Lecture II: Hamiltonian formulation of general relativity

(Courses in canonical gravity)

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1 Space-time foliation

The Hamiltonian formulation of ordinary mechanics is given in terms of a set of canonical variables q and p at a given instant of time t. In field theory, however, one is dealing with fields rather than a mechanical system then the canonical variables $\phi(x)$ are functions of position, and their canonical momenta are $\pi_\phi(x)$, both given at an instant of time. General relativity treats space and time on the same footing, that is not what is done in Hamiltonian formulations. Therefore, in order to discuss general relativity in a Hamiltonian fashion, one needs to break that equal footing. This requires a space-time splitting, since only time derivatives are transformed to momenta but not space derivatives.

We assume a foliation of space-time in terms of space-like three dimensional surfaces S of space-time manifold M. Thus, we consider the case of a Lorentzian manifold M diffeomorphic to $\mathbb{R} \times S$, where the manifold S represents 'space', and $t \in \mathbb{R}$ represents 'time'. It should be noted that, the particular slicing of space-time into 'instants of time' is an arbitrary choice, rather than something intrinsic to the world. In other words, there are lots of way to pick a diffeomorphism

$$\phi : M \longrightarrow \mathbb{R} \times S.$$

These give different ways to define a time coordinate τ on the space-time manifold M, namely the pull-back by ϕ of the standard time coordinate t on $\mathbb{R} \times S$:

$$\tau = \phi^* t.$$

For simplicity, we assume a submanifold $\Sigma \subset M$ is a slice of M if it equals $\{\tau = const\}$ for some time coordinate τ.

2 Geometry of hypersurfaces

Let us consider a surface $\Sigma \approx \Sigma_{t_0} : t_0 = const$ in a foliated space-time manifold $\mathbb{R} \times \Sigma$. This can be considered as a constraint surface characterized by $C_{t_0} = t - t_0 = 0$. Notice that, the geometry of the constraint surface Σ_{t_0} in a space-time is governed by a Riemannian geometry with the metric g_{ab} (with the inverse g^{ab}) rather than a Poisson or symplectic one on a phase space.

In analogy to the Poisson geometry, let us associate to the constraint C_{t_0} a (Hamiltonian-like) vector field as $g^\sharp dC_{t_0}$ on the Riemannian geometry (which is given by a metric tensor g^{ab} rather than a Poisson tensor $\mathcal{P}^{ij}$):

$$g^\sharp dC_{t_0} \;=\; g^{ab}\partial_b C_{t_0} \;=\; g^{ab}\partial_b t \;=\; g^\sharp dt \; . \tag{2.1}$$

This shows that, in the herein Riemannian geometry, the vector field $X^a = g^{ab}\partial_b t$ is normal[1] to the constraint surface Σ_{t_0}; this is opposite to the case happens in the Poisson geometry where the Hamiltonian vector field of a single, necessarily first class constraint must be tangent to the constraint surface. This is, indeed, because of the antisymmetric feature of the Poisson tensor that makes the Hamiltonian vector field of a single constraint C tangent to the constraint surface: $X_C C = \mathcal{P}^{ij}\partial_i C \partial_j C = 0$.

Using the definition of the normal vector X^a we can determine the normalized (time-like) normal vector to the surface as

$$n^a \;=\; \frac{X^a}{\sqrt{-g_{bc}X^b X^c}} \; , \tag{2.2}$$

such that $g_{ab}n^a n^b = -1$. Furthermore, for any vector field s^a tangent to Σ_{t_0} we have $g_{ab}s^a n^b = 0$.

The tangent space in TM at each point of Σ_t can be decomposed to a '*spatial* tangent space' spanned by vectors tangent to Σ_t, and a 'normal space' spanned by the unique unit future-pointing vector field n^a normal to Σ_t. For example, given a vector field $Z^a \in T_pM$ at any point $p \in \Sigma_t$, we can decompose it into a component tangent to Σ_t and a normal component proportional to n^b:

$$Z^a \;=\; \underbrace{-g_{ab}Z^a n^b}_{\perp} + \underbrace{(Z^a + g_{ab}Z^a n^b)}_{\parallel} \; . \tag{2.3}$$

Each spatial slice Σ_t is equipped with its own Riemannian structure. The *induced metric* h_{ab} on Σ_t can be uniquely determined by using the two conditions that

$$h_{ab}n^a \;=\; 0 \; , \qquad \text{and} \qquad h_{ab}s^a \;=\; g_{ab}s^a \; , \tag{2.4}$$

