Inflationary Cosmology

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Four Lectures

- Lecture I: The Problems of the Hot Big Bang Model
- Lecture II: The Inflationary Solution
- Lecture III: Inflationary Perturbations of Quantum-Mechanical Origin
- Lecture IV: Inflation and Planck
Lecture I: problems of the standard hot Big Bang model
The « hot Big Bang phase » is the standard cosmological model and provides a convincing description of the Universe on a wide range of energy scales.

The model is based on three assumptions:

1- Gravity is described by General Relativity

2- Cosmological principle: the Universe is homogeneous and isotropic (on large scales)

3- Matter/energy is given by different sources to be listed in the following
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1- Gravity is described by General Relativity

\[
G_{\mu\nu} [g_{\mu\nu}] = \frac{8\pi G}{c^4} \sum_{i=1}^{N} T_{\mu\nu}^{(i)}
\]

c, G: Relativistic theory of gravitation
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\[ G_{\mu\nu}[g_{\mu\nu}] = \frac{8\pi G}{c^4} \sum_{i=1}^{N} T_{\mu\nu}^{(i)} \]

This leads to the FLRW metric
Cosmological Principle: the Universe is (on large scales ...) homogeneous and isotropic. Technically, this implies that the metric is of the FLRW type.

\[ ds^2 = -c^2 dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] \]

\[ d\eta^2 = \gamma_{ij}^{(3)} dx^i dx^j = a^2(\eta) \left[ -d\eta^2 + \delta_{ij}^{(3)} dx^i dx^j \right] \]

In conformal time, the same metric reads

\[ \eta = \int \frac{dt}{a(t)} \]

Only one (time-dependent) undetermined function: the scale factor \( a(t) \)
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This leads to the FLRW metric

- The size of the Universe is described by a single function of time, the scale factor \( a(t) \)

- The spatial curvature of the Universe can be zero (flat) positive or negative
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The Einstein equations reduce to ordinary non-linear differential equations

\[
H^2 = \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi G}{c^4} \sum_{i=1}^{N} \rho^{(i)}
\]

\[
\dot{\rho}^{(i)} + 3H \left( \rho^{(i)} + p^{(i)} \right) = 0
\]

Hubble parameter= Expansion rate of the Universe

\( H_0 \approx 70 \text{ km/s/Mpc} \)
1- The last equation is in fact obtained from the conservation of the stress energy tensor of a perfect fluid

\[ T_{\mu\nu} = (\rho + p)u_{\mu} u_{\nu} + pg_{\mu\nu} \]

energy density

pressure

\[ \nabla_{\mu} T^{\mu\nu} = 0 \quad \Rightarrow \quad \dot{\rho} + 3H(\rho + p) = 0 \]

2- Conservation of the total stress energy tensor is equivalent to Einstein equations because of Bianchi identities.
The fact that $a = a(t)$ implies that the Universe is expanding. This expansion is now measured with very high accuracy.

The Universe is measured to be spatially flat.
The standard model

\[ p = \frac{1}{3} \rho, \quad \rho \propto \frac{1}{a^4} \]
\[ a(t) \propto t^{1/2} \]

Radiation
dominated era

Matter dominated era

Dark energy
dominated era

Dark energy

Pressureless matter
dominated era
Hubble radius: $c/H$
characteristic length beyond which the expansion matters

\[ a \propto t^{1/2} \]
\[ H = \frac{1}{2t} \]
\[ \frac{1}{H} = 2t \propto a^2 \]
\[ \ln \left( \frac{1}{H} \right) = \text{Cte} + 2 \ln a \]
The standard model, though it is a simple construction, can account for a large number of observations and/or experimental tests:

- Expansion (Hubble diagram)
- CMB (this school!)
- Nucleosynthesis
- etc ...
The standard model, despite its impressive achievements, suffers from a number of troubling puzzles

- Horizon problem
- Flatness problem
- Origin of the inhomogeneities in our Universe
- etc ...

All this issues are related to the initial conditions
Q1: what is the (proper) distance between the photon and Earth at time \(t\)?

\[
d_{\text{proper}}(t) = a(t) \left[ r_{\text{em}} - \int_{t_{\text{em}}}^{t} \frac{d\tau}{a(\tau)} \right]
\]

Q2: definition of the horizon. At a given reception time, what is the coordinate distance to the furthest point where a "message", sent to us from there (ie "causal contact"), could have reached Earth?

\[
r_{\text{em}} = \int_{t_{\text{em}}}^{t_{\text{reception}}} \frac{d\tau}{a(\tau)}
\]

\(t_{\text{em}}=0=\text{"Big-Bang"}\)

\[
d_H(t_{\text{reception}}) = a(t_{\text{reception}}) \int_{0}^{t_{\text{reception}}} \frac{d\tau}{a(\tau)}
\]
Reminder: the lss

- At the lss, z~1100, the Universe became transparent
- Therefore, this is the furthest event we can “see”
Q3: what is the angular diameter of the horizon at the lss?

