Modified Gravity and the dRGT Massive Gravity

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Outline

1. The cosmic acceleration and Modified Gravity
2. dRGT Massive Gravity
3. The Vainshtein mechanism in dRGT Massive Gravity
4. Conclusions and present status
1. The cosmic acceleration and Modified Gravity
Standard cosmology before 2000

- Homogeneity and isotropy
- General Relativity
- SM matter + CDM

Abundance of elements
CMB
Structure formation (given initial perturbations)

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) < 0
\]
The late time acceleration

- Adding a cosmological constant term, observational data on:
  - type IA supernovae
  - CMB anisotropies
  - large scale structure

are best fitted by:

- a universe whose energy density is dominated by the cosmological constant, and
- which has recently entered an accelerated phase

\[
\frac{\ddot{a}}{a} > 0
\]
Why is this a problem?

At a fundamental level, we don’t understand the $\Lambda g_{\mu\nu}$ term:

- $\Lambda$ may be a second characteristic scale of gravity . . .
  - then the universe is very fine tuned: coincidence problem

- . . . or the semiclassical manifestation of vacuum energy
  - observed and predicted values differ by $60 – 120$ orders of magnitude

- or there may be both a “bare” and a semiclassical $\Lambda$
  - extremely fine tuned cancellation needed $\frac{\Lambda - \Lambda_{\text{vac}}}{\Lambda} \sim 10^{-60}$
Alternative explanations

- Homogeneity and isotropy ➔ Lemaître-Tolman-Bondi, back-reaction
- General Relativity ➔ Modified Gravity (f(R), braneworlds, massive gravity, ...)
- SM matter + CDM ➔ Dark Energy (quintessence, k-essence, ...)

Possibly the solution to the problem is more fundamental...
The energy budget is dominated by a component which is undetected in lab and has exotic properties $\rho + 3\rho < 0 \ldots$

...or may the data simply suggest that GR is inadequate at cosmological scales?
Infrared modifications of gravity

- To test the latter idea: build theories of gravity whose predictions differ from GR’s on ultra-large scales
  - and reproduce GR predictions where they are well tested

- aim: find “standard” cosmological solutions which flow to de-Sitter
  - explain acceleration as self-acceleration, without fluids with negative pressure

- this may also help with the cosmological constant problem
  - t’Hooft naturalness, degravitation, . . .
Typical problems in modified gravity

Modifying GR always introduces new degrees of freedom

• they have to be screened on lab/astrophysical scales
  – Vainshtein, Chameleon, Symmetron, …

• quite often they are ghosts
  – e.g. BD ghost in massive GR, bending mode in s.a. DGP

• ghost ≡ mode with negative kinetic energy
  – catastrophic instability of the vacuum
2. dRGT Massive Gravity
Linearized GR as a free massless spin-2 field

Take free GR action, expand to 11 order around Minkowski:

\[
S^{(2)}_{GR} = \int d^4x \left( -\frac{1}{2} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda} \partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu} \partial_\nu h + \frac{1}{2} \partial_\lambda h \partial^\lambda h \right)
\]

Pretend \(h_{\mu\nu}\) is just a field on Minkowski, “forget” about GR

- \(h_{\mu\nu}\) transforms in the massless, spin-2 repr. of Lorentz
  - it has 2 propagating degrees of freedom, it is “unique”

- enjoys gauge invariance \(h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu} \xi_{\nu)}\)
  - “unique” quadratic, local, Lorentz inv. action with no more than two derivatives for a symmetric (0,2) tensor

\(S^{(2)}_{GR}\) has very distinct properties besides coming from \(S_{GR}\)!
GR as a self-interacting massless spin-2 field

Expand full free GR action around Minkowski in powers of $h_{\mu\nu}$

$$S_{GR} = S_{GR}^{(2)} + \int d^4x \left( \partial^2 h^3 + \partial^2 h^4 + \ldots + \partial^2 h^n + \ldots \right)$$

Start from $S_{GR}^{(2)}$, and add all possible interaction terms

- $S_{GR}$ “singled out” by imposing that gauge invariance holds
  - full gauge invariance $\equiv$ “deformation” of $h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial(\mu\xi_\nu)$

- Interacting theory still has 2 propagating d.o.f.
  $\rightarrow$ GR $\sim$ self-interacting theory of a massless spin-2 field

- Substitute $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$: Minkowski metric disappears
  - gauge inv. $\Rightarrow$ we can interpret $g_{\mu\nu}$ as a metric
Why considering massive gravity?