[1] Given any surface $\tilde{C}(x^b) = 0$, the normal vector to $\tilde{C}$ is determined as $d\tilde{C} = (\partial\tilde{C}/\partial x^a)\partial_a =: Y^a\partial_a$. Thus, $Y^a = g^{ab}\partial_b\tilde{C}$ is the components of the normal vector.

for any vector s^a tangent to Σ_t. So that, the induced metric h_{ab} reads

$$\boxed{h_{ab} \;=\; g_{ab} + n_a n_b} \tag{2.5}$$

Interestingly, in comparison to the Poisson geometry, the induced metric h_{ab} is analogous to the Dirac bracket, which subtracts from off the Poisson structure any contribution from the flow of the constraints transversal to the constraint surface. The inverse of the induced metric h_{ab} can be defined as $h^{ab} = g^{ab} + n^a n^b$.

In order to study the dynamics of the canonical formulation, we consider an interpretation of the induced metric h_{ab} as a time-dependent 3-dimensional tensor field on the family of manifolds Σ_t. Thus, the time-dependent fields h_{ab} will play a crucial role as the configuration variables of canonical gravity. In this way, it makes sense to define time derivatives of the induced metric or any other fields. Let us introduce a *time-evolution vector field* $t = t^a \nabla_a$ to define the direction of time derivatives, such that t^a is normalized: $t^a \nabla_a t = 1$. By introducing the *shift vector* $N^a := h^{ab} t_b$, and the *laps function* $N := -n_b t^b$, the time-evolution vector field t^a can be decomposed to the spatial and normal parts as

$$t^a \;=\; N n^a \;+\; N^a \; . \tag{2.6}$$

Using this relation, we can write the inverse space-time metric as

$$g^{ab} \;=\; h^{ab} - n^a n^b \;=\; h^{ab} - \frac{1}{N^2}(t^a - N^a)(t^b - N^b). \tag{2.7}$$

By inverting this matrix and writing it in coordinate basis we obtain the line element

$$\boxed{g_{ab} dx^a dx^b \;=\; -N^2 dt^2 + h_{ab}\left(dx^a + N^a dt\right)\left(dx^b + N^b dt\right)} \tag{2.8}$$

in coordinates x^a such that $t^a \nabla_a = \partial / \partial t$. This shows that, the space-time geometry is described not by a single metric but by the spatial geometry of slices, encoded in h_{ab}, together with deformations of neighboring slices with respect to each other as described by N and N^a.

Given a time-evolution vector field, we complete the interpretation of tensor fields on a foliated space-time as time-dependent tensor fields on space. Given two spatial slices in the foliation, in order to speak about time-dependence of the tensor fields, we need to show how tensor fields on these slices change. To do that, we are first required to uniquely associate a point on one slice with a point on the other slice, then, by evaluating the fields at the associated points we can show their changes when going from one slice to the next. A *time derivative* of a tensor field is defined as the Lie derivative along the time-evolution vector field t^a:

$$\boxed{\dot{T}^{a_1 \cdots a_n}{}_{b_1 \cdots b_m} \;:=\; \left(h^{a_1}_{c_1} \cdots h^{a_n}_{c_n} h^{b_1}_{d_1} \cdots h^{b_m}_{d_m}\right) \mathcal{L}_t T^{c_1 \cdots c_n}{}_{d_1 \cdots d_m}} \tag{2.9}$$

In the case of a 4-dimensional space-time Einstein's equation is really 10 different equations, since there are 10 independent components in the Einstein tensor. We will rewrite these equations in terms of the metric on the slice Σ, or 3-metric h_{ab}, and the 'extrinsic curvature' K_{ab} of the slice Σ, which describes the curvature of the way it sits in M. In what follows we shall see that the extrinsic curvature can also be thought of as representing the time derivative of the 3-metric. We can think of (h_{ab}, K_{ab}) as Cauchy data for the metric, just as we think of the vector potential on space and the electric field as Cauchy data for electromagnetism or the Yang-Mills field. We will see that of Einstein's 10 equations, 4 are constraint equations that the Cauchy data must satisfy, while 6 are evolutionary equations saying how the 3-metric changes with time. This is called the Arnowitt-Deser-Misner, or ADM, formulation of Einstein's equation.

