Group Theory and Harmonic Oscillators in the Plane

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Plan

1. Harmonic oscillators in the plane: main results
   - $SO(2)$ gauge invariant model (General spectrum)
   - Physical projector (Physical states)
2. $U(rd) = U_N(1) \times SU(rd)$ dynamical symmetry (to rmve dgnracy)
   - The model $G = SO(r = 2), \ d = 2$
   - Gauge invariant states: global $SU(2)$ dynamical symmetry

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Lagrange function

The Lagrange function describing such system can be written as follows

$$L = \frac{1}{2g^2}\left[\dot{q}_i^a - \lambda\varepsilon^{ab}q_i^b\right]^2 - V(q_i^a)\ , \tag{1}$$

where

$$i = 1, 2, \ldots, d\ ;\ a = 1, 2 = b, \tag{2}$$

$$V(q_i^a) = \frac{1}{2}\omega^2 q_i^a q_i^a. \tag{3}$$

$SO(2) \times SO(d)$

The model is then gauge invariant $SO(2)$ and admits a global $SO(d)$ symmetry associated to the *a priori* indiscerniblity of particles, justifying hence the name given to it : *model* $SO(2) \times SO(d)$ or simply $2 \times d$.

Yang-Mills Lagrangian density

In fact, the above model represents physically a dimensional reduction to $0 + 1$ space-time dimensions of some pure gauge theory of $SO(2)$ local symmetry (abelian) with addition of a mass term which is also properly gauge invariant. Indeed, let's consider the Yang-Mills Lagrangian density in some D-dimensional Minkowski space-time endowed with the metric structure $\eta_{\mu\nu} = \text{diag}(+, \underbrace{- - - \ldots -}_{D-1})$, given by

Yang-Mills Lagrangian density

Let's consider the Yang-Mills Lagrangian density in some D-dimensional Minkowski space-time endowed with the metric structure $\eta_{\mu\nu} = \text{diag}(+, \underbrace{- - - \ldots -}_{D-1})$, given by

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a , \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \, , \tag{4}$$

where a and μ are the Lie algebra index associated to some an *a priori* non-abelian group and the space-time index, respectively.

Abelian theory

The limits of the abelian theory the dimensional reduction transforms the variables as follows

$$A^a_\mu(\vec{x}, t) \rightarrow A^a_\mu(t) \left(\begin{array}{c} A^a_i(t) \equiv q_i{}^a(t) \\ A^a_o(t) \equiv \lambda^a(t) \end{array} \right). \tag{5}$$

Equations of motion

The equations of motion are established from Langrange-Euler formula

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{6}$$

Specifically, with the gauge condition $\lambda(t) = 0$, we obtain the following equations which characterize the dynamics of a set of d oscillators constrained to have a vanishing angular momentum in the plane,

$$\ddot{q}_i^a = -g^2\omega^2 q_i^a \ , \ \ \varepsilon^{ab} q_i^a \dot{q}_i^a = 0. \tag{7}$$

Dirac algorithm (suitable for constrained systems)

The classical hamiltonian formulation with the appropriated symplectic structure, using the Dirac algorithm for constrained or singular systems presents as follows

$$H = H_o + \lambda(t)\phi, \;\; H_o = \frac{1}{2}\left[g^2(p_i^a)^2 + \omega^2(q_i^a)^2\right], \tag{8}$$

$$\phi = \varepsilon^{ab} p_i^a q_i^b, \;\; \{q_i^a, p_j^b\} = \delta^{ab}\delta_{ij}. \tag{9}$$

Quantization procedure

The first step in a quantization procedure, having in hand the quantum cartesian basis, is to identify an appropriate Hilbert space (quantum space) on which the spectrum could be easily reached. The Fock basis is a natural choice for harmonic systems. Here, this basis is extended to his helicity sector exploiting the advantage to be in the plane. Moreover, for technical reasons, the coherent state helicity basis associated to that of Fock is used.