$$ds^2 = -c^2 dt^2 + a^2(t) \left( dr^2 + r^2 d\Omega^2 \right)$$

$$D = a(t_{em}) r(t_{em}) \Delta \Omega$$

$$d_H(t_{reception})$$

with t_reception=t_lss

$$r_{em} = \int_{t_{lss}}^{t_{now}} \frac{d\tau}{a(\tau)}$$

emitted at the lss received now

$$d_H(t_{reception}) = a(t_{reception}) \int_{0}^{t_{reception}} \frac{d\tau}{a(\tau)}$$

$$\Delta \Omega = \left[ \int_{0}^{t_{lss}} \frac{d\tau}{a(\tau)} \right] \times \left[ \int_{t_{lss}}^{t_{now}} \frac{d\tau}{a(\tau)} \right]^{-1}$$
- For the case of a radiation dominated era followed by a matter dominated era, the calculation is easy.

- NB: we identify the lss with the equivalence time (but leads to a small error)

\[ \Delta \Omega = \frac{1}{2} \left[ 1 - (1 + z_{\text{lss}})^{-1/2} \right]^{-1} (1 + z_{\text{lss}})^{-1/2} \]

\[ \Delta \Omega \approx 0.5 \times (1 + z_{\text{lss}})^{-1/2} \approx 0.85^\circ \text{ Size of the moon!} \]
What we have just computed ...

$\delta T / T \approx 1$

Low contrast map
What we observe ...

Low contrast map
Despite its impressive achievements, we therefore see that the standard models has "issues".

Why is the Universe so spatially flat??

\[ \Omega < 1 \]
\[ \Omega = 1 \]
\[ \Omega > 1 \]
Beyond the cosmological principle

The real Universe (ie on « small » scales) is of course not homogeneous and isotropic!
Beyond the cosmological principle

- The real Universe is not homogeneous and isotropic!
- The mechanism amplifying the perturbations is gravitational instability.
- Today the inhomogeneities are large but, in the early Universe, they were small. One can therefore work with a linear theory and study the various scales independently in Fourier space.

\[ g_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \bar{x}) \]
\[ \rho(t) + \delta \rho(t, \bar{x}) \]

The perturbations obey the perturbed Einstein equations

\[ \delta G_{\mu\nu} = \frac{8\pi G}{c^4} \sum_{i=1}^{N} \delta T_{\mu\nu}^{(i)} \]
Beyond the cosmological principle

- But the real Universe is not homogeneous and isotropic!

- The mechanism amplifying the perturbations is gravitational instability.

- Today the inhomogeneities are large but, in the early Universe, they were small. One can therefore work with a linear theory and study the various scales independently in Fourier space.

\[ \delta g_{\mu\nu} (t, \vec{x}) = \int d\vec{k} \, \delta g_{\mu\nu} (t, \vec{k}) \, e^{i\vec{k} \cdot \vec{x}} \]

The wavelength of each mode is increasing because the Universe is expanding:

\[ \lambda = \frac{2\pi}{k} a(t) \]
Fourier transform = analyzing “scale by scale”

All the wavelength are outside the Hubble radius in the early Universe

\[ \ln \frac{1}{H} \]
\[ \ln \lambda \]

\[ \lambda = \frac{2\pi}{k} a(t) \]

Slope 2
Slope 3/2
Slope 1

Radiation
Matter

Radiation dominated era
Matter dominated era
But what is the source of the fluctuations??

Observationally, we know that if the initial power spectrum (two point correlation function) is close to scale invariance, then one can reproduce the observations.

\[ \langle \delta g_{\mu\nu}(t_{ini}, \vec{k}) \delta g_{\mu\nu}(t_{ini}, \vec{k}) \rangle \sim k^0 \]

Scale outside the Hubble radius

In the standard model, this is just postulated.

But why is it so??
Lecture II: The inflationary solution
The standard model, despite its impressive achievements, suffers from a number of troubling puzzles

- Horizon problem
- Flatness problem
- Origin of the inhomogeneities in our Universe
- etc ...

All this issues are related to the initial conditions
Inflation is a phase of **accelerated expansion** taking place in the very **early Universe**. The scale factor is such that

\[
\frac{d^2a}{dt^2} > 0
\]

This assumption allows us to solve several problems of the standard hot Big Bang model:

- Horizon problem
- Flatness
- Monopoles problem ...
- Inflation does not replace the Hot Big Bang model. It is a new ingredient which completes the standard model. It takes place before the Hot Big Bang phase.

- The energy scale of inflation is poorly constrained.

\[
\rho_{\text{now}} \simeq 10^{-47}\text{GeV}^4
\]

\[
\rho_{\text{LHC}} \simeq (100\text{GeV})^4
\]

\[
\rho_{\text{inf}} \simeq (10^{16}\text{GeV})^4
\]
A simple model ...

