\mu\nu}\pi_{\Theta}^{\mu}\pi_{\Theta}^{\nu}}$$

Then the Lagrangian may be expressed as

$$\mathcal{L} = \frac{1}{2}\pi_{\Phi}^{2} - \frac{\lambda\psi^{2}}{\ell^{2}}\frac{1}{\sqrt{1+\ell^{2}\pi_{\Theta}^{2}/(\lambda^{2}\psi)}}$$

The corresponding energy momentum tensor

$$T_{\mu\nu} = 2\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} - \mathcal{L}g_{\mu\nu} = \pi_{\Phi\mu}\pi_{\Phi\nu} + \frac{\ell^2\psi}{\lambda}\frac{\pi_{\theta\mu}\pi_{\theta\nu}}{\sqrt{1+\ell^2\pi_\Theta^2/(\lambda\psi)}} - g_{\mu\nu}\mathcal{L}$$

may be expressed as a sum of two ideal fluids

$$T_{\mu\nu} = (p_1+\rho_1)u_{1\mu}u_{1\nu} + (p_2+\rho_2)u_{2\mu}u_{2\nu} - (p_1+p_2)g_{\mu\nu}$$

with $u_{1\mu} = \pi_{\Phi\mu}/\pi_\Phi$ $\quad u_{2\mu} = \pi_{\Theta\mu}/\pi_\Theta$

$$p_1 = \frac{1}{2}\pi_\Phi^2 \qquad p_2 = -\frac{\lambda\psi^2}{\ell^2}\frac{1}{\sqrt{1+\ell^2\pi_\Theta^2/(\lambda^2\psi)}}$$

$$\rho_1 = \frac{1}{2}\pi_\Phi^2 \qquad \rho_2 = \frac{\lambda\psi^2}{\ell^2}\sqrt{1+\ell^2\pi_\Theta^2/(\lambda^2\psi)}$$

The Hamiltonian may be identified with the total energy density

$$\mathcal{H} = T^{\mu}{}_{\mu} + 3\mathcal{L} = \rho_1 + \rho_2$$

which yields

$$\mathcal{H} = \frac{1}{2}\pi_{\Phi}^2 + \frac{\lambda\psi^2}{\ell^2}\sqrt{1+\ell^2\pi_{\Theta}^2/(\lambda^2\psi)}$$

It may be easily verified that $\mathcal{H}$ is related to $\mathcal{L}$ through the Legendre transformation

$$\mathcal{H}(\pi_{\Phi}^{\mu},\pi_{\Theta}^{\mu}) = \pi_{\Phi}^{\nu}\Phi_{,\nu} + \pi_{\Theta}^{\nu}\Theta_{,\nu} - \mathcal{L}(\Phi_{,\mu},\Theta_{,\mu})$$

Here the dependence on Φ and Θ is suppressed. The rest of the field variables are constrained by Hamilton's equations

Hamilton's equations

$$\Phi_{,\mu} = \frac{\partial \mathcal{H}}{\partial \pi^{\mu}_{\Phi}} \qquad \Theta_{,\mu} = \frac{\partial \mathcal{H}}{\partial \pi^{\mu}_{\Theta}}$$

$$\pi^{\mu}_{\Phi} = \frac{\partial \mathcal{L}}{\partial \Phi_{,\mu}} \qquad \pi^{\mu}_{\Theta} = \frac{\partial \mathcal{L}}{\partial \Theta_{,\mu}}$$

Now we multiply the first and second equations by u_1^{μ} and u_2^{μ}, respectively, and we take a covariant divergence of the next two equations