2.1 Intrinsic and extrinsic geometry

The spatial metric h_{ab} itself is an intrinsic quantity, and as a metric it allows one to define a unique covariant derivative operator D_a on Σ such that $D_a h_{bc} = 0$. This *covariant derivative* can be written in terms of the space-time covariant derivative ∇_a as

$$D_c T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} := (h^{a_1}{}_{d_1} \cdots h^{a_k}{}_{d_k} h_{b_1}{}^{e_1} \cdots h_{b_l}{}^{e_l}) h_c{}^f \nabla_f T^{c_1 \cdots c_n}{}_{d_1 \cdots d_m}. \quad (2.10)$$

Definition. Given the three dimensional covariant derivative D_a, we can define the *intrinsic-curvature tensors* as with any covariant derivative:

$$^{(3)}R_{abc}{}^d\, \omega_d \;=\; D_a D_b \omega_c - D_b D_a \omega_c \quad (2.11)$$

for all spatial 1-form ω_c, i.e., $\omega_a n^a = 0$. From this intrinsic Riemannian curvature, we can obtain the intrinsic Ricci tensor $^{(3)}R_{ab}$ and scalar $^{(3)}R$ by the usual contractions.

In contrast to the intrinsic geometry, which applies to a single (Σ, h_{ab}) no matter how it is embedded in a space-time manifold, the *extrinsic geometry* of Σ in $\mathbb{R} \times \Sigma$ refers to the bending of Σ in its neighborhood, which in general implies a changing normal vector field n^a along Σ. This notion is captured in the definition of the extrinsic-curvature tensor:

Definition. Given any normal vector n^a to the surface Σ, the *extrinsic-curvature tensor* is a spatial tensor on Σ by definition of D_a:

$$K_{ab} \;:=\; D_a n_b \;=\; h_a^c\, h_b^d \nabla_c n_d \;. \tag{2.12}$$

Thus, K_{ab} measures how much the surface Σ is curved in the way it sits in M, because it says how much a vector tangent to Σ will fail to be tangent if we parallel translate it a bit using the Levi-Civita connection ∇ on M.

The extrinsic-curvature tensor has some properties as follows:

(i) From the definition (2.12) we can write

$$\begin{aligned} K_{ab} \;&=\; h_b{}^d\, h_a{}^c \nabla_c n_d \;=\; (g_b{}^d + n_b n^d) h_a{}^c \nabla_c n_d \\ &=\; h_a{}^c \nabla_c n_b + n_b n^d h_a{}^c \nabla_c n_d \;. \end{aligned} \tag{2.13}$$

It can be seen that, in contrast to commuting h_{ab} with D_c, the space-time metric and its covariant derivative is always *commutative*.

(ii) The extrinsic curvature tensor is *symmetric*:

$$\boxed{K_{ab} \;=\; K_{ba}} \tag{2.14}$$

This also concludes that all spatial projections of $\nabla_a n_b$ are symmetric: $\nabla_a n_b - \nabla_b n_a = 0$.

(iii) From the symmetric property of K_{ab} we have that $K_{ab} = \frac{1}{2}(K_{ab}+K_{ba})$; using this together with (2.13) we can write

$$\begin{aligned} 2K_{ab} \;&=\; (g_a{}^c + n_a n^c) \nabla_c n_b + (g_b{}^c + n_b n^c) \nabla_c n_a \\ &=\; n^c \nabla_c (n_a n_b) + \nabla_a n_b + \nabla_b n_a \\ &=\; n^c \nabla_c h_{ab} + h_{cb} \nabla_a n^c + h_{ac} \nabla_b n^c \;=:\; \mathcal{L}_n h_{ab} \;. \end{aligned} \tag{2.15}$$

Thus, the extrinsic curvature is half of the Lie derivative of the intrinsic metric along the unit normal:

$$\boxed{K_{ab} \;=\; \frac{1}{2} \mathcal{L}_n h_{ab}} \tag{2.16}$$

(iv) From last identity in Eq. (2.15) we write that

$$\begin{aligned} K_{ab} \;&=\; \frac{1}{2} \left[n^c \nabla_c h_{ab} + h_{cb} \nabla_a n^c + h_{ac} \nabla_b n^c \right] \\ &=\; \frac{1}{2N} \left[N n^c \nabla_c h_{ab} + h_{cb} \nabla_a (N n^c) + h_{ac} \nabla_b (N n^c) \right] \\ &=\; \frac{1}{2N} h_a{}^c h_b{}^d \; \mathcal{L}_{t-N} h_{cd} \;=\; \frac{1}{2N} h_a{}^c h_b{}^d \; (\mathcal{L}_t h_{cd} - \mathcal{L}_N h_{cd}), \end{aligned} \tag{2.17}$$