New phase driven by an unknown fluid. Can be switched off by putting N to zero ...

\[ a(t) = a_i \left[ \frac{3}{2} (1 + \omega_X) H_i (t - t_i) + 1 \right]^{2/[3(1+\omega_X)]} \]

\[ \omega_X \equiv \frac{p_X}{\rho_X} \]

\[ \Delta \Omega = \frac{1}{2} \left[ 1 - \left( 1 + z_{lss} \right)^{-1/2} \right]^{-1} (1 + z_{lss})^{-1/2} \left\{ 1 + \frac{1 - 3\omega_X}{1 + 3\omega_X} \frac{1 + z_{lss}}{1 + z_{end}} \left[ 1 - e^{-N(1+3\omega_X)/2} \right] \right\} \]
We need a fluid with a negative pressure such that $p < -\rho/3$

We “just” need a solid angle larger than $4\pi$

$$\Delta \Omega = \frac{1}{2} \left[ 1 - (1 + \zeta_{\text{ls}})^{-1/2} \right]^{-1} (1 + \zeta_{\text{ls}})^{-1/2} \left\{ 1 + \frac{1 - 3\omega_x}{1 + 3\omega_x} \frac{1 + \zeta_{\text{ls}}}{1 + z_{\text{end}}} \left[ 1 - e^{-N(1+3\omega_x)/2} \right] \right\}$$

\[1 + 3\omega_x < 0\]

\[N > -4 + \ln z_{\text{end}}\]
We have acceleration if the pressure is negative!

\[ 1 + 3\omega_x < 0 \quad \rightarrow \quad p_x < -\frac{\rho_x}{3} \]

\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3}\rho
\]

\[-\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = \kappa p \]

\[
\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3p)
\]

\[ \ddot{a} > 0 \]
"Every form of energy weighs in General Relativity"
1- At high energies, field theory is the correct framework to describe matter

2- The Universe is homogeneous and isotropic: spin 0 particle
The action of a self interacting scalar field is given by

\[ S = - \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + V(\Phi) \right] \]

\[ T_{\mu\nu} = - \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \]

\[ T^{\mu}_{\ \nu} = g^{\mu\lambda} \partial_\lambda \Phi \partial_\nu \Phi - \delta^{\mu}_{\ \nu} \left[ \frac{1}{2} g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi + V(\Phi) \right] \]

\[ \rho = \frac{\dot{\Phi}^2}{2} + V(\Phi) \]

\[ p = \frac{\dot{\Phi}^2}{2} - V(\Phi) \]

- The potential energy must dominate
- The potential must be flat in order to have inflation
1- At high energies, field theory is the correct framework to describe matter

2- The Universe is homogeneous and isotropic: spin 0 particle

3- Energy density & pressure

\[ \rho = \frac{\dot{\phi}^2}{2} + V(\phi) \]

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The potential must be flat
1- At high energies, field theory is the correct framework to describe matter

2- The Universe is homogeneous and isotropic: spin 0 particle

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\[ \rho = \frac{\dot{\phi}^2}{2} + V(\phi) \]
\[ p = \frac{\dot{\phi}^2}{2} - V(\phi) \]

4- Inflation stops when the potential is no longer flat enough, ie when

\[ \frac{\ddot{a}}{a} = H^2 (1 - \epsilon_1) \quad \text{with} \quad \epsilon_1 \equiv -\frac{\dot{H}}{H^2} \simeq \frac{M_{\text{pl}}^2}{2} \left( \frac{V_\phi}{V} \right)^2 \]
Inflation is a quasi exponential expansion of spacetime

\[ \rho = \frac{\dot{\phi}^2}{2} + V(\phi) \]

\[ p = \frac{\dot{\phi}^2}{2} - V(\phi) \]

\[ p \simeq -\rho \]

\[ \frac{d\rho}{dt} + 3H(\rho + p) = 0 \simeq \frac{d\rho}{dt} \]

\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{\rho}{3M_{Pl}^2} \]

\[ a(t) \sim e^{Ht} \]

Flatness problem solved!
Inflation is a quasi exponential expansion of spacetime

\[ \rho = \frac{\dot{\phi}^2}{2} + V(\phi) \]

\[ p = \frac{\dot{\phi}^2}{2} - V(\phi) \]

\[ p \simeq -\rho \]

\[ \frac{d\rho}{dt} + 3H(\rho + p) = 0 \simeq \frac{d\rho}{dt} \]

\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{\rho}{3M_{Pl}^2} \]

\[ a(t) \sim e^{Ht} \]
Inflation: the Hubble parameter \( \ell_H \equiv \frac{1}{H} \) is almost constant (exact for de Sitter).

\[
\ln \frac{1}{H} \approx \text{constant}
\]
Inflation: the Hubble parameter of radiation is almost constant (exact for de Sitter).

\[ \ell_H \equiv \frac{1}{H} \sim \text{constant} \]
Inflation: the Hubble parameter $\ell_H \equiv \frac{1}{H} \sim \text{constant}$

One can now choose sensible initial conditions for the cosmological perturbations

$\ln \frac{1}{H}$
Field theory is the correct description at high energies.

A natural realization is a scalar field slowly rolling down its flat potential.

Inflation ends by violation of the slow-roll conditions or by instability.