Quantum cartesian basis

The quantum cartesian basis is obtained through the Heisenberg algebra spanned by the the following relations

$$(\hat{q}_i^a)^\dagger = \hat{q}_i^a, \quad (\hat{p}_i^a)^\dagger = \hat{p}_i^a, \quad \left[\hat{q}_i^a, \hat{p}_j^b\right] = i\hbar\delta^{ab}\delta_{ij}. \tag{10}$$

Quantum composite operators

The quantum composite operators associated to the classical phase space variables are given by

$$\widehat{H} = \widehat{H}_0 + \lambda(t)\widehat{\phi}, \tag{11}$$

where

$$\widehat{H}_0 = \frac{1}{2}g^2\widehat{p}_i^a\widehat{p}_i^a + \frac{1}{2}\omega^2\widehat{q}_i^a\widehat{q}_i^a, \tag{12}$$

$$\widehat{\phi} = \epsilon^{ab}\widehat{p}_i^a\widehat{q}_i^b. \tag{13}$$

Gauge invariance

The gauge invariance of the system is ensured since

$$[\widehat{H}_0, \widehat{\phi}] = 0. \tag{14}$$

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Helicity basis

The annihilation and creation operators in the helicity basis write as follows

$$\alpha_i^{\pm} = \frac{1}{\sqrt{2}} \left[\alpha_i^1 \mp i \alpha_i^2 \right], \quad \alpha_i^{\pm\dagger} = \frac{1}{\sqrt{2}} \left[\alpha_i^{1\dagger} \pm i \alpha_i^{2\dagger} \right], \tag{15}$$

$$\alpha_i^a = \sqrt{\frac{\omega}{2\hbar g}} \left[\widehat{q}_i^a + i \frac{g}{\omega} \widehat{p}_i^a \right]. \tag{16}$$

Hamiltonian and the gauge generator

The hamiltonian and the gauge generator are given in term of the helicity basis by

$$\widehat{H}_0 = \hbar g\omega \left[{\alpha_i^+}^\dagger \alpha_i^+ + {\alpha_i^-}^\dagger \alpha_i^- + d \right] = \hbar g\omega \left[\widehat{N} + d \right], \qquad (17)$$

$$\widehat{\phi} = -\hbar \left[{\alpha_i^+}^\dagger \alpha_i^+ - {\alpha_i^-}^\dagger \alpha_i^- \right]. \qquad (18)$$

Helicity orthonormalized basis

The Fock helicity orthonormalized basis is thus spanned by the following kets

$$|n_i^{\pm}>=\prod_i \frac{1}{\sqrt{n_i^{+}!n_i^{-}!}}(\alpha_i^{+\dagger})^{n_i^{+}}(\alpha_i^{-\dagger})^{n_i^{-}}|0>. \qquad (19)$$

Spectrum

The hamiltonian as well as the unique first class constraint are diagonalized, (suitably) as follows

$$\widehat{H}_0|n_i^{\pm}>= \hbar g\omega \left[\sum_i (n_i^+ + n_i^-) + d\right] |n_i^{\pm}>, \tag{20}$$

$$\widehat{\phi}|n_i^{\pm}>= -\hbar \sum_i (n_i^+ - n_i^-)|n_i^{\pm}> . \tag{21}$$

Matching condition

The states annihilated by the first class constraint $\widehat{\phi}$

$$\sum_i n_i^+ = n = \sum_i n_i^- , \tag{22}$$

whereas the energy levels of these states are given by

$$E_n = \hbar g\omega(2n + d), \quad n = 0, 1, 2, \ldots . \tag{23}$$

Coherent states basis

The coherent states basis allows to take better advantatge of the facilities offered by this operator. The helicity complexe variables to be used for the construction of the helicity coherent states are given by

$$z_i^{\pm} = \frac{1}{\sqrt{2}}\left[z_i^1 \mp i z_i^2\right], \quad \overline{z}_i^{\pm\dagger} = \frac{1}{\sqrt{2}}\left[\overline{z}_i^{1\dagger} \pm i \overline{z}_i^{2\dagger}\right], \tag{24}$$

$$z_i^a = \sqrt{\frac{\omega}{2\hbar g}}\left[q_i^a + i\frac{g}{\omega}p_i^a\right]. \tag{25}$$