where we substituted $Nn^a = t^a - N^a$ and smuggled in projections $h_a{}^c h_b{}^d$ since K_{ab} is spatial. Furthermore, $\mathcal{L}_N h_{cd} = D_a N_b + D_b N_a$. By using the definition (2.9) we have that $\dot{h}_{ab} = h_a{}^c h_b{}^d \; \mathcal{L}_t h_{cd}$, thus, Eq. (2.17) can be rewritten as

$$\boxed{K_{ab} \;=\; \frac{1}{2N}\left(\dot{h}_{ab} - D_a N_b - D_b N_a\right)} \tag{2.18}$$

Similar to splitting of the space-time metric (2.8) to spatial and temporal sections, intrinsic and extrinsic curvatures (2.11) and (2.12) can also describe together the space-time curvature.

One can show that the symmetries of the Riemann tensor $R^a{}_{bcd}$ reduces number of independent components from n^4, where n is the dimension of space-time, down to $n^2(n^2-1)/2$. For a 4 dimensional space-time, the number of space-time tensors is 20 and there are 6 components for the spatial Riemann tensor. As for the Ricci tensor R_{ab}, since it is symmetric one would expect that it has $n(n+1)/2$ independent components. In three dimension the Ricci tensor has 6 independent components, which is just as many as the Riemann tensor. In four dimension space-time it has 10 independent components. Consequently, using symmetry of the extrinsic curvature, it provides only 6 components more than the spatial Riemann tensor, which is 12 independent components. These components we introduced so far constitutes all curvature components necessary for a canonical decomposition.

2.2 The Gauss-Codazzi equations

We shall show that four of Einstein's equations are constraints that the the 3-metric h_{ab} and extrinsic curvature K_{ab} must satisfy (see next section). This is because that some components of the Riemann tensor depend only on the extrinsic and the intrinsic curvatures, that is, the curvature of h_{ab}. The formulas that describe the precise relations between curvature components are known as the Gauss-Codazzi equations, which we now derive.

We will compute the components $R_{efg}{}^h$ in terms of K_{ab} and ${}^{(3)}R_{abc}{}^d$. To do this we compute

$$\begin{aligned} D_a D_b \omega_c \;&=\; D_a(h_b{}^d h_c{}^e \nabla_d \omega_e) \;=\; h_a{}^f h_b{}^g h_c{}^h \nabla_f (h_g{}^d h_h{}^e \nabla_d \omega_e) \\ &=\; h_a{}^f h_b{}^d h_c{}^e \nabla_f \nabla_d \omega_e + h_c{}^e (h_a{}^f h_b{}^g \nabla_f h_g{}^d) \nabla_d \omega_e \\ &\quad + h_b{}^d (h_a{}^f h_c{}^h \nabla_f h_h{}^e) \nabla_d \omega_e \; . \end{aligned} \tag{2.19}$$

In the second term we have

$$h_a{}^f h_b{}^g \nabla_f h_g{}^d \;=\; h_a{}^f h_b{}^g \nabla_f (g_g{}^d + n_g n^d) = n^d h_b{}^g \nabla_a n_g = K_{ab} n^d \; ,$$

and in the last term

$$h_b{}^d (h_a{}^f h_c{}^h \nabla_f h_h{}^e) \nabla_d \omega_e \;=\; h_b{}^d K_{ac} n^e \nabla_d \omega_e = -K_{ac} h_b{}^d \omega_e \nabla_d n^e = -K_{ac} K_b^e \omega_e$$

where we have used $\omega_a n^a = 0$ for spatial ω_e. Thus, using definition (2.11) and the Eq. (2.19) we obtain

$$\begin{aligned} {}^{(3)}R_{abc}{}^{e}\omega_e &= D_a D_b \omega_c - D_b D_a \omega_c \\ &= h_a{}^{f} h_b{}^{d} h_c{}^{e} (\nabla_f \nabla_d \omega_e - \nabla_d \nabla_f \omega_e) \\ &\quad - K_{ac} K^e_b \omega_e + K_{bc} K^e_a \omega_e \ . \end{aligned} \tag{2.20}$$

This gives the so-called *Gauss equation*:

$$\boxed{h_a{}^{e} h_b{}^{f} h_c{}^{g} R_{efg}{}^{h} \ = \ {}^{(3)}R_{abc}{}^{d} + K_{ac} K_b{}^{d} - K_{bc} K_a{}^{d}} \tag{2.21}$$