After inflation, the field oscillates at the bottom of its potential: this is the reheating.
End of Inflation (I)

Slow-roll phase

Oscillatory phase

Violation of Slow-roll
The field oscillates much faster than the Universe expands

\[ \frac{d\rho_{\text{inf}}}{dt} = -\frac{6p}{p+2} H \rho_{\text{inf}} \]

Equation of state

\[ w = \frac{p-2}{p+2} \quad p > 1 \iff -\frac{1}{3} < w < 1 \]

For \( p=2 \)

\[ \phi(t) = \phi_{\text{end}} \left( \frac{a_{\text{end}}}{a} \right)^{3/2} \sin(mt) \]
The previous model cannot describe particle creation.

$\Gamma$ is the inflaton decay rate.

$$\frac{d\rho_{\text{inf}}}{dt} = -\frac{6p}{p+2} H \rho_{\text{inf}} - \frac{2p}{p+2} \Gamma \rho_{\text{inf}}$$

$$\frac{d\rho_{\text{rad}}}{dt} = -4H \rho_{\text{rad}} + \frac{2p}{p+2} \Gamma \rho_{\text{inf}}$$
\[ w_{\text{reh}} = \frac{\rho_{\text{inf}} + \rho_{\text{rad}}/3}{\rho_{\text{inf}} + \rho_{\text{rad}}} \]

End of Inflation (IV)
The first temperature in the Universe is

\[ T_{\text{RH}} \propto g_*^{-1/4} (\Gamma m_{\text{Pl}})^{1/2} \]
End of Inflation (VI)

- Oscillatory phase
- Radiation-dominated era
- Matter-dominated era

Diagram showing the evolution of the Hubble radius over time with different phases marked:
- Oscillatory phase
- Radiation-dominated era
- Matter-dominated era

Expressed in terms of potential $V(\phi)$, with $p=2$ and $p=4$.
Consequences of reheating

- So far we do not know so much on the reheating temperature, ie (can be improved - the upper bound- if gravitinos production is taken into account)

\[ \rho_{\text{end}} > \rho_{\text{reh}} > \rho_{\text{BBN}} \]

- The previous description is a naive description of the infaton/rest of the world coupling. It can be much more complicated.
  - Theory of preheating, thermalization etc ...

- How does the reheating affect the inflationary predictions?
  - It modifies the relation between the physical scales now and the number of e-folds at which perturbations left the Hubble radius
  - Can the oscillations of the inflaton affect the behaviour of the perturbations?
The common way to realize inflation is to assume that there is a scalar field (or several scalar fields) dominating in the early Universe.
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There are plenty of different models

1- Single field inflation with standard kinetic term

\[ \mathcal{L}(\dot{\phi}, \phi) = \frac{\dot{\phi}^2}{2} - V(\phi) \]
The common way to realize inflation is to assume that there is a scalar field (or several scalar fields) dominating in the early Universe.

There are plenty of different models

1- Single field inflation with standard kinetic term

2- Single field with non-standard kinetic term (K-inflation)

\[ \mathcal{L}_{\text{DBI}} (X, \phi) = -T(\phi) \sqrt{1 - \frac{2X}{T(\phi)}} + T(\phi) - V(\phi) \]

\[ X \equiv -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \]
The common way to realize inflation is to assume that there is a scalar field (or several scalar fields) dominating in the early Universe.

There are plenty of different models

1- Single field inflation with standard kinetic term

2- Single field with non-standard kinetic term (K-inflation)

3- Multiple field inflation

Different models are characterized by different potentials; the inflationary trajectory can be complicated.
Lecture III: inflationary perturbations of quantum-mechanical origin
The standard model, despite its impressive achievements, suffers from a number of troubling puzzles

- Horizon problem
- Flatness problem
- Origin of the inhomogeneities in our Universe
- etc ...

All this issues are related to the initial conditions
The mechanism amplifying the perturbations is gravitational instability

What is the source??

Gravitational collapse
- In order to have a more realistic description of the (early) universe (CMB, structure formation ...) one must go beyond the cosmological principle.

- In the early universe, the deviations are small since \( \delta T/T \approx 10^{-5} \). This allows us to use a linear theory.

- The source of these fluctuations will be the unavoidable quantum fluctuations of the coupled gravitational field and matter.

- This leads to inflation’s main success: the production of a scale invariant spectrum.
\[ \frac{\delta T}{T}(\vec{e}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{e}) \quad \rightarrow \quad \left\langle \frac{\delta T}{T}(\vec{e}_1) \frac{\delta T}{T}(\vec{e}_2) \right\rangle = \sum_{\ell=2}^{+\infty} \frac{2\ell + 1}{4\pi} C_\ell P_\ell(\cos \theta) \]

with

\[ C_\ell = \langle a_{\ell m} a_{\ell m}^* \rangle = \int_0^{+\infty} \frac{dk}{k} j_\ell^2(kr_{\text{iss}}) T(k; \theta_{\text{stand}}) \mathcal{P}_\zeta(k; \theta_{\text{reh}}, \theta_{\text{inf}}) \]