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Physical projector

The physical projector is an operator which, being applied onto any quantum space quantity, constructs a physical (gauge invariant) one by averaging over the manifold of the gauge symmetry group, all finite gauge transformations generated by the first-class constraint of a system.

Physical projection operator

The gauge group is simply $SO(2)$ for which the manifold is the unit circle parmetrised by the rotation angle $0 < \theta < 2\pi$, the physical projection operator is represented as follows

$$\boldsymbol{E} = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-(i/\hbar)\theta\widehat{\phi}}, \tag{26}$$

with the fundamental properties

$$\boldsymbol{E}^2 = \boldsymbol{E}, \quad \boldsymbol{E}^\dagger = \boldsymbol{E}. \tag{27}$$

Physical time propagator

The physical time propagator of the system then writes

$$U_{phys}(t_2, t_1) = U(t_2, t_1)\boldsymbol{E} = \boldsymbol{E}U(t_2, t_1)\boldsymbol{E}, \tag{28}$$

$$U(t_2, t_1) = e^{-(i/\hbar)\int_{t_1}^{t_2} dt[\widehat{H}_0 + \lambda(t)\widehat{\phi}]}. \tag{29}$$

Physical time propagator

By integrating over the rotation angle θ and after some computations, one gets

$$U_{phys}(t_2, t_1) = x^d x^{\widehat{N}} \boldsymbol{E}, \tag{30}$$

where

$$x = e^{-\frac{i}{\hbar}(t_2 - t_1)\hbar g\omega}. \tag{31}$$

Physical states

Hence denoting these physical states of energy $E_n = \hbar g\omega(2n + d)$ by $|E_n, \mu_n >$ with the degeneracy index μ_n, gives

$$\mathbf{E} = \sum_{E_n,\mu_n} |E_n, \mu_n >< E_n, \mu_n|, \quad < E_n, \mu_n | E_m, \mu_m >= \delta_{n,m}\delta_{\mu_n,\mu_m}. \tag{32}$$

Physical propagator

We have the following expression for the physical propagator

$$\begin{aligned} U_{phys}(t_2,t_1) &= \sum_{E_n,\mu_n} e^{-i/\hbar(t_2-t_1)E_n}|E_n,\mu_n><E_n,\mu_n| \\ &= e^{-i(t_2-t_1)g\omega d}\sum_{E_n,\mu_n} e^{-i(t_2-t_1)g\omega(2n)}\times \\ &\quad \times|E_n,\mu_n><E_n,\mu_n|. \end{aligned} \tag{33}$$

Consequently, the time dependence of $x^d x^{\widehat{N}} \boldsymbol{E}$ determines the energy levels, while the matrix elements of this operator give the associated wave function.

Trace of the operator $x^{\widehat{N}}\boldsymbol{E}$

In comparing equations (30) and (33), we obtain

$$\sum_{E_n,\mu_n} x^{(2n+d)}|E_n,\mu_n><E_n,\mu_n| = \boldsymbol{E}x^d x^{\widehat{N}}\boldsymbol{E} = x^d x^{\widehat{N}}\boldsymbol{E}. \quad (34)$$