By computing the relation

$$\begin{aligned} h^a{}_e h^b{}_f h^c{}_g R_{abcd} n^d &= h^a{}_e h^b{}_f h^c{}_g (\nabla_a \nabla_b - \nabla_b \nabla_a) n_c \\ &= h^a{}_e h^b{}_f h^c{}_g \left(\nabla_a (g_b{}^{d} \nabla_d n_c) - \nabla_b (g_a{}^{d} \nabla_d n_c) \right) \\ &= D_e K_{fg} - h^a{}_e h^b{}_f h^c{}_g \nabla_a (n_b n^d \nabla_d n_c) \\ &\quad - D_f K_{eg} + h^a{}_e h^b{}_f h^c{}_g \nabla_b (n_a n^d \nabla_d n_c) \\ &= D_e K_{fg} - D_f K_{eg} - h^a{}_e h^b{}_f h^c{}_g (n^d \nabla_d n_c)(\nabla_a n_b - \nabla_b n_a). \end{aligned}$$

Using the symmetry of the spatial projection of $\nabla_a n_b$, the last term in this equation vanishes. The result is the *Codazzi equation*:

$$\boxed{h^a{}_e h^b{}_f h^c{}_g R_{abcd} n^d \ = \ D_e K_{fg} - D_f K_{eg}} \tag{2.22}$$

Let us introduce the *Ricci equation* as

$$R_{acbd} n^c n^d \ = \ n^c (\nabla_a \nabla_c - \nabla_c \nabla_a) n_b \ . \tag{2.23}$$

This equation can be derived in terms of $\mathcal{L}_n$, the Lie derivative along the unit normal n^a, of the extrinsic curvature K_{ab}, and the normal acceleration $a_a := n^c \nabla_c n_a$ (satisfying $a_a n^a = 0$):

$$\boxed{R_{acbd} n^c n^d \ = \ -\mathcal{L}_n K_{ab} + K_{ac} K^c_b + D_{(a} a_{b)} + a_a a_b} \tag{2.24}$$

Using the Ricci equation (2.24), using the relation

$$R_{ab} n^a n^b \ = \ R_{acd}{}^{c} \ n^a n^b$$

we obtain the following equation

$$\boxed{R_{ab} n^a n^b \ = \ (K^a_a)^2 - K_a{}^{b} K_b{}^{a} + \nabla_a v^a} \tag{2.25}$$

where the vector field v^a is defined as $v^a := -n^a \nabla_c n^c + n^c \nabla_c n^a$.

From the Gauss-Codazzi equations together with the Ricci equation, the Ricci scalar R reads

$$\begin{aligned} R &= g^{ab}g^{cd}R_{abcd} = (h^{ab}-n^a n^b)(h^{cd}-n^c n^d)R_{abcd} \\ &= h^{ab}h^{cd}R_{abcd} - 2R_{ab}n^a n^b \; . \end{aligned} \tag{2.26}$$

Then, using the symmetry of the Riemann tensor, we find the Ricci scalar in terms of the extrinsic curvature

$$\boxed{R = {}^{(3)}R + K_{ab}K^{ab} - (K^a{}_a)^2 - 2\nabla_a v^a} \tag{2.27}$$

Up to a divergence $\nabla_a v^a$, we can thus decompose the Ricci scalar into a "kinetic" term quadratic in extrinsic curvature, and a potential term ${}^{(3)}R$ which depends only on the spatial metric and its spatial derivatives.

The extrinsic curvature, as shown by (2.18), plays the role of a "velocity" of the spatial metric and is thus a candidate for its momentum. In the next section, we discuss the Hamiltonian formalism of general relativity in terms of canonical variables.

3 The ADM formalism

The action of general relativity in metric variables is given by Einstein-Hilbert action

$$S_{\mathrm{EH}}[g] = \frac{1}{16\pi G}\int d^4x\sqrt{-\det g}R =: \int dt\; L_{\mathrm{grav}} \tag{3.1}$$

where $\det g$ is the determinant of the metric g_{ab}. Once the space-time is foliated, using (2.27) the gravitational Lagrangian in terms of the extrinsic curvature K_{ab} and the 3-curvature ${}^{(3)}R$ becomes

$$\boxed{L_{\mathrm{grav}} = \frac{1}{16\pi G}\int d^3x N\sqrt{-\det h}\left({}^{(3)}R + K_{ab}K^{ab} - (K_a{}^a)^2\right)} \tag{3.2}$$

up to boundary terms which do not affect local field equations. The determinant $\det g = -N^2 \det h$.