- Example: relativistic scalar field \( (\Box - m^2) \phi = 0 \)
  - massless case: \( \phi(r) \propto \frac{1}{r} \) (Coulomb)
  - massive case: \( \phi(r) \propto \frac{e^{-mr}}{r} \) (Yukawa)

  for \( r \ll r_c \equiv 1/m \) essentially no difference, “Coulombian” profile modified for \( r \gtrsim r_c \)

- Idea: give gravity a “finite range” \( r_c \simeq 1/H_0 \) by giving it a mass
  - program: formulate a self-interacting theory of a massive spin-2 field
Free massive spin-2 field: the Fierz-Pauli action

Action for a symmetric (0,2) tensor field on Minkowski

\[ S_{FP}^{(2)} = S_{GR}^{(2)} - \int d^4 x \, \frac{m^2}{2} \left( h_{\mu\nu} h^{\mu\nu} - h^2 \right) \]

- \( h_{\mu\nu} \) transforms in the massive, spin-2 repr. of Lorentz
  - it has 5 propagating degrees of freedom

- it is “unique”: altering \( S_{FP}^{(2)} \) changes number of d.o.f.
  - and generically introduces ghost instabilities

- the FP action does not enjoy gauge invariance
  - the mass term breaks \( h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_{(\mu} \xi_{\nu)} \)
The vDVZ discontinuity

Static, spherically symmetric source: test bodies feel

\[ ds^2 = -\left(1 + 2\, \phi(r)\right) dt^2 + \left(1 - 2\, \psi(r)\right) \delta_{ij} \, dx^i \, dx^j \]

- For \( r \ll r_c \) the grav. potentials correctly scale as \( 1/r \)
  - and decay exponentially for \( r \gtrsim r_c \)
- But: for \( r \ll r_c \), \( \phi(r) = 2\, \psi(r) \), while in GR \( \phi(r) = \psi(r) \)
  - either planet or photon trajectories are wrong (25%) \( \rightarrow \) FP does not reproduce GR’s results on small scales!
- \( m \rightarrow 0 \) sends \( r_c \rightarrow \infty \), but nothing changes for \( r \ll r_c \)
  - \( m \rightarrow 0 \) limit of FP theory is not GR: vDVZ discontinuity
Massive self-interactions: the Vainshtein mechanism

- Consider *non-linear* GR + FP mass term
  - just one of the possible non-linear extensions of FP

- Non-linearities become important at $r_v = \frac{5 \sqrt{GM}}{m^4}$
  - plug linear solution in interaction terms, compare amplitude
  - for $r_s \ll r \ll r_v$, linear approx. is valid in GR and not in MG

- For $r < r_v$, non-linearities restore agreement with GR
  - setting $m \sim H_0$, for the sun $r_v$ bigger than the Milky Way!

- $m \to 0$ implies $r_v \to \infty$: vDVZ discontinuity disappears
Non-linear extensions of FP and the BD ghost

However:

- A generic non-linear extension of FP propagates 6 d.o.f.
  - the sixth degree of freedom is associated with a ghost

- Need to choose interaction terms such that
  - the number of degrees of freedom remains 5
  - Vainshtein mechanism is effective ($r_v$ is model dependent)

- How to do this in practice?
  - theory is not gauge invariant $\rightarrow$ cannot proceed as in GR

- Implement absence of 6th d.o.f., but it is hard. . .
dRGT Massive Gravity

\[ S = \int d^4x \sqrt{-g} \left[ \frac{M_p^2}{2} \left( R[g] - \frac{m^2}{2} \left( \mathcal{U}_2[K] + \alpha_3 \mathcal{U}_3[K] + \alpha_4 \mathcal{U}_4[K] \right) \right) + \mathcal{L}_M[g, \psi_i] \right] \]

\[ \mathcal{K}_{\mu \nu} = \delta_{\mu \nu} - \left[ \sqrt{g^{-1} \cdot \eta} \right]_{\nu}^{\mu} \quad \quad \mathcal{U}_i[K] \supset tr(K^n)^m \]