Translate 3d into 2d

Inflationary power spectrum

\[ k^3 \left| \zeta_{\vec{k}} \right|^2 \]

Describes the evolution of the perturbations when they re-enter the Hubble radius
From COBE to Planck …
Quantum fluctuations as seeds of CMB anisotropy and large scale structures

\[
\log \left( H^{-1} \right), \log \left( \lambda_k \right)
\]

Initial Quantum State

Squeezed State

Cosmic Microwave Background

\[
\log \left( \frac{a}{a_{in}} \right) \equiv N
\]
Structure formation and inflation

Inflationary power spectrum

\[ \log \left( \frac{1}{H} \right) \]

\[ \log (\lambda_k) \]

\[ \mathcal{P}_\zeta(k; \theta_{\text{reh}}, \theta_{\text{inf}}) \sim k^{n-1} \]

Must be (almost) scale invariant \( n \sim 1 \)

\[ \log \left( \frac{a}{a_{\text{in}}} \right) \equiv N \]
Production of cosmological perturbations in the Early universe is very similar to pair creation in a static electric field $E$

$$S = - \int d^4x \left( \frac{1}{2} \eta^{\alpha\beta} \mathcal{D}_\alpha \phi \mathcal{D}_\beta \phi^* + \frac{1}{2} m^2 \phi \phi^* \right)$$

$\mathcal{D}_\alpha \phi = \partial_\alpha \phi + iq \phi A_\alpha$

One works in the Fourier space

$$\phi(t, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \, \phi_k(t) e^{i\vec{k} \cdot \vec{x}}$$

$$\gamma \equiv \frac{k_\perp^2 + m^2}{qE}$$

$$k_\perp^2 = k_x^2 + k_y^2$$

$$\tau \equiv \sqrt{qEt} - \frac{k_z}{\sqrt{qE}}$$

The frequency is time-dependent: one has to deal with a parametric oscillator.

The inflationary mechanism is a conservative one: similar to the Schwinger effect in QFT

**Schwinger effect**
- Scalar field
- Classical electric field
- Amplitude of the effect controlled by $E$

**Inflationary cosmological perturbations**
- Perturbed metric
- Background gravitational field: scale factor
- Amplitude controlled by the Hubble parameter $H$

\[ g_{\mu\nu} = g_{\mu\nu}^{\text{FLRW}} + \delta g_{\mu\nu}(t, \vec{x}) \]
What are the equations of motion?

Perturbed Einstein equations

\[ \delta G_{\mu\nu} = \kappa \delta T_{\mu\nu} \]

\[ \delta \Gamma = \delta g (\partial g + \cdots) + g (\partial \delta g + \cdots) \]

\[ \delta R = \partial \delta g - \partial \delta g + \delta \Gamma \Gamma + \Gamma \delta \Gamma + \cdots \]

\[ \delta G = \delta R - \frac{1}{2} \delta R g - \frac{1}{2} R \delta g \]

\[ \phi + \delta \phi \]

\[ \dot{\phi} \delta \phi + \frac{dV}{d\phi} \delta \phi + \cdots \]
On top of the classical FLRW background, we have small quantum perturbations of the matter fields, and through Einstein equations, of the metric tensor. We have “gravitational phonons”.

\[
ds^2 = a^2(\eta) \left\{ - (1 - 2\phi) \, d\eta^2 + 2 (\partial_i B) \, dx^i \, d\eta + \left[ (1 - 2\psi) \delta_{ij} + 2 \partial_i \partial_j E + h_{ij} \right] \, dx^i \, dx^j \right\}
\]

**Scalar perturbations**

\[
\Phi_B = \phi + \frac{1}{a} \left[ a (B - E') \right]'
\]

**Mukhanov-Sasaki variable**

\[
v = a \left[ \delta \phi (g^i) + \phi' \frac{\Phi_B}{\mathcal{H}} \right]
\]

\[
\zeta = -\frac{\mu}{2a\sqrt{\epsilon_1}}
\]

**Tensor perturbations or gravitational wave**

\[
\delta^{ij} h_{ij} = 0
\]

\[
\partial^i h_{ij} = 0
\]

**Curvature perturbations**
One uses the fact that the spacelike sections are flat and we Fourier transform the previous quantities.

**Scalar perturbations**

\[
v(\eta, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \int d\vec{k} \mu_s(\eta) e^{i\vec{k} \cdot \vec{x}}
\]

**Gravitational waves**

\[
h_{ij}(\eta, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \frac{1}{a(\eta)} \int d\vec{k} \sum_{s=+,-,\times} p_{ij}^s(\vec{k}) \mu_T^s(\eta) e^{i\vec{k} \cdot \vec{x}}
\]

**Key result:** The amplitudes of scalar and tensor perturbations in GR obey the equation of a parametric oscillator.
The effective frequency is controlled by the scale factor, i.e., by the background gravitational field.