We obtain a set of four 1st order diff. equations

$$\dot{\Phi} = \pi_\Phi \qquad \dot{\Theta} = \frac{\psi}{\lambda}\frac{\pi_\Theta}{\sqrt{1+\ell^2\pi_\Theta^2/(\lambda^2\psi)}}$$

$$\dot{\pi}_\Phi + 3H_1\pi_\Phi = -\frac{3}{\ell^2\lambda}\frac{4\lambda^2\psi + 3\ell^2\pi_\Theta^2}{\sqrt{1+\ell^2\pi_\Theta^2/(\lambda^2\psi)}}\phi'$$

$$\dot{\pi}_\Theta + 3H_2\pi_\Theta = -\frac{1}{\ell^2\lambda}\frac{4\lambda^2\psi + 3\ell^2\pi_\Theta^2}{\sqrt{1+\ell^2\pi_\Theta^2/(\lambda^2\psi)}}$$

where $\dot{\Phi} \equiv u_1^\mu \Phi_{,\mu},\ \dot{\Theta} \equiv u_2^\mu \theta_{,\mu},\ \dot{\pi}_\Phi = u_1^\mu \pi_{\Phi,\mu},\ \dot{\pi}_\Theta = u_1^\mu \pi_{\Theta,\mu}$

$$\phi' = \sqrt{\frac{4\pi G}{3}}\sinh\left(2\sqrt{\frac{4\pi G}{3}}\Phi\right) \qquad 3H_1 = u^\mu_{1\,;\mu} \qquad 3H_2 = u^\mu_{2\,;\mu}$$

4. Isotropic Homogeneous Evolution

To exhibit the main features we solve our equations assuming spatially flat FRW spacetime with line ellement