This shows that the trace of this operator is nothing but the partition function of the spectrum :

$$Tr x^{\widehat{N}}\boldsymbol{E} = \sum_{n=0}^{\infty} d_n x^{2n}. \quad (35)$$

where the coeficients d_n, $n \in \mathbb{N}$ specify the degeneracies of energy levels E_n of physical states,

$$E_n = \hbar g\omega(2n+d). \quad (36)$$

Physical states

The representations of the physical states, in the configuration space in terms of wave functions are generated by the diagonal matrix elements of the operator $x^{\widehat{N}}\boldsymbol{E}$:

$$< z_i^{\pm}|x^{\widehat{N}}\boldsymbol{E}|z_i^{\pm} >= \sum_{n,\mu_n} x^{2n}| < z_i^{\pm}|E_n, \mu_n > |^2. \tag{37}$$

Spectrum

Coming back to the spectrum, we have

$$Trx^{\widehat{N}}\boldsymbol{E} = \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{[1 - xe^{i\theta}]^d [1 - xe^{-i\theta}]^d} \ . \qquad (38)$$

The degeneracies appear immediately in comparing (35) and (38) :

$$d_n = \left[\frac{(d-1+n)!}{(d-1)!n!}\right]^2 , \ E_n = \hbar g\omega(2n+d) \ \ n = 0, 1, 2, \ldots . \qquad (39)$$

Generetors of $SO(d)$

In terms of quantum helicity degrees of freedom previously defined, the $d(d-1)/2$ generetors of $SO(d)$ are given by

$$\begin{aligned}\widehat{L}_{ij} &= i\hbar[\alpha_i^{a\dagger}\alpha_j^a - \alpha_j^{a\dagger}\alpha_i^a] \\ &= i\hbar[\alpha_i^{+\dagger}\alpha_j^+ + \alpha_i^{-\dagger}\alpha_j^- - \alpha_j^{+\dagger}\alpha_i^+ - \alpha_j^{-\dagger}\alpha_i^-], \end{aligned} \quad (40)$$

with the following algebra

$$[\widehat{L}_{ij}, \widehat{L}_{kl}] = -i\hbar[\delta_{ik}\widehat{L}_{jl} - \delta_{il}\widehat{L}_{jk} - \delta_{jk}\widehat{L}_{il} + \delta_{jl}\widehat{L}_{ik}]. \quad (41)$$

$d \times d$ rotation matrix

Denoting by (T_{ij}) the tensors which allows the matrix represention in the d-dimensional space of the generetors of the $SO(d)$ global symmetry, we can write $\widehat{L}_{ij}$ as follows :

$$\widehat{L}_{ij} = \alpha^{\dagger} \cdot (T_{ij}) \cdot \alpha, \quad (T_{ij})_{kl} = i\hbar[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}], \tag{42}$$

with the $d \times d$ rotation matrix in $SO(d)$ parametrised by the hyperangle ω_{ij} given by

$$R_{kl}(\omega_{ij}) = (e^{-(i/2\hbar)\omega_{ij}T_{ij}})_{kl}. \tag{43}$$

Helicity coherent states and the creation operators

These operators act onto the helicity coherent states and the creation operators as follows

$$e^{-(i/2\hbar)\omega_{ij}\widehat{L}_{ij}}|z_i^{\pm}>=|R_{ij}(\omega_{ij})z_j^{\pm}>, \tag{44}$$

$$e^{-(i/2\hbar)\omega_{ij}\widehat{L}_{ij}}\alpha_i^{\pm\dagger}e^{(i/2\hbar)\omega_{ij}\widehat{L}_{ij}}=\alpha_j^{\pm\dagger}R_{ji}(\omega_{ij}). \tag{45}$$

$SO(d)$-valued partition function

Having set the required elements, the evaluation of the partition function extended to $SO(d)$ becomes possible. We have

$$Trx^{\widehat{N}}e^{-(i/2\hbar)\omega_{ij}\widehat{L}_{ij}}\boldsymbol{E} = \int_0^{2\pi}\frac{d\theta}{2\pi}\frac{1}{det[\delta_{ij} - xe^{i\theta}R_{ij}(\omega_{ij})]det[1 - xe^{-i\theta}R_{ij}(\omega_{ij})]}\,. \quad (46)$$

$SO(d)$-valued partition function : case $d = 1$

Hence the expression (46) reduces to

$$Trx^{\widehat{N}}\boldsymbol{E} = \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{[1 - xe^{i\theta}][1 - xe^{-i\theta}]} = \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1 - x^2}. \quad (47)$$

Here there is no global symmetry since there is only one particle.