3.1 Constraints

Eq. (3.2) shows that the ten independent components of the space-time metric g_{ab} are replaced by the six components of the induced Riemannian metric h_{ab} on the slice Σ, plus the three components of the shift vector N_a and the lapse function N. The action of general relativity depends on $\dot{h}_{ab}$ via extrinsic curvature K_{ab}, thus, one can obtain the momentum p^{ab} conjugate to h_{ab}:

$$\begin{aligned} p^{ab}(x) &= \frac{\delta L_{\rm grav}}{\delta \dot{h}_{ab}(x)} = \frac{1}{2N}\frac{\delta L_{\rm grav}}{\delta K_{ab}} \\ &= \frac{\sqrt{\det h}}{16\pi G}\left(K^{ab} - K^c_c h^{ab}\right) . \end{aligned} \tag{3.3}$$

However, the action does not depend on time derivatives of the remaining space-time metric component N and N^a; therefore, momenta conjugate to N and N^a, are given, respectively, by

$$p_N(x) = \frac{\delta L_{\rm grav}}{\delta \dot{N}(x)} = 0 \qquad \text{and} \qquad p_a(x) = \frac{\delta L_{\rm grav}}{\delta \dot{N}^a(x)} = 0 , \tag{3.4}$$

presenting two constraints on the gravitational phase space. Since the relation (3.3) can be inverted for

$$\dot{h}_{ab} = \frac{16\pi G N}{\sqrt{\det h}}\left(2p_{ab} - p^c_c h_{ab}\right) + 2D_{(a}N_{b)} , \tag{3.5}$$

thus, relations in (3.4) present two *primary* constraints. Then one can work out the total Hamiltonian by the formula

$$H_{\rm grav} = \int d^3x \left(\dot{h}_{ab}p^{ab} + \lambda p_N + \mu^a p_a\right) - L_{\rm grav} , \tag{3.6}$$

which, by substituting $\dot{h}_{ab}$ from (3.5), leads to

$$\begin{aligned} H_{\rm grav} = \int d^3x \Bigg[& \frac{16\pi G N}{\sqrt{\det h}}\left(p_{ab}p^{ab} - \frac{1}{2}(p^a_a)^2\right) + 2p^{ab}D_a N_b \\ & - \frac{N\sqrt{\det h}}{16\pi G}\,{}^{(3)}R + \lambda p_N + \mu^a p_a \Bigg] . \end{aligned} \tag{3.7}$$

The primary constraints (3.4) imply *secondary* constraints:

$$0 = \dot{p}_N = \{p_N, H_{\rm grav}\} =: -C_{\rm grav}(h_{ab}, p^{ab}) , \tag{3.8}$$

$$0 = \dot{p}_{N^a} = \{p_{N^a}, H_{\rm grav}\} =: -C^{\rm grav}_a(h_{ab}, p^{ab}) . \tag{3.9}$$

By working out the Poisson bracket in these relations we obtain the so-called *Hamiltonian constraint*:

$$\boxed{C_{\rm grav} = \frac{16\pi G}{\sqrt{\det h}}\left(p_{ab}p^{ab} - \frac{1}{2}(p^a_a)^2\right) - \frac{\sqrt{\det h}}{16\pi G}\,{}^{(3)}R \approx 0} \tag{3.10}$$

and the *diffeomorphism constraint*:

$$\boxed{C^{\rm grav}_a = -2D_b p^b_a \approx 0} \tag{3.11}$$

We have integrated by parts, using $2\int d^3x\sqrt{\det h}D_a(p^{ab}N_b/\sqrt{\det h})$ as a boundary term for any vector field N^a, in derivation of $C_{\rm grav}$.

The lapse function N and the shift vector N^a now play the role of Lagrange multipliers of the secondary constraints. Then, with these notations, the total Hamiltonian, as a linear combination of the constraints, can be written as where we identify the Hamiltonian density as follows

$$\boxed{H_{\rm grav} \;=\; \int d^3x\,(NC_{\rm grav} + N^a C_a^{\rm grav} + \lambda p_N + \mu^a p_{N^a}) + H_{\partial\Sigma}} \tag{3.12}$$

where $H_{\partial\Sigma}$ is the Hamiltonian of the boundary term.