- At linear order in \( h_{\mu \nu} = g_{\mu \nu} - \eta_{\mu \nu} \), \( S \) reduces to \( S_{FP} \)
- The theory propagates 5 degrees of freedom
  - the Boulware-Deser mode is absent
- The background structure \( \eta_{\mu \nu} \) do not disappear
- The theory does not enjoy gauge invariance
  - \( g_{\mu \nu} \) is a tensor field on Minkowski \( \Leftrightarrow \) absolute space
The Stückelberg trick

“Restore diff. invariance”: construct a different action which is diff. invariant and physically equivalent to the previous one.

- Substitute $\eta_{\mu\nu} \rightarrow \Sigma_{\mu\nu} \equiv \frac{\partial \phi^\alpha}{\partial x^\mu} \frac{\partial \phi^\beta}{\partial x^\nu} \eta_{\alpha\beta}$, $\phi^\alpha$: scalar functions
  - $\Sigma_{\mu\nu}$ transforms as a (0,2) tensor under reparametrizations

- $\phi^\alpha$ encodes how we are positioned in relation to the absolute reference
  - usual to write $\phi^\alpha(x) = x^\alpha - \pi^\alpha(x)$. If $\pi^\alpha(x) = 0$, we are in the absolute reference (called “unitary gauge”)

- Changing coordinates now leave the action invariant in form
  - but excites the Stückelberg fields $\pi^\alpha(x)$
3. The Vainshtein mechanism in dRGT Massive Gravity
Spherically symmetric solutions

- Most general static, spherically symmetric ansatz
  \[ ds^2 = -C^2(r)\, dt^2 + A^2(r)\, dr^2 + 2D(r)\, dt\, dr + B^2(r)\, d\Omega^2 \]

- Two branches of solutions: diagonal \((D = 0)\) and non-diagonal \((D \neq 0)\) (in unitary gauge)
  - the Birkhoff theorem does not hold

- The two branches are physically distinct
  - without Stückelbergs: no reparametrization invariance
  - or: mapping one branch into the other excites Stückelbergs

- Non-diagonal branch has asymptotically non-flat solutions
  - we consider only the diagonal branch
Focusing on the Vainshtein mechanism

- Rescale $r \rightarrow \rho(r)$: $ds^2 = -(1 + n(\rho)) \, dt^2 + (1 - f(\rho)) \, d\rho^2 + \rho^2 d\Omega^2$
  
  - $H(\rho) \equiv B(r(\rho))$ remains in the equations of motion
  - define $1 + h(\rho) \equiv r(\rho)/H(\rho)$

- At the Vainshtein radius $\rho_v = \sqrt[3]{GM/m^2}$ only $h$ goes non-linear ($\sim$ Vainshtein paper)
  
  - if $r_c \gg r_s$, we get the hierarchy $r_s \ll \rho_v \ll r_c$

- We then focus on scales $GM \ll \rho \ll 1/m$
  
  - and take the linear approximation for $n, f$, while keeping all non-linear terms in $h$
The quintic equation

\[ f = -2 \frac{GM}{\rho} - (m\rho)^2 \left( h - \alpha h^2 + \beta h^3 \right) \quad \alpha = 1 + 3 \alpha_3 \]

\[ \dot{h} = 2 \frac{GM}{\rho^2} - m^2 \rho \left( h - \beta h^3 \right) \quad \beta = \alpha_3 + 4 \alpha_4 \]

\[ q(h, \rho; \alpha, \beta) = \frac{3}{2} \beta^2 h^5 - \left( \alpha^2 + 2\beta \right) h^3 + 3 \left( \alpha + \beta \left( \rho_v/\rho \right)^3 \right) h^2 - \frac{3}{2} h \left( \rho_v/\rho \right)^3 = 0 \]