$SO(d)$-valued partition function : case $d = 2$

Hence the expression (46) reduces to

$$
\begin{aligned}
Tr \quad & x^{\widehat{N}} e^{-(i/\hbar)\omega_{12}\widehat{L}_{12}} \boldsymbol{E} = \\
= \quad & \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{[1 - xe^{i(\theta+\omega_{12})}][1 - xe^{i(\theta-\omega_{12})}]} \times \\
& \times \frac{1}{[1 - xe^{-i(\theta-\omega_{12})}][1 - xe^{-i(\theta+\omega_{12})}]} \\
= \quad & \sum_{n=0}^{\infty} x^{2n} \sum_{p=-n}^{+n} [(n+1) - |p|] e^{2ip\omega_{12}}. \qquad (48)
\end{aligned}
$$

$SO(d)$-valued partition function : case $d = 2$

We note that, for the $d = 2$, all the $d_n = (n+1)^2$ physical states sharing the same energy level E_n may be listed in the one dimensional representations of the global symmetry $SO(2) = U(1)$ indexed by the whole helicity p so that $-n \leq p \leq n$ with however a persistent degeneracy given by

$$d(n,p) = n + 1 - |p|, \tag{49}$$

for each of these helicity representations, i.e. for each p. Obviously we have the following verification

$$\sum_{p=-n}^{n} d(n,p) = (n+1)^2 = d_n, \quad n = 0, 1, \ldots \tag{50}$$

$U(rd) = U_N(1) \times SU(rd)$ dynamical symmetry

It appears clairly that quantized, the system admits a symmetry even more wider than the global symmetry $SO(d)$. This is the dynamical global unitary symmetry $U(rd) = U_N(1) \times SU(rd)$ of wich gauge invariant states we are going to identify in the system.

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Group $SU(2)$

It is well known that the group $SU(2)$ possesses three generators T_1, T_2 and T_3 in the cartesian basis. It is common to redefine the two firsts generators to obtain the helicity generators $T_\pm$ associated to the remaining, T_3,

$$[T_a, T_b] = i\epsilon_{abc}T_c, \quad T_\pm = T_1 \pm iT_2, \quad T_1 = \frac{1}{2}\left[T_+ + T_-\right],$$

$$T_2 = \frac{1}{2i}\left[T_+ - T_-\right], [T_+, T_-] = 2T_3, \quad [T_3, T_\pm] = \pm T_\pm, \tag{51}$$

$$\vec{T}^2 = \frac{1}{2}(T_+T_- + T_-T_+) + T_3^2\,,$$

where $\vec{T}^2 = T_1^2 + T_2^2 + T_3^2$ is the Casimir operator.

Spin representation

$$\begin{aligned}
&\vec{T}^2 :\ t(t+1),\quad t \in \mathbb{N}, \mathbb{N}+\tfrac{1}{2}\,,\quad T_3|m\rangle = m|m\rangle,\\
&\langle m|m\rangle = \delta_{mm'}\,,\quad m = -t, -t+1, \cdots, t-1, t\,,\\
&T_\pm|m\rangle = \sqrt{t(t+1) - m(m\pm 1)}\,|m\pm 1\rangle,\\
&\vec{T}^2|m\rangle = t(t+1)|m\rangle\,.
\end{aligned} \tag{52}$$

This clearly means that starting from the highest weight state $m = t$ and by application of T_-, one falls imediately onto the previous state in the weight diagram and so on. The same considerations is absolutely possible starting from the states of lowest weight by successive applications of the operator T_+. These facts are fundamental, since it is henceforth possible to identify all the representatives of this symmetry.