The fact that the Hamiltonian involves terms proportional to the lapse and shift should not be surprising, since the role of the Hamiltonian is to generate time evolution, and in general relativity we need to specify the lapse and shift to know the meaning of time evolution. However, if we express the quantities $C_{\rm grav}$ and $C_a^{\rm grav}$ in terms of the extrinsic curvature using the formula following Gauss-Codazzi equations, we find that

$$C_{\rm grav} \;=\; -2G_{ab}n^a n^b \qquad \text{and} \qquad C_a^{\rm grav} \;=\; -2G_{ai}n^a$$

This implies that the Hamiltonian density for general relativity must vanish by the vacuum Einstein equation: $H_{\rm grav} = 0$. In other words, the constraints $C_{\rm grav} = C_a^{\rm grav} \approx 0$ are precisely the 4 Einstein equations that are constraints on the initial data. Therefore, there is no proper Hamiltonian which would be non-trivial on the constraint surface. This is in agreement with the fact that there is no absolute time in general relativity, since a non-vanishing Hamiltonian would generate time evolution in an external time parameter. Instead, dynamics is determined by the constraints, such that evolution as a gauge flow can be parametrized arbitrarily. In this way, we see the reparameterization invariance of coordinates in a generally covariant thoery.

Since in the herein canonical formalism, the configuration space of general relativity is $Met(\Sigma)$, thus, it is natural to expect that the phase space Γ is the space of all pairs $(h_{ab}, N, N^a; p^{ab}, p_N, p_{N^a})$, or the cotangent bundle $T^*Met(\Sigma)$. However, not all points of this phase space represent allowed states. The Einstein equations that are constraints must be satisfied, and this restriction picks out a subspace of the phase space called the *physical phase space*:

$$\Gamma_{\rm phys} \;=\; \{C_{\rm grav} \approx 0;\; C_a^{\rm grav} \approx 0\} \;\subset\; T^*Met(\Sigma)\;. \tag{3.13}$$

The Hamiltonian (3.12) vanishes on this subspace. In section 3.3 we will see that these constraints are *first class* and thus generate gauge transformations which do not change the physical information in solutions. The Hamiltonian constraint does this for time, and the diffeomorphism constraint for spatial coordinates. Once these constraints are satisfied, we make sure that the formulation is space-time covariant even though we started the canonical formulation with slicing of space-time determined by any time function t.

3.2 Equations of motion

In the presence of the total Hamiltonian of general relativity, we can obtain the evolutionary part of Einstein's equations by this means. These are really just the equations $G_{ab} = 0$ in disguise, which are equations for the second time derivative of the 3-metric, but rewritten so as to give twice as many first-order equations. Then, Hamiltonian equations give $\dot{N}(x) = \lambda(x)$ and $\dot{N}^a(x) = \mu^a(x)$, which tells us that these functions can change arbitrarily due to reparameterizations. Moreover,

$$\dot{h}_{ab} \;=\; \{h_{ab}, H_{\text{grav}}\} \qquad \text{and} \qquad \dot{p}^{ab} \;=\; \{p^{ab}, H_{\text{grav}}\}\ .$$

The first relation just reproduces the equation (3.5) in terms of the momentum. Finally the last relation is a non-trivial evolution equation which can be computed in several steps. We write only the final equation here (for details of calculation see [2]):

$$\begin{aligned}
\dot{p}^{ab} \;&=\; \{p^{ab}, H_{\text{grav}}\} \;=\; -\frac{\delta H_{\text{grav}}}{\delta h_{ab}} \\
&=\; -\frac{N\sqrt{\det h}}{16\pi G}\left({}^{(3)}R^{ab} - \frac{1}{2}\,{}^{(3)}Rh^{ab}\right) + \frac{8\pi G N}{\sqrt{\det h}}h^{ab}\left(p^{cd}p_{cd} - \frac{1}{2}(p^c_c)^2\right) \\
&\quad - \frac{32\pi G N}{\sqrt{\det h}}\left(p^{ac}p_c{}^b - \frac{1}{2}p^{ab}p^c_c\right) + \frac{\sqrt{\det h}}{16\pi G}\left(D^aD^bN - h^{ab}D_cD^cN\right) \\
&\quad + \sqrt{\det h}D_c\left(p^{ab}N^c/\sqrt{\det h}\right) - 2p^{c(a}D_cN^{b)}\ .
\end{aligned} \tag{3.14}$$

The point is that, even on the physical phase space where $H_{\text{grav}} = 0$, the time evolution given by Hamilton's equations is nontrivial.