It is different for scalar and tensor. Hence one can expect a different result for $S$ and $T$.

\[
\frac{d^2 \mu_k}{d\eta^2} + \omega^2(k, \eta) \mu_k = 0
\]

Gravitational waves

\[
\omega_T^2(k, \eta) = k^2 - \frac{a''}{a}
\]

scalar perturbations

\[
\omega_S^2(k, \eta) = k^2 - \frac{(a \sqrt{\epsilon_1})''}{a \sqrt{\epsilon_1}}
\]

Fig. 1. Parametric amplification. a) variation of the length of the pendulum, b) increased amplitude of oscillations.
Calculating the two-point correlations: slow-roll parameters

The slow-roll parameters are the “small parameter” of a perturbative calculation of the power spectrum

$$\epsilon_0 \propto H^{-1} \simeq \text{constant}$$

$$\epsilon_{n+1} = \frac{d \ln |\epsilon_n|}{dN}, \quad n \geq 0$$

$$\epsilon_1 = \frac{M_{\text{Pl}}^2}{2} \left( \frac{V_\phi}{V} \right)^2$$

$$\epsilon_2 = 2M_{\text{Pl}}^2 \left[ \left( \frac{V_\phi}{V} \right)^2 - \frac{V_{\phi\phi}}{V} \right]$$
What is the typical time-dependence of the frequency?

- The time-dependence is different from that of the Schwinger effect.

- It encodes the microphysics of inflation.

The effective frequency depends on the detailed shape of the inflaton potential.

\[
\omega_s^2(k, \eta) = k^2 - \frac{2 + 3\epsilon_1 + 3\epsilon_2/2}{\eta^2}
\]

\[
\omega_T^2(k, \eta) = k^2 - \frac{2 + 3\epsilon_1}{\eta^2}
\]
Initial conditions

\[ \frac{\omega_s^2(k, \eta)}{a^2 H^2} = \frac{k^2}{a^2 H^2} - \left( 2 - \epsilon_1 + \frac{3}{2} \epsilon_2 \right) = \left( 2\pi \frac{\ell_H}{\lambda} \right)^2 - \left( 2 - \epsilon_1 + \frac{3}{2} \epsilon_2 \right) \]

Small scales

\[ \omega_s^2(k, \eta) \sim k^2 \]

Large scales

\[ \omega_s^2(k, \eta) \sim \frac{1}{\eta^2} \left( 2 + 3\epsilon_1 + \frac{3}{2} \epsilon_2 \right) \]
Which initial conditions?

\[ \mu''_k + k^2 \mu_k = 0 \]

\[ \mu_k(\eta) \approx \frac{\alpha_k}{\sqrt{2k}} e^{-ik\eta} + \frac{\beta_k}{\sqrt{2k}} e^{ik\eta} \]

Key assumption: one choose the adiabatic vacuum initially

\[ \alpha_k = 1, \quad \beta_k = 0 \]

\[ \mu_k(\eta) \approx \frac{1}{\sqrt{2k}} e^{-ik\eta} \]
\[ k^3 P_\zeta = \frac{H^2}{\pi \epsilon_1 m_{\text{pl}}^2} \left[ 1 - 2(C + 1)\epsilon_1 - C\epsilon_2 - (2\epsilon_1 + \epsilon_2) \ln \frac{k}{k_\text{p}} \right] \]

\[ k^3 P_h = \frac{16 H^2}{\pi m_{\text{pl}}^2} \left[ 1 - 2(C + 1)\epsilon_1 - 2\epsilon_1 \ln \frac{k}{k_\text{p}} \right] \]

- The amplitude is controlled by \( H \)
- For the scalar modes, the amplitude also depends on \( \epsilon_1 \)

The ratio of dp to gw amplitudes is given by

\[ r = \frac{P_h}{P_\zeta} = 16\epsilon_1 \]

Gravitational waves are subdominant

The spectral indices are given by

\[ n_S - 1 \equiv \frac{d \ln P_\zeta}{d \ln k}, \quad n_T \equiv \frac{d \ln P_h}{d \ln k} \]

\[ n_S - 1 = -2\epsilon_1 - \epsilon_2, \quad n_T = -2\epsilon_1 \]

The running, i.e. the scale dependence of the spectral indices, of dp and gw are

\[ \alpha_S \equiv \frac{d^2 \ln P_\zeta}{d (\ln k)^2}, \quad \alpha_T \equiv \frac{d^2 \ln P_h}{d (\ln k)^2} \]

\[ \alpha_S = \mathcal{O} \left( \epsilon^2, \cdots \right), \quad \alpha_T = \mathcal{O} \left( \epsilon^2, \cdots \right) \]
Slow-roll predictions: the example of LF

\[ V(\phi) = M^4 \left( \frac{\phi}{m_{\text{Pl}}} \right)^p \]

\[ \epsilon_1 = \frac{p^2}{2} \frac{M^2}{m_{\text{Pl}}} \frac{\phi^2}{\phi^2} \]

\[ \epsilon_2 = 2p \frac{M^2}{m_{\text{Pl}}} \frac{\phi^2}{\phi^2} \]

\[ \epsilon_1 = \frac{M^2}{m_{\text{Pl}}} \left( \frac{V_\phi}{V} \right)^2 \]

\[ \epsilon_2 = 2M^2 \left[ \left( \frac{V_\phi}{V} \right)^2 - \frac{V_{\phi\phi}}{V} \right] \]