One harmonic oscillator

For an harmonic oscillator corresponding to the case $d = 1$, we know that at the excitation level n, the quantum numbers t and m caracterising $SU(2)$ are given in the helicity basis by

$$|n_+, n_- >= \frac{1}{\sqrt{n_+! n_-!}} (\alpha_+^\dagger)^{n_+} (\alpha_-^\dagger)^{n_-} |0 >, \tag{53}$$

where

$$n = n_+ + n_-, \quad m = \frac{1}{2}(n_+ - n_-), \quad t = \frac{1}{2}(n_+ + n_-). \tag{54}$$

Case $d = 2$ of our model

In this case, the following choices may be done to facilitate the identification of the physical states. The inclusions of the gauge group $SO(2)$ and that of the global symmetry group $SO(d = 2)$ into $SU(r = 2)$ and $SU(d = 2)$ respectively are chosen such that

- $\widehat{\phi}$ coincides with the generator T_3 of the Cartan subagebra of $SU(r = 2)$,
- $\widehat{L}_{12}$ coincides with the generator T_3 of the Cartan subagebra of $SU(d = 2)$.

Case $d = 2$ of our model

Consequently, the physical states are such that the eigenvalues of T_3 for $SU(r = 2)$ vanish and that of T_3 for $SU(d = 2)$ corresponds to the helicity quantum number p of $SO(d = 2)$. Finally, in addition to the excitation quantum number, the physical states are charcterized by the quantum numbers of $SU(d = 2)$ in other words the value of the spin t and that of T_3 in $SU(d = 2)$ which represented here by m.

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Gauge invariant states

Let keep ourselves to the concrete determination of the gauge invariant states within the representation of the global dynamical symmetry $SU(2) = SU(d = 2)$. Let us note these states

$$|n, t, p = m > . \tag{55}$$

Generators of the symmetry $SU(2)$

The generators of this symmetry $SU(2)$ are given in the appropriated basis by

$$\begin{aligned} T_1 &= \tfrac{1}{2}[\alpha_+^{+\dagger}\alpha_-^+ + \alpha_-^{+\dagger}\alpha_+^+] + \tfrac{1}{2}[\alpha_+^{-\dagger}\alpha_-^- + \alpha_-^{-\dagger}\alpha_+^-], \\ T_2 &= -\tfrac{1}{2}i[\alpha_+^{+\dagger}\alpha_-^+ - \alpha_-^{+\dagger}\alpha_+^+] - \tfrac{1}{2}i[\alpha_+^{-\dagger}\alpha_-^- - \alpha_-^{-\dagger}\alpha_+^-] \\ T_3 &= \tfrac{1}{2}[\alpha_+^{+\dagger}\alpha_+^+ - \alpha_-^{+\dagger}\alpha_-^+] + \tfrac{1}{2}[\alpha_+^{-\dagger}\alpha_+^- - \alpha_-^{-\dagger}\alpha_-^-], \\ T_\pm &= T_1 \pm iT_2 = \alpha_\pm^{+\dagger}\alpha_\mp^+ + \alpha_\pm^{-\dagger}\alpha_-^\mp, \end{aligned} \tag{56}$$

while the excitation levels operator also called number operator is given by

$$\widehat{N} = \alpha_+^{+\dagger}\alpha_+^+ + \alpha_-^{+\dagger}\alpha_-^+ + \alpha_+^{-\dagger}\alpha_+^- + \alpha_-^{-\dagger}\alpha_-^-. \tag{57}$$

Physical states

The physical states may be represented as follows

$$\left[\frac{1}{(n_{++}+n_{+-})!(n_{-+}+n_{--})!(n_{++}+n_{-+})!(n_{+-}+n_{--})!}\right]^{1/2}\times$$
$$\times\left(\alpha_+^{+\dagger}\alpha_+^{-\dagger}\right)^{n_{++}}\left(\alpha_+^{+\dagger}\alpha_-^{-\dagger}\right)^{n_{+-}}\left(\alpha_-^{+\dagger}\alpha_+^{-\dagger}\right)^{n_{-+}}\left(\alpha_-^{+\dagger}\alpha_-^{-\dagger}\right)^{n_{--}}|0>, (58)$$