If matter sources are present, they, too, contribute to the action and thus to the canonical constraints. In particular, the matter Hamiltonian C_{matt} will be added to the Hamiltonian constraint C_{grav}, and energy flows of matter will be added to the diffeomorphism constraint C_a^{grav}. Then, the combined Hamiltonian constraint is $C = C_{\text{grav}} + C_{\text{matt}}$ and the total diffeomorphism constraint reads $C_a = C_a^{\text{grav}} + C_a^{\text{matt}}$. It is convenient to exhibit the structure of the system for constraints in smeared form, integrated with respect to the multipliers N and N^a: the smeared Hamiltonian and diffeomorphism constraints are given, respectively, by

$$H[N] \;:=\; \int d^3xN(x)C(x) \;=\; \int d^3xN\left(C_{\text{grav}} + C_{\text{matt}}\right), \tag{3.15}$$

$$D[N^a] \;:=\; \int d^3xN^a(x)C_a(x) \;=\; \int d^3xN^a\left(C_a^{\text{grav}} + C_a^{\text{matt}}\right). \tag{3.16}$$

Recall that the lapse and shift measure how much time evolution pushes the slice Σ in the normal direction and the tangent direction, respectively. In particular, if we set the shift equal to zero, the Hamiltonian for general relativity is equal to $H[N]$, and it generates time evolution in a manner that corresponds to pushing Σ forwards in the normal direction. On the other hand, if we set the lapse equal to zero, the Hamiltonian becomes $D[N^a]$, which generates a funny sort of 'time evolution' that pushes Σ in a direction tangent to itself. More precisely, this quantity generates transformations of (total) physical phase space $\Gamma = \Gamma_{\rm grav} \times \Gamma_{\rm matt}$ corresponding to the flow on Σ generated by N^a. This flow is a 1-parameter family of diffeomorphisms of Σ. For this reason, C_a or $D[N^a]$ is called the diffeomorphism constraint, while C or $H[N]$ is called the Hamiltonian constraint. It is actually no coincidence that C and C_a, play a dual role as both constraints and terms in the Hamiltonian. This is, in fact, a crucial special feature of field theories with no fixed background structures.

3.3 General constraint algebra

So far, we have seen that in general relativity, there are four *primary* constraints $p_N \approx 0$ and $p_{N^a} \approx 0$, whose time derivatives lead to more four *secondary* constraints $C = C_{\rm grav} + C_{\rm matt} \approx 0$ and $C_a = C_a^{\rm grav} + C_a^{\rm matt} \approx 0$. Since the eight constraints $p_\mu := (p_N, p_{N^a})$ and $C_\mu := (C, C_a)$ are independent, thus, they form eight *first-class* constraints for general relativity. Thus, these constraints form a first-class algebra in which there are *four* independent gauge transformations by changing space-time coordinates, exactly the number of secondary constraints on phase-space functions. (The primary constraints only generate change of N and N^a.) The total Hamiltonian (3.12) is a linear combination of (first class) constraints, i.e., it vanishes identically on solutions of equations of motion. This is a generic property of generally covariant systems.

In addition to the first-class nature, which tells us that Poisson brackets of the constraints vanish on the constraint surface, there is a specific "off-shell algebra" of constraints, satisfied by the constraint functions on the whole phase space including the part off the constraint surface: this shows what kinds of transformation the constraints generate, and how they are related to space-time properties. In the presence of the matter contribution it is interesting to see the Poisson brackets of the Hamiltonian and diffeomorphism constraints. The full constraint algebra is then given as

$$\{D[N^b], D[M^a]\} = D[\mathcal{L}_{N^b} M^a] \,, \tag{3.17}$$

$$\{D[N^a], H[N]\} = H[\mathcal{L}_{N^a} N] \,, \tag{3.18}$$

$$\{H[N], H[M]\} = -D[h^{ab}(N\partial_b M - M\partial_b N)]. \tag{3.19}$$

The relations above are know as *Dirac algebra*. In the first two lines, we simply have the expected action of infinitesimal spatial diffeomorphisms, with multipliers on the right-hand side given by Lie derivatives $\mathcal{L}_{N^b} M^a = [N, M]^a$ and $\mathcal{L}_{N^a} N = N^a \partial_a N$. The last line, includes the phase space function h^{ab}, which is the so-called structure function, rather than just phase-space independent structure constants.

References

[1] John C. Baez, Javier P. Muniain, *Gauge Fields, Knots and Gravity*, (Series on Knots and Everything, Vol 4, 1994).

[2] Martin Bojowald, Canonical gravity and applications (Cambridge University Press, 2011).