- gravitational potentials trivial to find once we know \( h(\rho) \)
- At fixed \( \rho \), \( h \) obeys an algebraic equation (quintic)
  - impossible to write the general solution in radicals
- Remarkable symmetry connects behaviour at different \( \alpha, \beta \)

\[ q \left( \frac{h}{k}, \sqrt[3]{k} \rho; k \alpha, k^2 \beta \right) = \frac{1}{k} q(h, \rho; \alpha, \beta) \]
**Local solutions, asymptotic and inner**

Local existence of solutions $\iff$ **Implicit Function theorem**:

$$q(h_0, \rho_0) = 0, \quad \frac{\partial q}{\partial h}(h_0, \rho_0) \neq 0 \quad \Rightarrow \quad q(h, \rho) = 0 \iff h = f(\rho) \quad ; \quad f'(\rho_0) = \left. \frac{\partial q}{\partial \rho} \right|_0 \left( \left. \frac{\partial q}{\partial h} \right|_0 \right)^{-1}$$

- Define **shape function** $s_\rho(h) \equiv q(h, \rho)$ (1 parameter family)
  - simple root of $s_\rho(h) \iff$ local solution $h(\rho)$ of the quintic in a neighbourhood of $\rho = \bar{\rho}$

- **Asymptotic solutions** $\rho \to \infty$: two types of solutions
  - “Coulombian” vDVZ solution $(1/\rho)$ ; non-flat solution

- **Inner solutions** $\rho \gtrsim 0$: two types of solutions
  - “Coulombian” GR solution $(1/\rho)$ ; self-shielding solution
Solution matching in the phase space

- Solutions $h(\rho)$ of the quintic $\Leftrightarrow$ flow of simple roots of $s_\rho(h)$
  - Local solution extends until the root remains simple

- Global solutions: root corresponding to an asymptotic sol. flows to the root corresponding to an inner sol.
  - remaining a simple root all the way down

- Strategy: instead of trying to find solutions of the quintic, we study the flow of roots of $s_\rho(h)$
  - study analytically the evolution of $s_\rho(h)$ with $\rho$
  - ask Mathematica for which $\alpha, \beta$ double roots appear
  - plot $s_\rho(h)$ and change $\rho$ “continuously” (Mathematica)
Three asymptotic solutions and one inner solutions, but no global solutions!
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Phenomenology: creation and annihilation of solutions

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Three asymptotic solutions and one inner solutions, but no global solutions!
Shape function and flow of the roots
Phase space diagram of global solutions

- Asymptotically flat Vainshtein solutions: 4 & 5
  - Non asymptotically flat Vainshtein solutions: 5, 6, 7, 8 & 9
- Self-shielding, asympt. flat solutions: all but 2 and 7
  - region 2: NO global solutions
Numerical solutions: Vainshtein mechanism (flat)

Ratio with the gravitational potentials in GR, asymptotically flat spacetime
Numerical solutions: Vainshtein mechanism (non flat)

Ratio with the gravitational potentials in GR, non asymptotically flat spacetime

\[ \frac{f}{f_{GR}} \]

\[ \frac{n^i}{n^i_{GR}} \]

\[ h \]

\[ \rho/\rho_U \]
Numerical solutions: self-shielding

Non asymptotically flat with self-shielding at the origin

Ratio with the GR potentials

The potentials for $\rho \gtrsim 0$
4. Conclusions and present status
Conclusions and present status: sph. symm. sol.

Diagonal branch:

- The Vainshtein mechanism works in a wide part of the phase space
  - we characterized exactly where this happens

- However, all these solutions are unstable
  - self-shielding solutions are ruled out

Non-diagonal branch:

- Also here, solutions are unstable
Conclusions and present status: cosmology

- FLRW solutions exist, mass terms *mimics exactly a CC* in the entire cosmic history
  - background evolution is marvellous

- All these solutions are *unstable*
  - vanishing kinetic terms for vector and scalar perturbations,
    higher order analysis reveals energy unbounded from below
  - ghost-like instability: but no BD ghost (5 d.o.f.)!

- Stable cosmological solutions: inhomogeneous and/or anisotropic
  - our universe may be in a “Vainshtein region” of an inhomogeneous universe until $r_V \sim r_H$
Conclusions and present status: extensions

Plenty of ways to extend the theory. Some examples:

- Non-flat absolute geometry
  - usually de Sitter or Anti-de Sitter

- Massive bi-gravity models
  - by promoting the absolute metric to a dynamical object

- Add scalar fields
  - mass varying Massive Gravity, quasi-dilaton Massive Gravity

- Modify the coupling to matter
  - matter can couple to a mixture of the physical and absolute metrics
Thank you very much!