Physical states

so that the quantum number associated to the operator $\widehat{N}$ is given by

$$\begin{aligned} N &= (n_{++} + n_{+-}) + (n_{-+} + n_{--}) + \\ &\quad + (n_{++} + n_{-+}) + (n_{+-} + n_{--}) \\ &= 2n, \end{aligned} \tag{59}$$

and that associated to T_3 writes

$$\begin{aligned} m = p &= \frac{1}{2}\left[(n_{++} + n_{+-}) - (n_{-+} + n_{--}) + (n_{++} + n_{-+}) - \right. \\ &\quad \left. -(n_{+-} + n_{--})\right] = (n_{++} - n_{--}). \end{aligned} \tag{60}$$

Physical states

Using the following usefull formula, we can identify explicitly the physical states

$$T_-|t, p> = [(t - p + 1)(t + p)]^{1/2} |t, p - 1 > . \qquad (61)$$

The fundamental level $n = 0 = N$

The highest weight state which stands at the same time of the singlet of the representation in this case is given by

$$|0, 0, 0>, \tag{62}$$

such that

$$T_3|0, 0, 0>= 0, \quad T_+|0, 0, 0>= 0 = T_-|0, 0, 0> . \tag{63}$$

The level $n = 1$, $N = 2$

- Maximal weight state $t = p = n = 1$

$$|1, 1, 1 >= {\alpha^{+}_{+}}^{\dagger} {\alpha^{-}_{+}}^{\dagger} |0 > . \qquad (64)$$

This state is normalized such that $T_+|1, 1, 1 >= 0$, as it should.

The level $n = 1$, $N = 2$

- The state before $|1, 1, 1>$ is $|1, 1, 0>$ such that $T_-|1, 1, 1>= \sqrt{2}|1, 1, 0>$. We have

$$|1,1,0>= \frac{1}{\sqrt{2}}\left[\alpha_-^{+\dagger}\alpha_+^{-\dagger} + \alpha_-^{-\dagger}\alpha_+^{+\dagger}\right]|0,0,0> . \tag{65}$$

- The previous state is $|1, 1, -1>$ such that $T_-|1, 1, 0>= \sqrt{2}|1, 1, -1>$. Consequently, we have

$$|1,1,-1>= \alpha_-^{+\dagger}\alpha_-^{-\dagger}|0,0,0> . \tag{66}$$

This state is the last of the subgroup of states characterized by the spin $t = 1$, since we have

$$T_-|1,1,-1>= 0. \tag{67}$$

The level $n = 1$, $N = 2$

- Following state of highest weight : $t = p = 0$
  It is given by

$$|1, 0, 0 >= \frac{1}{\sqrt{2}} \left[\alpha_-^{+\dagger} \alpha_+^{-\dagger} - \alpha_-^{-\dagger} \alpha_+^{+\dagger} \right] |0, 0, 0 > . \qquad (68)$$

In conclusion at the level $n = 1$ the set of $4 = (1 + 1)^2$ states sharing the energy level E_n presents as follows

$$\{ |1, 1, 1 > |1, 1, 0 >, |1, 1, -1 >, |1, 0, 0 > \} . \qquad (69)$$

The level $n = 2$, $N = 4$

The construction of the corresponding states follows strictly the same principle as above.