So we get a trajectory in the sr parameters plane
The slow-roll parameters should be evaluated at (pivot scale) Hubble radius crossing:

\[ \Delta N_* = -\frac{1}{M_{Pl}^2} \int_{\phi_*}^{\phi_{end}} \frac{V}{V_\phi} d\phi \]

\[ \frac{\phi_*^2}{M_{Pl}^2} = 4 \left( \Delta N_* + \frac{1}{2} \right) \]

\[ \epsilon_1 = \frac{1}{2(\Delta N_* + 1/2)} \]

\[ \epsilon_2 = \frac{1}{(\Delta N_* + 1/2)} \]
Slow-roll predictions: the example of LFI2

The slow-roll parameters should be evaluated at (pivot scale) Hubble radius crossing

$$\epsilon_1 = \frac{1}{2(\Delta N_* + 1/2)} \quad \epsilon_2 = \frac{1}{(\Delta N_* + 1/2)}$$

1- Changing the expansion during reheating, i.e. $a_{\text{reh}}/a_{\text{end}}$ will change $\Delta N_*$

2- But the possible variation of $a_{\text{reh}}/a_{\text{end}}$ are limited because

$$\rho_{\text{end}} > \rho_{\text{reh}} > \rho_{\text{BBN}}$$

3- Hence the variations of $\Delta N_*$ are also limited, typically $40 < \Delta N_* < 70$

4- A given value of $\Delta N_*$ corresponds to a fixed value of the reheating temperature and of the equation of state during reheating
The inflationary predictions can be represented in the slow-roll plane.

Different values of the parameter “p”

Same parameter “p” but different reheating temperature
Instead of working in the slow-roll plane, one can also work in the observable plane

\[ r = 16\epsilon_1 \]
\[ n_s - 1 = -2\epsilon_1 - \epsilon_2 \]

Different values of the parameter “p”
The amplitude of the fluctuations determine the mass scale of the potential

\[ C_\ell = \langle a_{\ell m} a_{\ell m}^* \rangle = \int_0^{+\infty} \frac{dk}{k} J_\ell^2 (k r_{\text{Ia}}) T (k; \theta_{\text{stand}}) \mathcal{P}_\zeta (k; \theta_{\text{re}}, \theta_{\text{inf}}) \]

\[ C_\ell \simeq \frac{2H^2}{25 \epsilon_1 m_{\text{Pl}}^2} \frac{1}{\ell (\ell + 1)} \]

\[ C_2 = \mathcal{O}(1) \left( \Delta N_* + \frac{1}{2} \right)^2 \left( \frac{m}{m_{\text{Pl}}} \right)^2 \]

\[ m \simeq 10^{-5} m_{\text{Pl}} \]

\[ \mathcal{P}_\zeta \simeq \frac{H^2}{\pi \epsilon_1 m_{\text{Pl}}^2} \]

**NB:**

\[ C_2 = \frac{4\pi Q^2}{5 T^2} \quad \text{with} \quad Q = 18 \times 10^{-6} K \]

\[ T = 2.7 K \]
Lecture IV: inflation after Planck
Planck results in brief:

$$100 \Omega_K = -0.05^{+0.65}_{-0.66}$$

$$\alpha_{RCIDI}^{(2,2500)} \in [-0.093, 0.014]$$

$$n_S = 0.9603 \pm 0.0073$$

$$\frac{d n_S}{d \ln k} = -0.0134 \pm 0.009$$

$$f_{NL}^{\text{loc}} = 2.7 \pm 5.8$$

$$f_{NL}^{\text{eq}} = -42 \pm 75$$

$$f_{NL}^{\text{ortho}} = -25 \pm 39$$

Flat universe with adiabatic, Gaussian and almost scale invariant fluctuations
<table>
<thead>
<tr>
<th>Physical Models</th>
<th>Single Field slow-roll</th>
<th>Single Field with Features (ie non slow-roll)</th>
<th>Single Field with non-canonical kinetic terms</th>
<th>Multi field</th>
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<td>Scalar power spectrum</td>
<td>$n_S \sim 1$</td>
<td>![Green Check Mark]</td>
<td>![DANGER]</td>
<td>![DANGER]</td>
</tr>
<tr>
<td>$\alpha_S \sim 0$</td>
<td>![Green Check Mark]</td>
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<td>Entropic &amp; adiabatic perturbations</td>
<td>$\mathcal{I} \ll \mathcal{R}$</td>
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<td>Gravity waves</td>
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<td>Non-Gaussianities compatible with zero</td>
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<td>![DANGER]</td>
<td>![DANGER]</td>
</tr>
</tbody>
</table>
Understanding the \((n_s, r)\) space

\[
\frac{d \left( \frac{\dot{\phi}^2}{2} \right)}{dt} = H \frac{\dot{\phi}^2}{2} (\epsilon_2 - 2\epsilon_1)
\]