ArXiv:1406.4550 [hep-th] (review on ghosts)
Scalar-tensor sector of dRGT massive gravity in the decoupling limit ($\Lambda_3 = \sqrt[3]{M_P m^2}$)

$$\mathcal{L} = -\frac{1}{2} h^{\mu\nu} \varepsilon_{\mu\nu}^{\rho\sigma} h_{\rho\sigma} + h^{\mu\nu} X^{(1)}_{\mu\nu} - \frac{\alpha}{\Lambda_3^3} h^{\mu\nu} X^{(2)}_{\mu\nu} - \frac{\beta}{\Lambda_3^6} h^{\mu\nu} X^{(3)}_{\mu\nu} + \frac{1}{M_P} h_{\mu\nu} T_{\mu\nu}$$

where, indicating $\Pi_{\mu\nu} = \partial_\mu \partial_\nu \pi$ and with $[ ]$ the cyclic trace

$$X^{(1)}_{\mu\nu} = [\Pi] \eta_{\mu\nu} - \Pi_{\mu\nu}$$

$$X^{(2)}_{\mu\nu} = ( [\Pi]^2 - [\Pi^2] ) \eta_{\mu\nu} - 2 [\Pi] \Pi_{\mu\nu} + 2 \Pi^2_{\mu\nu}$$

$$X^{(3)}_{\mu\nu} = ( [\Pi]^3 - 3 [\Pi] [\Pi^2] + 2 [\Pi^3] ) \eta_{\mu\nu} - 3 \left( [\Pi]^2 - [\Pi^2] \right) \Pi_{\mu\nu} + 6 [\Pi] \Pi^2_{\mu\nu} - 6 \Pi^3_{\mu\nu}$$

Note that $\pi$ does not have a kinetic term of its own, but is kinetically mixed with $h_{\mu\nu}$. The field redefinition

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \pi \eta_{\mu\nu} - \frac{\alpha}{\Lambda_3^3} \partial_\mu \pi \partial_\nu \pi$$

almost diagonalizes the Lagrangian, leaving one coupling (which cannot be removed by a local field redefinition).
We get

\[ \mathcal{L} = \mathcal{L}_\hat{h} + \mathcal{L}_\pi + \mathcal{L}_{\text{coup}} \]

where

\[ \mathcal{L}_\hat{h} = -\frac{1}{2} \hat{h}^{\mu\nu} \varepsilon_{\rho\sigma} \hat{h}^{\rho\sigma} + \frac{1}{M_P} \hat{h}_{\mu\nu} T^{\mu\nu} \]

\[ \mathcal{L}_\pi = 6 \mathcal{L}_2^{\text{gal}} + \frac{a(\alpha, \beta)}{\Lambda_3^3} \mathcal{L}_3^{\text{gal}} + \frac{b(\alpha, \beta)}{\Lambda_6^6} \mathcal{L}_4^{\text{gal}} + \frac{c(\alpha, \beta)}{\Lambda_9^9} \mathcal{L}_5^{\text{gal}} + \]

\[ + \frac{1}{M_P} \pi T - \frac{\alpha}{M_P \Lambda_3^3} \partial_\mu \pi \partial_\nu \pi T^{\mu\nu} \]

\[ \mathcal{L}_{\text{coup}} = -\frac{\beta}{\Lambda_6^6} \hat{h}^{\mu\nu} X^{(3)}_{\mu\nu}(\pi) \]
Small perturbations around non-trivial spherically symmetric background:

\[ \pi(t, r, \Omega) = \pi_0(r) + \varphi(t, r, \Omega) \]

Effective kinetic term for the perturbations

\[ S_\varphi = \frac{1}{2} \int d^4x \left[ K_t(r)(\partial_t \varphi)^2 - K_r(r)(\partial_r \varphi)^2 - K_\Omega(r)(\partial_\Omega \varphi)^2 \right] \]