- State of highest weight $t = p = 2$

$$|2, 2, 2 >= \frac{1}{2}\left(\alpha_+^{+\dagger}\alpha_+^{-\dagger}\right)^2 |0, 0, 0 > . \qquad (70)$$

The level $n = 2$, $N = 4$

♣ Previous state to $|2, 2, 2>$: $\quad |2, 2, 1>= \frac{1}{2} T_{-} |2, 2, 2>$

$$|2, 2, 1>= \frac{1}{2} \left(\alpha_{-}^{+\dagger} \alpha_{+}^{-\dagger} + \alpha_{+}^{+\dagger} \alpha_{-}^{-\dagger} \right) \alpha_{+}^{+\dagger} \alpha_{+}^{-\dagger} |0, 0, 0> . \quad (71)$$

♣ State before $|2, 2, 1>$: $\quad |2, 2, 0>= \frac{1}{\sqrt{6}} T_{-} |2, 2, 1>$

$$\begin{aligned} |2, 2, 0> \;\; &= \;\; \frac{1}{2\sqrt{6}} \left((\alpha_{-}^{+\dagger})^2 (\alpha_{+}^{-\dagger})^2 + (\alpha_{+}^{+\dagger})^2 (\alpha_{-}^{-\dagger})^2 + \right. \\ & \left. + 4 \alpha_{+}^{+\dagger} \alpha_{+}^{-\dagger} \alpha_{-}^{+\dagger} \alpha_{-}^{-\dagger} \right) |0, 0, 0> . \end{aligned} \quad (72)$$

The level $n = 2$, $N = 4$

- Previous state to $|2,2,0>$

$$
\begin{aligned}
|2,2,-1> &= \frac{1}{\sqrt{6}} T_-|2,2,0> \\
&= \frac{1}{2}\left(\alpha_-^{+\dagger}\alpha_+^{+\dagger}(\alpha_-^{-\dagger})^2 + \alpha_+^{-\dagger}\alpha_-^{-\dagger}(\alpha_-^{+\dagger})^2\right)\times \\
&\quad \times|0,0,0> . \qquad (73)
\end{aligned}
$$

- Previous state : $|2,2,-2> = \frac{1}{2}T_-|2,2,-1>$

$$
|2,2,-2> = \frac{1}{2}\left(\alpha_-^{+\dagger}\alpha_-^{-\dagger}\right)^2|0,0,0> . \qquad (74)
$$

The level $n = 2$, $N = 4$

- Following state of highest weight : $t = p = 1$

$$|2, 1, 1 >= \frac{1}{2} \left(\alpha_-^{+\dagger} \alpha_+^{-\dagger} - \alpha_+^{+\dagger} \alpha_-^{-\dagger} \right) \alpha_+^{+\dagger} \alpha_+^{-\dagger} |0, 0, 0 > . \quad (75)$$

The level $n = 2$, $N = 4$

Previous state to $|2,1,1>$: $\quad |2,1,0> = \frac{1}{\sqrt{2}} T_- |2,1,1>$

$$|2,1,0> = \frac{1}{2\sqrt{2}} \left((\alpha_-^{+\dagger})^2 (\alpha_+^{-\dagger})^2 - (\alpha_+^{+\dagger})^2 (\alpha_-^{-\dagger})^2 \right) |0,0,0> . \tag{76}$$

- Following state of highest weight : $t = p = 0$

$$|2,0,0> \;=\; \frac{1}{2\sqrt{3}}\Big((\alpha_-^{+\dagger})^2(\alpha_+^{-\dagger})^2 + (\alpha_+^{+\dagger})^2(\alpha_-^{-\dagger})^2 - 2\alpha_+^{+\dagger}\alpha_-^{+\dagger}\alpha_+^{-\dagger}\alpha_-^{-\dagger}\Big)\,|0,0,0> . \qquad (77)$$

One can easily check that $T_-|2,0,0> = 0$, showing that there is no state beyond.

Conclusion

In conclusion, the set of 9 physical states at the excitation level $n = 2$ is given by

$$\{|2,2,2>, |2,2,1>, |2,2,0>, |2,2,-1>, |2,2,-2>,$$

$$|2,1,1>, |2,1,0>, |2,1,-1>, \ |2,0,0>\}\,. \tag{78}$$

THANKS ! ! !