$$ds^2 = dt^2 - a^2(dr^2 + r^2 d\Omega^2)$$

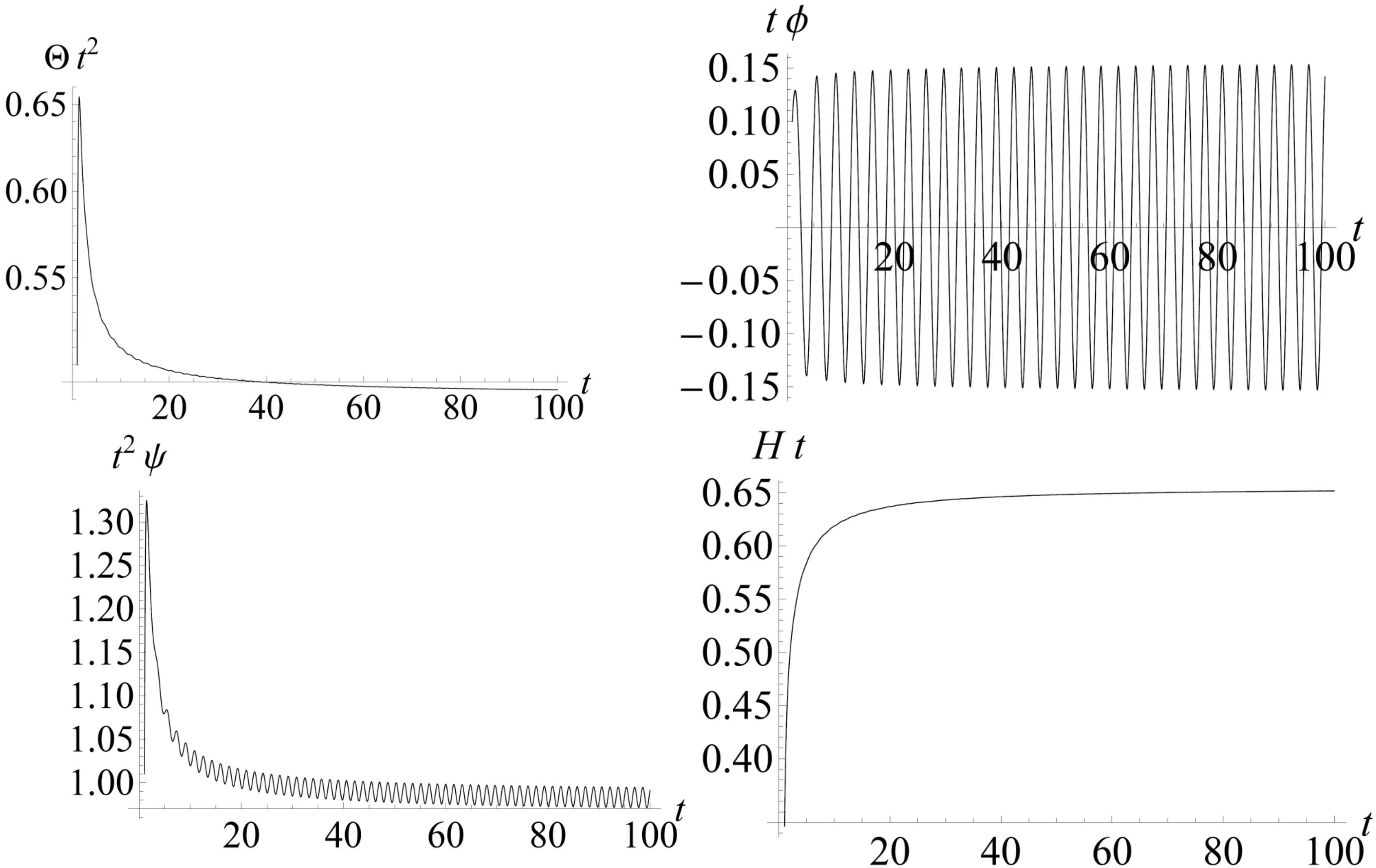
in which case

$$H_1 = H_2 = H \equiv \frac{\dot{a}}{a}$$

and we have in addition the equation for the scale *a*(*t*)

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3}\mathcal{H}}$$

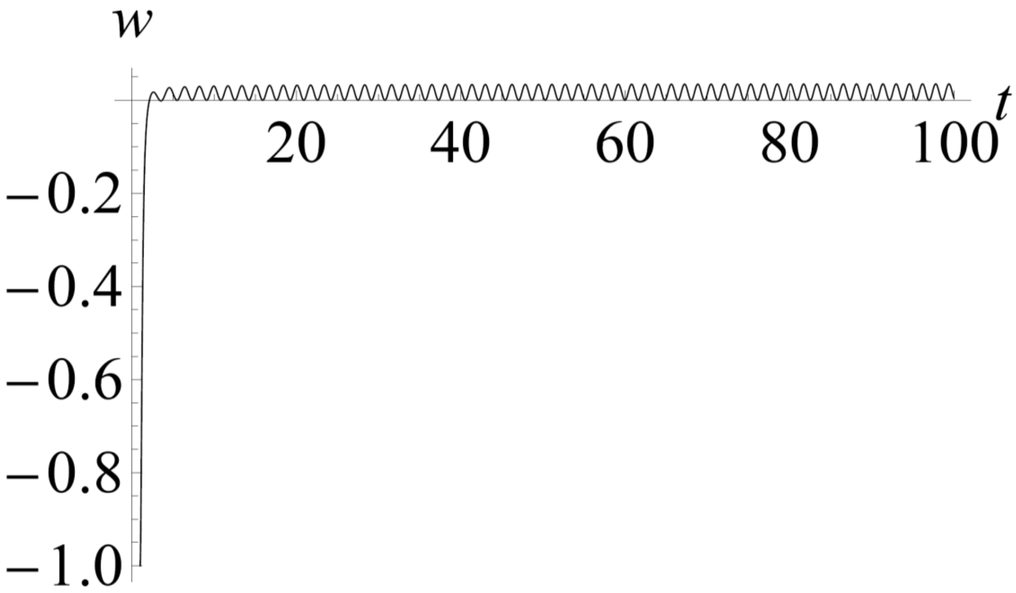
We evolve the radion-tachyon system from t=0 with some suitably chosen initial conditions



Evolution of the radion-tachyon system for $\lambda\ell^2$=1/3 with the initial conditions at t=0: $\Theta = 1.01,\ \ \ell\Phi = 0.1,\ \ \pi_\Theta = \pi_\Phi = 0$

Time is taken in units of ℓ

After the transient period the equation of state $w=p/\rho$ becomes positive and oscillatory