\[\epsilon_2 > 2\epsilon_1 : \quad \frac{\dot{\phi}^2}{2} \uparrow\]

\[\epsilon_2 < 2\epsilon_1 : \quad \frac{\dot{\phi}^2}{2} \downarrow\]

\[
\frac{d}{dt} \left( \frac{\dot{\phi}^2/2}{\rho} \right) = \frac{2H}{3} \epsilon_1 \epsilon_2
\]

\[\epsilon_2 < 0 : \quad \frac{\dot{\phi}^2}{\rho} \downarrow\]

\[\epsilon_2 > 0 : \quad \frac{\dot{\phi}^2}{\rho} \uparrow\]
Understanding the \((n_s, r)\) space
Constraining models: from WMAP to Planck

\[ V(\phi) = M^4 \left( \frac{\phi}{M_{Pl}} \right)^p \]
Constraining models: from WMAP to Planck
Constraining models: from WMAP to Planck
Constraining models: from WMAP to Planck
Category 1 is the category chosen by Planck

Plateau inflation
So … where do we stand?

- Define “model 1 is better than model 2”: Bayesian evidence.
- Apply this definition to the complete slow-roll landscape, ie we have to scan all single field slow-roll models, one by one, in an industrial way and study their predictions and how they perform: Planck data = big data era
- Establish a complete ranking of all these models: model comparison

- Single field slow-roll models is the favored class of models given the Planck data and the data prefers category 1.
- But this still leaves us with hundreds of scenarios and this does not tell us what is THE best model among those scenarios?
arXiv:1303.3787

≈ 74 models

≈ 700 slow roll formulas

≈ 365 pages

*Encyclopedia Inflationaris*

The encyclopedia contains the slow-roll treatment and comparison to the Planck data for all slow-roll models: **this is not a review paper!**
The ASPIC library provides all the numerical codes for all models.
A few examples
A few examples
A few examples
A few examples
A few examples
A few examples
A few examples
A few examples
A few examples
A few examples
A few examples
A few examples
A few examples
A few examples

SSBI4

$w = 0$

$\beta = -10^{-4}$
A few examples
A few examples
For model comparison, we compute the Bayesian evidence (integral of the likelihood over all parameter priors—probability of a model), i.e., the probability of a model, for each inflationary scenario.

How to do model comparison?

Bayesian evidence of the model “i”

$$\frac{p(M_i|D)}{p(HI|D)} = B_{i-HI}$$

Bayesian evidence of the reference model = Starobinsky model

- Posterior odds
  - If $B_{i-HI} > 1$, Model “i” is better than HI
  - If $B_{i-HI} < 1$, HI is better than model “i”
First calculation of inflationary Bayesian evidence in 2010

\[ \frac{p(M_i|D)}{p(HI|D)} = B_{i-HI} \quad \ln(B_{i-HI}) = 0 \]

Weak evidence

\[ \ln(B_{i-HI}) < 0 \quad B_{i-HI} < 1 \]

Model “i” is better than HI

\[ \ln(B_{i-HI}) > 0 \quad B_{i-HI} > 1 \]

Model “i” is better than HI

Bad \rightarrow Good

HI: Starobinsky model

Jeffrey’s scale

Different models

Strong evidence

Moderate evidence

Weak evidence

Inconclusive
Planck team calculation of Bayesian evidence

Bayesian Evidences $\log(\mathcal{E}/\mathcal{E}_{HI})$ for the inflationary models

One more model. Natural inflation
Bayesian evidences for all models

Bayesian Evidences $\log(\mathcal{E}/\mathcal{E}_{HI})$

Schwarz-Terrero-Escalante Classification:

1 2 3 4 5 6 7

J. Martin, C. Ringeval, R. Trotta, V. Vennin

ASPIC project
ASPIC evidence: good models

Bayesian Evidences $\ln(\mathcal{E}/\mathcal{E}_{\text{HI}})$ and $\ln(L_{\text{max}}/\mathcal{E}_{\text{HI}})$
Statistics

Summary

26 % inconclusive zone

21 % weak zone

18 % moderate zone

34 % strong zone

15 different potentials in the inconclusive zone
Planck has identified the shape of the inflaton potential:

Plateau inflation

NB: the difference between these models is “inconclusive”.

And the winners are ...

- MHI
- RGI
- HI
- SFI
- ESI
- KMIII
inflation is in good shape after Planck 2013

The data indicate that we deal with the simplest, i.e. non exotic, version of inflation: single field slow roll model with minimal kinetic term

The shape of the potential is constrained: plateau inflation

There are ~9 models that have a very good Bayesian evidence and a number of unconstrained parameters between zero and one. We have come a long road ... hundreds of models, Planck has identified ~9 favored scenarios!

Interestingly enough, most of these models are well justified from high energy physics & string theory
And the winners are …

Displayed Models: 182/182

Number of unconstrained parameters

\[ \ln(\varepsilon/\varepsilon_{\text{th}}) \]