In the asymptotic regime ($t \to \infty$)

$$w = \frac{p}{\rho} \simeq \frac{\dot{\Phi}^2}{\dot{\Phi}^2 + 2\psi^{3/2} |\pi_\Theta| / \ell}$$

Approximate asymptotic solution to second order in the amplitude A of $t\Phi$

$$\Phi \simeq \frac{A}{t} \cos\left(2\frac{t}{\ell}\right)$$

$$\psi \simeq \frac{\ell^2}{t^2}\left(1 - \frac{3A^2}{2} + A^2 \cos^2\left(2\frac{t}{\ell}\right)\right)$$

$$-\sqrt{\psi}\,\pi_\Theta \ell^3 \simeq \frac{4}{3} - 2A^2 + \frac{A^2}{2}\cos^2\left(\frac{2t}{\ell}\right)$$

so

$$w \simeq \frac{3}{2} A^2 \sin^2\left(2\frac{t}{\ell}\right)$$

Since the oscillations in w are rapid on cosmological timescales, it is most useful to time average co-moving quantities. The effective equation of state is then

$$\langle p \rangle = \langle w \rangle \langle \rho \rangle$$

By averaging the asymptotic w over long timescales we find

$$\langle w \rangle \simeq \frac{3A^2}{4} \simeq 0.017$$

where we have estimated A by comparing the exact and approximate solutions for $t\Phi$

Warm dark matter

CDM is rather successful in explaining the large-scale power spectrum and CMB spectrum.
However there is some inconsistency of many body simulations with observations:

1. Overproduction of satellite galaxies
2. Modelling haloes with a central cusp.

It is believed that these problems may be alleviated to some extent with the so called *Warm DM*

Besides, it has been recently argued that cosmological data favour a dark matter equation of state $w_{DM} \approx 0.01$

A. Avelino, N. Cruz and U. Nucamendi, arXiv:1211.4633

To demonstrate that our asymptotic equation of state is associated with WDM, we will show that the horizon mass at the time when the equivalent DM particles just become nonrelativistic is typically of the order of a small galaxy mass.

We assume that our equation of state $w \approx 0.017$ corresponds to that of (nonrelativistic) DM thermal relics of mass $m_{\rm DM}$ at the time of radiation-matter equality $t_{\rm eq}$

$$w_{\mathrm{D}M} = \frac{T}{m_{\mathrm{D}M}}$$

Hence, we take $T = T_{\mathrm{e}q} = 7.4\ \mathrm{eV}$ at $t = t_{\mathrm{eq}}$ and identify

$$w_{\mathrm{D}M}\,|_{\mathrm{e}q} = \langle w \rangle = 0.017$$

This gives $m_{\rm DM} \approx 0.43\ \mathrm{keV}$

.

These DM particles have become non relativistic at the temperature $T \equiv T_{NR} = m_{DM}$, corresponding to the scale

$$a_{NR} \simeq \frac{T_{eq}}{T_{NR}} a_{eq} = < w > a_{eq}$$

The horizon mass before equality evolves as

$$M_H \simeq M_{eq} \left(\frac{a}{a_{eq}} \right)^3 ,$$

where $M_{eq} \simeq 2 \times 10^{15} M_{\odot}$ for a spatially flat universe. Thus, at $a = a_{NR}$ we obtain

$$M_H \simeq 10^{10} M_{\odot}$$

the mass scale typical of a small galaxy and therefore the DM may be qualified as warm.

Conclusions&Outlook

- We have demonstrated that back-reaction causes the brane – radion system to behave as "warm" tachyon matter with a linear barotropic equation of state
- The ultimate question regards the clustering properties of the model. At the linear level one expects a suppression of small-scale structure formation: initially growing modes undergo damped oscillations once they enter the co-moving acoustic horizon
- Perturbation theory is not the whole story – it would be worth studying the nonlinear effects, e.g., using the Press-Schechter formalism as in the pure tachyon model of N. B., G. Tupper, R.Viollier, Phys. Rev. D 80 (2009)

Thank you